G-design with Six Vertices and Eight Edges

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Abstract: Let K_v be a complete graph with v vertices, G be a simple subgraph without isolate vertices of K_v. A G-design of K_v, denoted by $(v, G, 1) - GD$, is a pair (X, B), where X is the vertex set of K_v, and B is the collection of sub-graphs (called blocks) of K_v, such that each block is isomorphic to G, and any edge in K_v occurs in exactly one block. The existence of three types of G_i-design $(i = 1,2,3)$ where G_i with six vertices and eight edges will be discussed, that is, $(v, G_i, 1) - GD$ $(i = 1, 2, 3)$ exists iff $v \equiv 0, 1 \bmod (16)$ and $\geqslant 16$.

Key words: graph design, holey graph design, PBD-closure

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0 Introduction

Let K_v be a complete graph with v vertices, G be a simple subgraph without isolate vertices of K_v. A G-design of K_v, denoted by $(v, G, 1)$-GD, is a pair (X, B), where X is the vertex set of K_v, and B is the collection of sub-graphs (called blocks) of K_v, such that each block is isomorphic to G, and any edge in K_v occurs in exactly one block.

Let $K_{n_1, n_2, \cdots, n_h}$ be a complete graph with vertex set $X = \bigcup_{1 \leqslant i \leqslant h} X_i$, where X_i $(1 \leqslant i \leqslant h)$ are disjoint sets with $|X_i| = n_i$ and where two vertices x and y from different sets X_i and X_j are joined by exactly one edge from x and y. A holey G-design (briefly denoted by $(v, G, 1) - HD$) is a pair (X, A) where X is a v-set and A is a set of sub-graphs (called blocks) isomorphic to G, which form an edge-disjoint decomposition of $K_{n_1, n_2, \cdots, n_h}$. Each X_i $(1 \leqslant i \leqslant h)$ is called a hole (or group) of design and the multi-set $\{n_1, n_2, \cdots, n_h\}$ is called the type of the design. For a $(v, G, 1) - HD$, we use an "exponential" notation to describe its type: a type $1^i 2^r 3^k \cdots$ denotes i occurrences of 1, r occurrences of 2, etc.

In particulary, a $(v, \omega, G, 1) - ID$ can be viewed as an HD with type of $1^{v-\omega}\omega^1$ and a $(v, G, 1) - HD$ of type 1^v is essentially a $(v, G, 1) - GD$.

Let K be a set of positive integers. Then a pairwise balanced design $PBD[v, K]$ of order v with block sizes from K is a pair (V, B), where V is a finite set (the point set) of cardinality v and B is a family of subsets (called blocks) of V which satisfy the following properties:

(i) if $B \in B$, then $|B| \in K$;

(ii) every pair of distinct elements of V occurs in exactly one block of B.

A set S of positive integers is said to be PBD-closed if the existence of a PBD$[v, S]$ implies that v belongs to S. Let K be a set of positive integers and let $B(K) = \{v \mid \exists\, PBD[v, K]\}$. Then $B(K)$ is a PBD-closed set called the closure of K.

The necessary conditions for the existence of a $(v, G, 1) - GD$ were developed in [3], namely,

Lemma 0.1 Let G be a graph with k vertices and e edges. If a $(v, G, 1) - GD$ exists, then

(i) $v \geqslant k$,

(ii) $v(v-1) \equiv 0 (\bmod 2e)$,

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(iii) $v-1\equiv 0 \pmod d$,

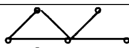
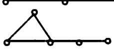
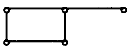
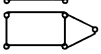
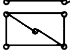
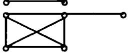
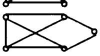
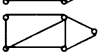
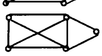
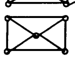
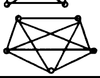
where d is the greatest common divisor of the degrees of the vertices of G.

The problem of existence $(v,G,1)-GD$ has some results in the form of the following theorems (see [3,4]).

Theorem 0.1 The necessary conditions for the existence of $(v,G,1)-GD$ are also sufficient for $k\leqslant 4$.

Theorem 0.2 The necessary conditions for the existence of $(v,G,1)-GD$ are also sufficient for $k=5$ except for some subgraphs (listed in Table 1).

Table 1

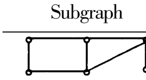
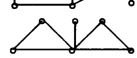
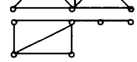
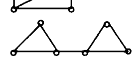
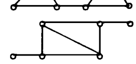
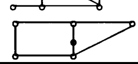
Subgraph	Necessary Condition	Result on Sufficient Condition
	$v\equiv 0$ or $1 \pmod 5$	$v\equiv 0$ or $1 \pmod 5$, $v\geqslant 6^{+}$
	$v\equiv 0$ or $1 \pmod 5$	$v\equiv 0$ or $1 \pmod 5$, $v\geqslant 6^{+}$
	$v\equiv 0$ or $1 \pmod 5$	$v\equiv 0$ or $1 \pmod 5$, $v>6^{+}$
	$v\equiv 0,1,4$ or $9 \pmod{12}$	$v\equiv 0,1,4$ or $9 \pmod{12}$ except possibly when $v=24^{x}$
	$v\equiv 0,1,4$ or $9 \pmod{12}$	$v\equiv 0,1,4$ or $9 \pmod{12}$, $v\geqslant 13^{+}$
	$v\equiv 0$ or $1 \pmod 7$	$v\equiv 0$ or $1 \pmod 7$ $v\neq 7,8$, except possibly when $v=119,120,147,203,204^{x}$
	$v\equiv 0$ or $1 \pmod 7$	$v\equiv 0$ or $1 \pmod 7$ $v\neq 8,14$, except possibly when $v=36,42,56,92,98,120^{x}$
	$v\equiv 0$ or $1 \pmod 7$	$v\equiv 0$ or $1 \pmod 7$ $v\neq 8$
	$v\equiv 0$ or $1 \pmod 7$	$v\equiv 17,33,49,97,113$ and 117^{x}
	$v\equiv 0$ or $1 \pmod{16}$	$v\equiv 1 \pmod{16}$ $v\neq 16$, $v=64^{x}$
	$v\equiv 0$ or $1 \pmod{16}$	
	$v\equiv 0$ or $1 \pmod 9$	$v\neq 9,10,18$, $v=19^{x}$

Note: +: The necessary condition are not always sufficient. x: Complete solution has not been obtained.

For the case $k=6$, Jianxing Yin and Busheng Gong (1998) first investigated the existence of G-design in [6] and gave the existence spectrum for $3\leqslant e(G)\leqslant 6$. For $e(G)=7$, Zihong Tian and Qingde Kang (2002) in [7] discussed the existence of $(v,G,\lambda)-GD$ where $G=K_{2,3}+e$, and the complete solution was given by Aiqing Xu (2003) in [1] and her paper [2]. So we have the following theorem.

Theorem 0.3 For $e(G)\leqslant 7$ the necessary conditions for the existence of $(v,G,\lambda)-GD$ where $k=6$ are sufficient except for some subgraphs (listed in Table 2).

Table 2

Subgraph	Result
	$v\equiv 0$ or $1 \pmod 7$, $v\geqslant 7$ and $v\neq 8$
	$v\equiv 0$ or $1 \pmod 7$, $v\geqslant 7$ and $v\neq 8$
	$v\equiv 0$ or $1 \pmod 7$, $v\geqslant 7$ and $v\neq 8$
	$v\equiv 0$ or $1 \pmod 7$, $v\geqslant 7$ and $v\neq 7$
	$v\equiv 0$ or $1 \pmod 7$, $v\geqslant 7$ and $v\neq 7$
	$v\equiv 0$ or $1 \pmod 7$, $v\geqslant 7$ and $v\neq 7$
	$v\equiv 0$ or $1 \pmod 7$, $v\geqslant 7$ and $v\neq 7$

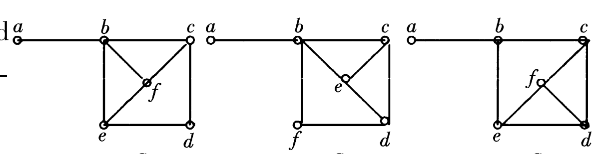
In this article, we shall show the necessary conditions $v\equiv 0,1 \bmod (16)$ and $v\geqslant 16$ for the existence of three types of G_i-design $(i=1,2,3)$ (as the followings) where G_i with six vertices and eight edges are also sufficient.

1 Some direct constructions

Abelian groups play an important role in our direct constructions. For most of our direct constructions, we adapt the familiar method using difference sets as in the construction of BIBDs, where we use finite abelian groups to generate the set of blocks for a given design. That is, instead of listing all the blocks of the design, we shall list a set of base blocks and generate the others by an additive group and perhaps some further automorphisms. If G is the additive group under consideration, then we shall adapt the following convention:

$\operatorname{dev} B=\{B+g: B\in B$ and $g\in G\}$,

where B is the collection of base blocks of the design. And for convenience we use the notation (a,b,c,d,e,f) to denote the graph G_i for $i=1,2,3$.



G_1 G_2 G_3

Lemma 1.1 There exists $(v,G_i,1)-GD$ where $i=$

$1,2,3$ for $v = 16,17$.

Proof For G_1, when $v = 16$, take $X = Z_{15} \cup \{\infty\}$ and develop the following base block under Z_{15}:

$(\infty,0,1,3,7,10)$.

when $v = 17$, take $X = Z_{17}$ and develop the following base block under Z_{17}:

$(0,1,3,6,13,9)$.

For G_2, when $v = 16$, take $X = Z_{15} \cup \{\infty\}$ and develop the following base block under Z_{15}:

$(\infty,0,1,3,6,7)$.

when $v = 17$, take $X = Z_{17}$ and develop the following base block under Z_{17}:

$(0,1,3,6,14,11)$.

For G_3, when $v = 16$, take $X = Z_{15} \cup \{\infty\}$ and develop the following base block under Z_{15}:

$(\infty,0,1,3,7,13)$.

when $v = 17$, take $X = Z_{17}$ and develop the following base block under Z_{17}:

$(0,1,3,9,4,13)$.

Lemma 1.2 There exists $(v, G_i, 1) - GD$ where $i = 1,2,3$ for $v = 32,33$.

Proof For G_1, when $v = 32$, take $X = Z_{31} \cup \{\infty\}$ and develop the following base blocks under Z_{31}:

$(\infty,7,2,11,30,1)$, $(0,15,5,23,12,19)$.

when $v = 33$, take $X = Z_{33}$ and develop the following base blocks under Z_{33}:

$(0,11,32,2,6,26)$, $(0,2,25,11,19,26)$.

For G_2, when $v = 32$, take $X = Z_{31} \cup \{\infty\}$ and develop the following base blocks under Z_{31}:

$(\infty,29,15,19,9,11)$, $(0,29,26,21,14,30)$.

when $v = 33$, take $X = Z_{33}$ and develop the following base blocks under Z_{33}:

$(0,27,16,8,11,7)$, $(0,4,11,13,23,22)$.

For G_3, when $v = 32$, take $X = Z_{31} \cup \{\infty\}$ and develop the following base blocks under Z_{31}:

$(\infty,5,11,21,8,6)$, $(0,9,23,24,16,12)$.

when $v = 33$, take $X = Z_{33}$ and develop the following base blocks under Z_{33}:

$(0,11,32,2,6,26)$, $(0,2,25,11,19,26)$.

Lemma 1.3 There exists $(v, G_i, 1) - HD$ type of (16^k) where $i = 1, 2,3$ for $k = 3,4,5,6,8$.

Proof For G_1, when $k = 3$, take $X_i = \{0 + i, 3 + i, \cdots, 45 + i\}$ for $0 \leqslant i \leqslant 2$ and develop the following base blocks under Z_{48}:

$(0,37,39,35,42,20)$, $(0,28,38,6,14,15)$.

when $k = 4$, take $X_i = Z_{16} \times \{i\}$ for $0 \leqslant i \leqslant 2$, $X_3 = Z_{16} \times \{\infty\}$ and develop the following base blocks under $Z_{16} \times Z_3$:

$((8,0),(4,2),(11,\infty),(6,0),(2,1),(5,0))$,
$((14,0),(4,1),(7,\infty),(13,0),(13,2),(5,0))$,
$((5,0),(0,\infty),(8,2),(9,\infty),(12,2),(1,1))$,
$((0,0),(0,\infty),(2,2),(13,1),(7,2),(4,1))$.

when $k = 5$, take $X_i = \{0 + i, 5 + i, \cdots, 75 + i\}$ for $0 \leqslant i \leqslant 4$ and develop the following base blocks under Z_{80}:

$(0,79,56,20,38,72)$, $(0,22,33,45,49,16)$,
$(0,9,75,21,58,37)$, $(0,29,5,8,10,77)$.

when $k = 6$, take $X_i = Z_{16} \times \{i\}$ for $0 \leqslant i \leqslant 4$, $X_5 = Z_{16} \times \{\infty\}$ and develop the following base blocks under $Z_{16} \times Z_5$:

$((0,0),(13,2),(12,3),(8,2),(13,3),(4,4))$,
$((0,0),(11,\infty),(4,1),(14,2),(15,3),(2,0))$,
$((0,0),(4,3),(8,\infty),(6,0),(0,4),(3,0))$,

$((0,0),(8,2),(3,3),(9,1),(7,\infty),(1,4))$,
$((0,0),(1,\infty),(4,0),(7,\infty),(7,0),(9,3))$,
$((0,0),(10,3),(1,2),(1,4),(13,2),(12,0))$.

when $k=8$, take $X_i = Z_{16} \times \{i\}$ for $0 \leqslant i \leqslant 6$, $X_7 = Z_{16} \times \{\infty\}$ and develop the following base blocks under $Z_{16} \times Z_7$:

$((8,3),(1,4),(9,\infty),(5,4),(11,6),(14,1))$,
$((3,1),(5,3),(4,6),(9,4),(0,1),(4,4))$,
$((1,4),(15,\infty),(6,2),(12,\infty),(10,1),(14,3))$,
$((13,\infty),(14,6),(1,3),(15,2),(0,5),(1,0))$,
$((7,4),(12,0),(12,6),(0,0),(6,3),(0,4))$,
$((15,4),(10,3),(1,6),(3,2),(12,0),(9,2))$,
$((6,5),(15,4),(4,3),(10,4),(8,6),(12,5))$,
$((0,2),(3,\infty),(12,5),(6,\infty),(6,3),(7,1))$.

For G_2, when $k=3$, take $X_i = \{0+i, 3+i, \cdots, 45+i\}$ for $0 \leqslant i \leqslant 2$ and develop the following base blocks under Z_{48}:

$(0,43,27,34,5,26)$, $(0,46,33,22,47,42)$.

when $k=4$, take $X_i = Z_{16} \times \{i\}$ for $0 \leqslant i \leqslant 2$, $X_3 = Z_{16} \times \{\infty\}$ and develop the following base blocks under $Z_{16} \times Z_3$:

$((10,1),(4,\infty),(2,2),(7,1),(7,0),(13,2))$,
$((11,2),(8,0),(2,1),(3,\infty),(0,2),(10,1))$,
$((10,\infty),(10,0),(6,2),(13,0),(5,\infty),(1,2))$,
$((10,1),(13,2),(3,\infty),(15,2),(14,1),(11,\infty))$.

when $k=5$, take $X_i = \{0+i, 5+i, \cdots, 75+i\}$ for $0 \leqslant i \leqslant 4$ and develop the following base blocks under Z_{80}:

$(0,53,60,66,17,75)$, $(0,77,65,57,36,75)$,
$(0,26,58,30,72,13)$, $(0,47,23,42,46,58)$.

when $k=6$, take $X_i = Z_{16} \times \{i\}$ for $0 \leqslant i \leqslant 4$, $X_5 = Z_{16} \times \{\infty\}$ and develop the following base blocks under $Z_{16} \times Z_5$:

$((1,1),(7,0),(3,3),(11,1),(14,\infty),(4,\infty))$,
$((8,1),(14,3),(0,0),(11,3),(4,1),(0,4))$,
$((15,4),(6,2),(2,\infty),(8,1),(4,3),(5,3))$,
$((4,4),(11,3),(2,4),(4,2),(10,\infty),(8,1))$,
$((6,4),(14,0),(2,\infty),(0,0),(1,1),(11,1))$,
$((0,3),(0,1),(0,2),(5,0),(15,4),(5,\infty))$.

when $k=8$, take $X_i = Z_{16} \times \{i\}$ for $0 \leqslant i \leqslant 6$, $X_7 = Z_{16} \times \{\infty\}$ and develop the following base blocks under $Z_{16} \times Z_7$:

$((9,6),(14,\infty),(1,0),(3,5),(0,3),(15,1))$,
$((14,\infty),(11,6),(8,5),(5,2),(15,\infty),(4,1))$,
$((6,1),(11,6),(13,1),(5,6),(4,4),(15,0))$,
$((7,2),(15,3),(12,5),(6,1),(3,2),(15,\infty))$,
$((0,3),(1,2),(2,\infty),(6,1),(0,6),(14,5))$,
$((15,0),(10,1),(4,5),(9,2),(11,6),(14,4))$,
$((1,3),(11,2),(13,5),(8,\infty),(2,1),(0,3))$,
$((0,5),(14,6),(2,4),(2,3),(2,1),(12,5))$.

For G_3, when $k=3$, take $X_i = \{0+i, 3+i, \cdots, 45+i\}$ for $0 \leqslant i \leqslant 2$ and develop the following base blocks under Z_{48}:

(0,46,32,13,8,33),　(0,8,24,13,30,17).

when $k=4$, take $X_i=Z_{16}\times\{i\}$ for $0\leqslant i\leqslant 2$, $X_3=Z_{16}\times\{\infty\}$ and develop the following base blocks under $Z_{16}\times Z_3$:

((0,1),(0,∞),(1,1),(2,∞),(10,2),(0,0)),
((0,∞),(5,1),(1,∞),(3,0),(0,2),(7,1)),
((0,1),(4,∞),(1,1),(8,∞),(11,2),(3,0)),
((0,0),(0,1),(2,2),(6,1),(5,2),(15,0)).

when $k=5$, take $X_i=\{0+i,5+i,\cdots,75+i\}$ for $0\leqslant i\leqslant 4$ and develop the following base blocks under Z_{80}:

(0,8,72,58,32,76),　(0,47,8,17,45,51),
(0,51,24,2,70,1),　(0,59,62,30,72,79).

when $k=6$, take $X_i=Z_{16}\times\{i\}$ for $0\leqslant i\leqslant 4$, $X_5=Z_{16}\times\{\infty\}$ and develop the following base blocks under $Z_{16}\times Z_5$:

((0,3),(0,∞),(1,4),(2,∞),(4,1),(9,0)),
((0,3),(0,1),(2,∞),(5,2),(11,3),(12,0)),
((0,1),(5,3),(1,4),(0,1),(0,2),(14,3)),
((0,4),(9,2),(1,4),(0,0),(2,1),(4,3)),
((0,1),(10,4),(0,3),(4,4),(15,∞),(7,2)),
((0,∞),(13,0),(12,4),(0,∞),(9,3),(6,1)).

when $k=8$, take $X_i=Z_{16}\times\{i\}$ for $0\leqslant i\leqslant 6$, $X_7=Z_{16}\times\{\infty\}$ and develop the following base blocks under $Z_{16}\times Z_7$:

((9,0),(13,5),(8,1),(2,5),(8,6),(10,2)),
((6,5),(2,3),(1,0),(13,4),(4,1),(15,∞)),
((15,4),(5,2),(6,∞),(10,1),(7,4),(3,3)),
((14,6),(9,3),(5,6),(6,0),(3,4),(11,∞)),
((14,2),(8,0),(8,∞),(14,1),(13,2),(15,6)),
((2,4),(5,5),(2,6),(12,2),(12,0),(3,1)),
((14,6),(15,3),(12,1),(7,0),(15,2),(15,6)),
((0,∞),(1,4),(8,∞),(4,3),(15,5),(11,6)).

2 Recursive construction method

In this section, we use PBD-closure and filling in holes (or groups) technics to obtain our result.

Lemma 2.1[5] If $K_3=\{v\mid v\geqslant 3\}$, $E=\{3,4,5,6,8\}$, then $B(E)=K_3$.

Lemma 2.2[2] (Filling in groups) Suppose that the following designs exist:

(i) a $(v,G,1)-HD$ of type $(g_1,g_2,\cdots,g_h)$;

(ii) a $(g_i,G,1)-GD$ and a $(g_i+1,G,1)-GD$ for $(1\leqslant i\leqslant h)$.

Then there exists a $(v,G,1)-GD$ and a $(v+1,G,1)-GD$.

Combining the above results we have

Theorem 2.1 The necessary conditions $v\equiv 0,1 \bmod (16)$ and $v\geqslant 16$ for the existence of $(v,G_i,1)-GD$ for $v=1,2,3$ are also sufficient.

Proof By applying Lemma 0.1, 1.1, 1.2, 1.3, 2.1 and 2.2, we can obtain the result.

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The Discrimination Method to Optimization Design of Light Line Illumination

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Abstract: In this paper, we discuss an important problem of vehicle manipulation the optimization design of light illumination. Using discrimination method, we get the optimization length. On the basis of this result, we simulate the reflection point on the test screen, which is the reflection of the light line illumination. Then draw the area of the whole reflection points by Matlab. Finally,we analysis the feasibility of the model and give the suggestion to the modification of the model.

Key words: Discrimination,C++,Matlab,Mathematica,Maple

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关于六点八边图的图设计

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[摘要] 设 K_v 是一个 v 点的完全图. G 为一个不含孤立点的简单图. K_v 的一个 G-设计,常记为 $(v,G_i,1)-GD$,是指一个二元组 (X,B),其中 X 为 K_v 的顶点集,B 是 K_v 的一些子图(亦称为区组)构成的集合,使得每一个区组与 G 同构,且 K_v 的任何一条边恰在 B 的一个区组中出现.本文讨论了三类六点八边图 $(v,G_i,1)-GD$ $(i=1,2,3)$ 的图设计存在问题,即 $(v,G_i,1)-GD$ $(i=1,2,3)$ 存在的充要条件是 $v\equiv 0,1 \bmod (16)$ 且 $v\geqslant 16$.

[关键词] 图设计,带洞图设计,PBD-闭集

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