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Limit Cycle Problem of Quadratic System of Type $(\mathrm{III})_{m=0}$, (II)

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Abstract: Continuing the study in paper [1], the limit cycle problem for the quadratic system $(\mathrm{III})_{m=0}$ is discussed in this paper for the case $0 < l < \frac{1}{2}$, $n < 0$, and we prove that the system has at most one limit cycle under certain conditions.

Key words: Quadratic system, Limit cycle, Uniqueness

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0 Introduction

Continuing the study of paper [1], we consider the quadratic system of type $(\mathrm{III})_{m=0}$. Without loss of generality, we may assume that $b = -1$, then the system to be considered is taken in the form

$$\frac{dx}{dt} = -y + \delta x + lx^2 + ny^2, \quad \frac{dy}{dt} = x(1 + ax - y). \tag{1}$$

As in [1], without loss of generality, we fix $a < 0$. The cases of $l < \frac{1}{2}$, $0 < n < 1$ and $l < 0$, $n > 1$ has been considered in [1]. We now consider the case

$$0 < l < \frac{1}{2}, \quad n < 0. \tag{2}$$

Assume further that

$$na^2 + l < 0. \tag{3}$$

System (1) has four singular points $O(0,0)$, $N(0,\frac{1}{n})$, $S_1(x_1, y_1)$ and $S_2(x_2, y_2)$, where $y_1 < 0 < y_2$, and x-coordinates of $S_i\,(i = 1,2)$ satisfy the equation

$$F(x) = (na^2 + l)x^2 + (\delta + 2na - a)x + n - 1 = 0.$$

Hence $N(0,\frac{1}{n})$ and $S_2(x_2, y_2)$ are saddles, but $O(0,0)$ and $S_1(x_1, y_1)$ are antisaddles. O is a focus if $|\delta| < 2$ and a node if $|\delta| \geqslant 2$. Thus we only need to discuss $|\delta| < 2$, since no limit cycles around the node.

The focal values of system $(1)_{\delta=0}$ at $O(0,0)$ are

$$W_1 = -a(2l - 1), \quad W_2 = 0, \quad W_3 = 0. \tag{4}$$

Since $a < 0$ and $l < \frac{1}{2}$, O is a stable weak focus of order one. It is easy to see that if $\delta \leqslant 0$, system (1) has no limit cycles (see [1], Th.1), and for $0 < \delta \ll 1$, $O(0,0)$ becomes to be unstable and a stable limit cycle bifurcates from O by the Andronov-Hopf-bifurcation. The aim of this article is to prove that system (1) has at most one limit cycle for all $\delta > 0$.

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1 Main Result

In this section we will prove that under certain conditions system (1) has at most one limit cycle surrounding O.

Lemma 1 System (1) can be reduced to a system of Liénard type

$$\frac{dx}{dt}=y,\ \frac{dy}{dt}=-g(x)-f(x)y, \tag{5}$$

with

$$f(x)=\frac{\alpha}{n}\cdot\frac{R_1(x)}{|f_1(x)|^{1+q}},\ g(x)=\frac{r(x)R_2(x)}{|f_1(x)|^{1+2q}},$$

where

$$\left.\begin{aligned}
&R_1(x)=\left(x+\frac{\alpha}{n}\right)[(2l-1)x+\delta(\alpha^2-\alpha\delta+1)]-\delta\left[\left(1-\frac{1}{n}\right)\alpha^2-\delta\alpha-1\right]x=\left(x+\frac{\alpha}{n}\right)\bar{R}_1(x)+\sigma x,\\
&R_2(x)=-(a^2\alpha^2-al\alpha-l)x^2-(2a\alpha^2-2l\alpha-\delta+a)x-(\alpha^2-\delta\alpha+1),\\
&r(x)=\frac{\alpha}{n}x\left(x+\frac{\alpha}{n}\right),\ f_1(x)=-(2a\alpha^2-2l\alpha-\alpha)x-(\alpha^2-\delta\alpha+1),\\
&q=\frac{2l\alpha-a\alpha^2}{-(2a\alpha^2-2l\alpha-\alpha)},
\end{aligned}\right\} \tag{6}$$

and α is the unique positive root of $a\alpha^3-(l+1)\alpha^2-n=0$.

Proof We use a series transformations as in [5]. Let $x=\bar{x}+\alpha y$, $y=\bar{y}$ and $dt=\alpha^2 d\bar{t}$ in system (1). Then it is reduced to (omitting the bars for simplicity)

$$\frac{dx}{dt}=f_0(x)+f_1(x)y,\ \frac{dy}{dt}=g_0(x)+g_1(x)y+g_2(x)y^2 \tag{7}$$

with

$$f_0(x)=(\delta-\alpha)x+(l-a\alpha)x^2,\ f_1(x)=-(2a\alpha^2-2l\alpha-\alpha)x-(\alpha^2-\delta\alpha+1),$$
$$g_0(x)=ax^2+x,\ g_1(x)=(2a\alpha-1)x+\alpha,\ g_2(x)=a\alpha^2-\alpha.$$

It is obvious that $x=\hat{x}$ with $f_1(\hat{x})=0$ is a line without contact for the system (7), unless $f_0(\hat{x})=0$ also holds. In the later case $x=\hat{x}$ is a straight line solution for system (7).

Because $\hat{x}=\dfrac{\alpha^2-\delta\alpha+1}{-(2a\alpha^2-2l\alpha-\alpha)}>0$, obviously, system (7) can only have limit cycles surrounding the origin. So we can restrict our attention to $x<\hat{x}$. Note that $f_1(x)<0$ for $x<\hat{x}$.

Because limit cycles can not cross the line $x=\hat{x}$, it is allowed to apply the transformation $\eta=x$, $\xi=f_0(x)+f_1(x)y$, which changes the system (7) into

$$\frac{d\eta}{dt}=\xi,\ \frac{d\xi}{dt}=-\psi_0(\eta)-\psi_1(\eta)\xi-\psi_2(\eta)\xi^2 \tag{8}$$

with

$$\psi_0(\eta)=\frac{r(\eta)R_2(\eta)}{f_1(\eta)},\ \psi_1(\eta)=\frac{R_1(\eta)}{f_1(\eta)},\ \psi_2(\eta)=-\frac{2l\alpha-a\alpha^2}{f_1(\eta)},$$

where $R_1(\eta)$, $R_2(\eta)$, $r(\eta)$ and $f_1(\eta)$ are given in (6).

Finally by the transformation

$$\bar{x}=\eta,\ \bar{y}=\frac{\xi}{|f_1(\eta)|^q},\ d\bar{t}=|f_1(\eta)|^q dt$$

with

$$q=\frac{2l\alpha-a\alpha^2}{-(2a\alpha^2-2l\alpha-\alpha)},$$

the system (8) is reduced to the Liénard system (5) (omitting the bars), where $g(x)$, $f(x)$ are given as above.

Remark 1 We define $\sum=n-\dfrac{n\delta}{\alpha}-\dfrac{n}{\alpha^2}-1$ as in [1].

Lemma 2 For $0<\delta<2$, $a<0$, $0<l<\dfrac{1}{2}$, $n<0$ and α is the positive root of the equation $ax^3-(l+1)x^2$

$-n=0$, which corresponds to the critical point at infinity of system (1). Then

(i) $\alpha^2-\delta\alpha+1>0$;

(ii) $a\alpha^2-\alpha<0$;

(iii) $a^2\alpha^2-al\alpha-l>0$, $2a\alpha^2-2l\alpha-\delta+a<0$, $2a\alpha^2-2l\alpha-\alpha<0$;

(iv) $\sum>0$ as $(1-\frac{1}{n})\alpha^2-\delta\alpha-1<0$.

Proof We now prove $a^2\alpha^2-al\alpha-l>0$, and the others are similar. Since $(a\alpha-1)(a^2\alpha^2-al\alpha-l)=na^2+l<0$, we get the result from $a\alpha-1<0$.

Notice that if $l+n>0$, the condition $\sum>0$ always holds.

Lemma 3 [2] Let $f(x)$, $g(x)$ be continuously differentiable functions for $k_1<x<k_2$, where $k_1<0<k_2$ such that for $k_1<x<k_2$ the following conditions are satisfied:

(i) $g(x)>0(<0)$ for $x>0(<0)$;

(ii) there exists a x_0 such that $f(x_0)=0$ and $f(x)>0(<0)$ for $x>x_0(<x_0)$;

(iii) $\frac{f(x)}{g(x)}$ is an increasing function both for $x<0$ and for $x>x_0$.

Then the Liénard system (5) has at most one periodic orbit. If exists it must be a limit cycle with negative characteristic exponent.

Theorem For $n<0$, $0<l<\frac{1}{2}$ and $(1-\frac{1}{n})\alpha^2-\delta\alpha-1<0$, system (1) has at most one limit cycle, and if exists it has a negative characteristic exponent.

Proof It will be shown that for system (5) with above conditions, by using Lemma 2, all the conditions of Lemma 3 are satisfied.

Let us take the nonzero root of $r(x)=0$ as $x^*=-\frac{\alpha}{n}>0$.

Some elementary calculations show that

$$R_1(0)=\frac{\delta\alpha}{n}(\alpha^2-\delta\alpha+1)<0,\quad R_2(0)=-(\alpha^2-\delta\alpha+1)<0,$$

$$R_1(\hat{x})=\frac{(\alpha^2-\delta\alpha+1)(2l-1)}{(2a\alpha^2-2l\alpha-\alpha^2)^2}\left(\frac{\alpha^2}{n}\sum\right)>0,\quad R_2(\hat{x})=\sum(1-a\alpha)>0,$$

$$x^*-\hat{x}=\frac{\alpha^2\sum}{n(2a\alpha^2-2l\alpha-\alpha)}>0.$$

In the following we will denote x_h as the smallest root of $R_2(x)=0$ and x_b as the smallest root of $R_1(x)=0$, then we get the Fig.1.

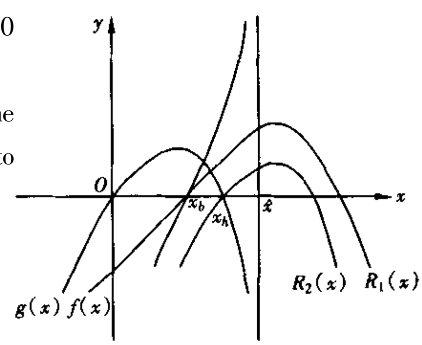

Fig.1 Pictures of $f(x)$, $g(x)$

By taking $k_1=-\infty$, $k_2=x_h$, from above discussion and Fig.1, the conditions (i) and (ii) of Lemma 3 are satisfied. Then we need only to check the condition (iii).

Consider

$$\frac{f(x)}{g(x)}=|f_1(x)|^q\left[\frac{\bar{R}_1(x)}{xR_2(x)}+\frac{\sigma}{(x+\frac{\alpha}{n})R_2(x)}\right],$$

$$\left[\frac{f(x)}{g(x)}\right]'=|f_1(x)|^q\frac{2l\alpha-a\alpha^2}{f_1(x)}\left[\frac{\bar{R}_1(x)}{xR_2(x)}+\frac{\sigma}{(x+\frac{\alpha}{n})R_2(x)}\right]$$

$$+|f_1(x)|^q\left[\frac{xR_2(x)\bar{R}'_1(x)-xR'_2(x)\bar{R}_1(x)-R_2(x)\bar{R}_1(x)}{x^2R_1^2(x)}-\frac{\sigma R_2(x)+(x+\frac{\alpha}{n})R'_2(x)}{(x+\frac{\alpha}{n})^2R_2^2(x)}\right].$$

Simplifying, we obtain

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{-\sigma}{(x+\frac{\alpha}{n})^2 R_2(x)} + \frac{-\delta(\alpha^2-\delta\alpha+1)}{x^2 R_2(x)} + \frac{R_1(x)u(x)}{x(x+\frac{\alpha}{n})f_1(x)R_2^2(x)},$$

where $u(x)=(2l\alpha - a\alpha^2)R_2(x) - f_1(x)R'_2(x)$.

For $x\in(-\infty,0)\cup(x_b,x_h)$, we get $x+\frac{\alpha}{n}<0$, $f_1(x)<0$, $R_2(x)<0$, $xR_1(x)>0$ and $\sigma>0$. Thus it is enough to show that $u(x)>0$. $u'(x)=(a\alpha^2-\alpha)R'_2(x)+2(a^2\alpha^2-al\alpha-l)f_1(x)<0$. since $R'_2(x)>0$, $f_1(x)<0$, $a\alpha^2-\alpha<0$ and $a^2\alpha^2-al\alpha-l>0$, the condition (iii) is satisfied. The proof is complete.

Remark 2 As pointed in [1] and [3], the number of limit cycles around O depends upon the separatrix cycle outside O passing through S_2 or N. If the separatrix cycle passing through S_2 (see Fig.2), denoted by Hoc(S_2), which is stable since div $|_{S_2}<0$, and has the same stability with the limit cycle created from O, which corresponds the uniqueness case we have proved in the theorem (see also [4]) and now the case $n<0$ was not discussed in [2]. But some times, the separatrix cycle is formed passing through N (see Fig.3). For example, as a, l fixed and $|n|$ big enough, then appears the case. Since div $|_N>0$, the Hoc (N) must be formed for certain $\delta^*>0$ with the stable limit cycle created from O inside. Then as δ increases from δ^*, an unstable limit cycle creates from Hoc (N) and we obtain two cycles around O.

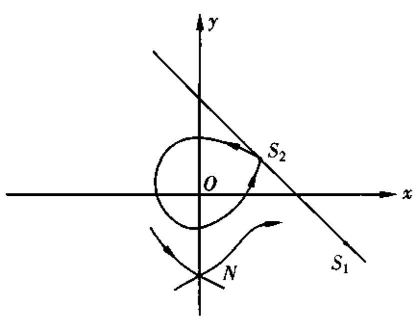

Fig.2 Hoc(S_2)

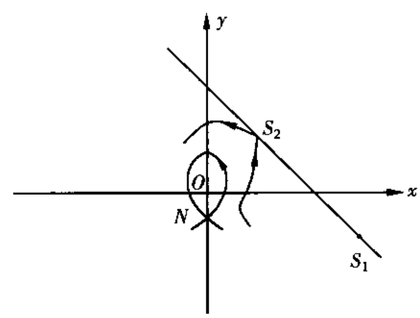

Fig.3 Hoc(N)

[References]

[1] Ali Elamin M Saeed, Luo Dingjun. Limit Cycle Problem of Quadratic System of Type. $(\mathrm{III})_{m=0}$, (I) (to appear).
[2] Coppel W A. A new class of quadratic system[J]. J Diff Eqs, 1991, 92(2): 360—370.
[3] Luo Dingjun. Limit Cycle Bifurcation of Planar Vector Fields with Several Parameters and Applications[J]. Ann of Diff Eqs, 1985, 1(1): 91—105.
[4] Sun Jianhua. Uniqueness and Bifurcation of Limit Cycles for Quadratic System $(\mathrm{III})_{m=0}$ $(0<n<1)$ [J]. Ann of Diff Eqs, 1992, 8(4): 463—468.
[5] Ye Yanqian. Qualitative Theory of Polynomial Systems[M]. Shanghai: Shanghai Scientific & Technical Publishers, 1995.

$(\mathrm{III})_{m=0}$型二次系统的极限环问题,(Ⅱ)

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[摘要] 本文讨论了在条件 $0<l<\frac{1}{2}$, $n<0$ 下,$(\mathrm{III})_{m=0}$型二次系统的极限环问题,证明了该系统在某些条件下最多只有一个极限环.

[关键词] 二次系统,极限环,细焦点

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