A Complete Classification of OGDDs of Type 4^4

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Abstract: An orthogonal group divisible design (OGDD) $(X, \mathcal{G}, \mathcal{A}, \mathcal{B})$ is a set X and a partition $\mathcal{G}$ of X into classes (usually called groups), and two disjoint sets $\mathcal{A}$ and $\mathcal{B}$ of 3-subsets of X, so that each pair $\{x, y\}$ of elements of X appears once in a 3-subset of $\mathcal{A}$ and once in a 3-subset of $\mathcal{B}$ if x and y are from different groups, and does not appear in a 3-subset of either if x and y are from the same groups. Moreover, (a) if $\{x,y,a\} \in \mathcal{A}$ and $\{x,y,b\} \in \mathcal{B}$, then a and b are in different groups; and (b) for two distinct intersecting triples $\{x,y,z\}$ and $\{u,v,z\}$ of $\mathcal{A}$, the triples $\{x,y,a\}$ and $\{u,v,b\}$ of $\mathcal{B}$ satisfy $a \neq b$. In this article, we will show that there is a unique isomorphism class of OGDDs of type 4^4.

Key words: orthogonal, group-divisible design, isomorphism

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型为 4^4 的 OGDD 的同构分类

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[摘要] 正交可分组设计是一个四元组 $(X,\mathcal{G},\mathcal{A},\mathcal{B})$,其中 X 是一个点集,$\mathcal{G}$ 是 X 的一个划分(称为组集),$\mathcal{A}$ 和 $\mathcal{B}$ 是两个不交的三元子集簇,满足对不在同一组的任一点对 $\{x,y\}$ 恰好出现在 $\mathcal{A}$ 的一个三元集中,也恰好出现在 $\mathcal{B}$ 的一个三元集中. 进一步有(a) 如果 $\{x,y,a\} \in \mathcal{A}$ 且 $\{x,y,b\} \in \mathcal{B}$,那么 a 和 b 不在同一组,且(b) 如果 $\{x,y,z\}$, $\{u,v,z\} \in \mathcal{A}$ 且 $\{u,v,b\}$, $\{x,y,a\} \in \mathcal{B}$,那么 $a \neq b$,在这篇文章中,将证明只有一个型为 4^4 的 OGDD 的同构类.

[关键词] 正交的,可分组区组设计,同构

1 Preliminary Results

OGDD is a generalization of OSTS suggested by Stinson and Zhu [5], and it is not only useful tool in the construction of OSTS (see [3]), but finding when they exists is an interesting question itself (see [2,4,6]).

For the existence problem for OGDDs of type g^u, Colbourn and Gibbons [4] have settled it with few possible exceptions for each group size g. The following were their concluding remarks.

The main question that remains open is whether there is any value of g for which an OGDD of type g^4 exists. On the basis of the nonexistence when $g=2$ and $g=4$, one might be tempted to conjecture that the answer is negative.

Dukes in [2] constructed an OGDD of type g^4 for $g=8,12$ and the author constructed an OGDD of type 4^4.

In this article, we will show that there is a unique isomorphism class of OGDDs of type 4^4.

Definition 1.1 A group-divisible design with block size 3 (briefly, 3-GDD) $(X,\mathcal{G},\mathcal{A})$ is a set X and a partition $\mathcal{G}$ of X into classes (usually called groups), and a set $\mathcal{A}$ of 3-subsets of X, so that each pair $\{x, y\}$ of elements

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of X appears once in a 3-subset of $\mathscr{A}$ if x and y are from different groups, and does not appear in a 3-subset of $\mathscr{A}$ if x and y are from the same groups.

Definition 1.2 An orthogonal group-divisible design (briefly, OGDD) $(X,\mathscr{G},\mathscr{A},\mathscr{B})$ is a pair of 3-GDDs $(X,\mathscr{G},\mathscr{A})$ and $(X,\mathscr{G},\mathscr{B})$ satisfying two orthogonality conditions:

(i) if $\{x,y,z\}\in\mathscr{A}$ and $\{x,y,w\}\in\mathscr{B}$, then z and w are in different groups; and

(ii) for two distinct intersecting triples $\{x,y,z\}$ and $\{u,v,z\}$ of $\mathscr{A}$, the triples $\{x,y,w\}$ and $\{u,v,t\}$ of $\mathscr{B}$ satisfy $w\neq t$.

Definition 1.3 Let $(X,\mathscr{G},\mathscr{A},\mathscr{B})$ and $(X,\mathscr{G},\mathscr{C},\mathscr{D})$ be two OGDDs of type g^u. We say they are isomorphic if there exists a bijection α from X to X such that $\mathscr{G}=\mathscr{G}\alpha$, $\mathscr{A}=\mathscr{C}\alpha$ and $\mathscr{B}=\mathscr{D}\alpha$.

Definition 1.4 A transversal design (briefly, TD) TD$(3,4)$ is a 3-GDD of type 4^3.

2 Special Properties

In this article we always let

$G_0=\{0,3,6,9\}$, $G_1=\{1,4,7,10\}$, $G_2=\{2,5,8,11\}$,

$H=\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}$, $\mathscr{G}=\{G_0,G_1,G_2,H\}$ and $X=G_0\cup G_1\cup G_2\cup H$.

Definition 2.1 Let $(X,\mathscr{G},\mathscr{B})$ be a 3-GDD of type 4^4. We define

$$\mathscr{B}_g=\{B\in\mathscr{B}:B\cap H=\varnothing\},\ \mathscr{B}_h=\{B\in\mathscr{B}:B\cap H\neq\varnothing\},$$

$$\mathscr{P}_{\mathscr{B},i}=\{\{x,y\}:\{\alpha_i,x,y\}\in\mathscr{B}\}\text{ and }\mathscr{P}_{\mathscr{B},i,j}=\{\{x,y\}\in\mathscr{P}_{\mathscr{B},i}:x,y\notin G_j\},$$

where $i=0,1,2,3$ and $j=0,1,2$.

By the definition of 3-GDD, we have

Lemma 2.2 If $(X,\mathscr{G},\mathscr{B})$ is a 3-GDD of type 4^4 then

(i) each $\mathscr{P}_{\mathscr{B},i}$ is a partitions of $X\backslash H$; (ii) each point of $X\backslash H$ appears exactly twice in $\mathscr{B}_g$.

By the definition of OGDD, we have

Lemma 2.3 If $(X,\mathscr{G},\mathscr{A},\mathscr{B})$ is an OGDD of type 4^4 then $\mathscr{B}_g\cup\mathscr{A}_g$ are the blocks of a TD$(3,4)$.

Lemma 2.4 If $(X,\mathscr{G},\mathscr{A},\mathscr{B})$ is an OGDD of type 4^4 then $\{\mathscr{P}_{\mathscr{B},i}:i=0,1,2,3\}$ is a partition of $\{\{x,y\}:\{x,y,z\}\in\mathscr{A}_g\}$.

Lemma 2.5 Let $(X,\mathscr{G},\mathscr{A},\mathscr{B})$ be an OGDD of type 4^4. If $\{x,y\}\in\mathscr{P}_{\mathscr{B},i,j}$ and $\{z,t\}\in\mathscr{P}_{\mathscr{A},i,j}$, then $\{x,y\}\cap\{z,t\}=\varnothing$, where $i=0,1,2,3$, $j=0,1,2$.

Definition 2.6 Let $\mathscr{P}_{\mathscr{B},i,1}\cup\mathscr{P}_{\mathscr{B},i,2}=\{0,b_{i,0}\},\{3,b_{i,1}\},\{6,b_{i,2}\},\{9,b_{i,3}\}$, for $i=0,1,2,3$. We define a matrix $M_{\mathscr{B}}=(b_{i,j})_{4\times4}$.

By the definition of OGDD, we have

Lemma 2.7 Let $(X,\mathscr{G},\mathscr{A},\mathscr{B})$ be an OGDD of type 4^4, $M_{\mathscr{B}}=(b_{i,j})_{4\times4}$ and $M_{\mathscr{A}}=(a_{i,j})_{4\times4}$. If $b_{i,j}=b_{s,t}$ and $a_{i,j}=a_{s,t}$, then $i=s$ and $j=t$.

Lemma 2.8 Let $(X,\mathscr{G},\mathscr{A},\mathscr{B})$ be an OGDD of type 4^4, and $\{z,x,y\},\{z,u,v\}\in\mathscr{A}_g$. If $\{x,y\}\in\mathscr{P}_{\mathscr{B},i}$ and $\{u,v\}\in\mathscr{P}_{\mathscr{B},j}$, then $i\neq j$.

3 Two Non-isomorphic TD(3,4)

It is easy to see that there are only two non-isomorphic Latin squares of side 4 (can be found in [1]), that is,

$L_1=$

0	1	2	3
1	0	3	2
2	3	1	0
3	2	0	1

$L_2=$

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

Hence we have

Lemma 3.1 There are only two non-isomorphic TD$(3,4)$, namely, $\mathscr{T}_1$ and $\mathscr{T}_2$ as follows.

$\mathcal{T}_1 = \{\{0,1,2\},\{0,4,5\},\{0,7,8\},\{0,10,11\},\{3,1,5\},\{3,4,2\},\{3,7,11\},\{3,10,8\},$
$\{6,1,8\},\{6,4,11\},\{6,7,5\},\{6,10,2\},\{9,1,11\},\{9,4,8\},\{9,7,2\},\{9,10,5\}\}.$

$\mathcal{T}_2 = \{\{0,1,2\},\{0,4,5\},\{0,7,8\},\{0,10,11\},\{3,1,5\},\{3,4,2\},\{3,7,11\},\{3,10,8\},$
$\{6,1,8\},\{6,4,11\},\{6,7,2\},\{6,10,5\},\{9,1,11\},\{9,4,8\},\{9,7,5\},\{9,10,2\}\}.$

4 No OGDD Based on $\mathcal{T}_1$

Lemma 4.1 There are only six ways to partition $\mathcal{T}_1$ into two parts $\mathcal{A}_g$ and $\mathcal{B}_g$ such that they satisfy the condition (ii) of Lemma 2.2.

Proof It is easy to see that there are only the following six ways.

(i) $\mathcal{A}_g = \{\{0,1,2\},\{0,4,5\},\{3,7,11\},\{3,10,8\},\{6,1,8\},\{6,4,11\},\{9,7,2\},\{9,10,5\}\}$,
$\mathcal{B}_g = \{\{0,7,8\},\{0,10,11\},\{3,1,5\},\{3,4,2\},\{6,7,5\},\{6,10,2\},\{9,1,11\},\{9,4,8\}\}$.

(ii) $\mathcal{A}_g = \{\{0,1,2\},\{0,4,5\},\{3,7,11\},\{3,10,8\},\{6,7,5\},\{6,10,2\},\{9,1,11\},\{9,4,8\}\}$,
$\mathcal{B}_g = \{\{0,7,8\},\{0,10,11\},\{3,1,5\},\{3,4,2\},\{6,1,8\},\{6,4,11\},\{9,7,2\},\{9,10,5\}\}$.

(iii) $\mathcal{A}_g = \{\{0,1,2\},\{0,7,8\},\{3,1,5\},\{3,7,11\},\{6,4,11\},\{6,10,2\},\{9,4,8\},\{9,10,5\}\}$,
$\mathcal{B}_g = \{\{0,4,5\},\{0,10,11\},\{3,4,2\},\{3,10,8\},\{6,1,8\},\{6,7,5\},\{9,1,11\},\{9,7,2\}\}$.

(iv) $\mathcal{A}_g = \{\{0,1,2\},\{0,7,8\},\{3,4,2\},\{3,10,8\},\{6,4,11\},\{6,7,5\},\{9,1,11\},\{9,10,5\}\}$,
$\mathcal{B}_g = \{\{0,4,5\},\{0,10,11\},\{3,1,5\},\{3,7,11\},\{6,1,8\},\{6,10,2\},\{9,4,8\},\{9,7,2\}\}$.

(v) $\mathcal{A}_g = \{\{0,1,2\},\{0,10,11\},\{3,1,5\},\{3,10,8\},\{6,4,11\},\{6,7,5\},\{9,4,8\},\{9,7,2\}\}$,
$\mathcal{B}_g = \{\{0,4,5\},\{0,7,8\},\{3,4,2\},\{3,7,11\},\{6,1,8\},\{6,10,2\},\{9,1,11\},\{9,10,5\}\}$.

(vi) $\mathcal{A}_g = \{\{0,1,2\},\{0,10,11\},\{3,4,2\},\{3,7,11\},\{6,1,8\},\{6,7,5\},\{9,4,8\},\{9,10,5\}\}$,
$\mathcal{B}_g = \{\{0,4,5\},\{0,7,8\},\{3,1,5\},\{3,10,8\},\{6,4,11\},\{6,10,2\},\{9,1,11\},\{9,7,2\}\}$.

Lemma 4.2 In the sense of isomorphism, there is a unique way to partition $\mathcal{T}_1$ into two parts $\mathcal{A}_g$ and $\mathcal{B}_g$ such that they satisfy the condition (ii) of Lemma 2.2.

Proof Let $\tau_1 = (6,9)(1,4)(2,5)$, $\tau_2 = (6,9)(1,2)(4,5)(7,8)(10,11)$, $\tau_3 = \tau_2$, $\tau_4 = (1,0)(4,3)(7,6)(10,9)$ and $\tau_5 = (6,9)(7,10)(8,11)$. It is easily checked that there are isomorphic mappings τ_1 from (i) to (ii), τ_2 from (iii) to (iv), τ_3 from (v) to (vi), τ_4 from (i) to (iii), and τ_5 from (iii) to (v).

Theorem 4.3 There is no OGDD of type 4^4 based on $\mathcal{T}_1$.

Proof From Lemma 4.1 and Lemma 4.2, we only consider the case (i) of Lemma 4.1. It follows from Lemma 2.4 that

$\mathcal{P}_{\mathcal{B},0,0} \subset \{\{1,2\},\{4,5\},\{7,11\},\{10,8\},\{1,8\},\{4,11\},\{7,2\},\{10,5\}\}$ and $\mathcal{P}_{\mathcal{A},0,0} \subset \{\{7,8\},\{10,11\},\{1,5\},\{4,2\},\{7,5\},\{10,2\},\{1,11\},\{4,8\}\}$.

Let $\{1,2\} \in \mathcal{P}_{\mathcal{B},0,0}$. From Lemma 2.8, $\{4,5\} \notin \mathcal{P}_{\mathcal{B},0,0}$. Hence it is impossible to arrange $\mathcal{P}_{\mathcal{B},0,0}$ and $\mathcal{P}_{\mathcal{A},0,0}$ such that they satisfy Lemma 2.5.

5 Partition $\mathcal{T}_2$ into $\mathcal{A}_g$ and $\mathcal{B}_g$

Lemma 5.1 There are only six ways to partition $\mathcal{T}_2$ into two parts $\mathcal{A}_g$ and $\mathcal{B}_g$ such that they satisfy the condition (ii) of Lemma 2.2.

Proof It is easy to see that there are only the following six ways.

(i) $\mathcal{A}_g = \{\{0,1,2\},\{0,4,5\},\{3,7,11\},\{3,10,8\},\{6,1,8\},\{6,4,11\},\{9,7,5\},\{9,10,2\}\}$,
$\mathcal{B}_g = \{\{0,7,8\},\{0,10,11\},\{3,1,5\},\{3,4,2\},\{6,7,2\},\{6,10,5\},\{9,1,11\},\{9,4,8\}\}$.

(ii) $\mathcal{A}_g = \{\{0,1,2\},\{0,4,5\},\{3,7,11\},\{3,10,8\},\{6,7,2\},\{6,10,5\},\{9,1,11\},\{9,4,8\}\}$,
$\mathcal{B}_g = \{\{0,7,8\},\{0,10,11\},\{3,1,5\},\{3,4,2\},\{6,1,8\},\{6,4,11\},\{9,7,5\},\{9,10,2\}\}$.

(iii) $\mathcal{A}_g = \{\{0,1,2\},\{0,7,8\},\{3,1,5\},\{3,7,11\},\{6,4,11\},\{6,10,5\},\{9,4,8\},\{9,10,2\}\}$,
$\mathcal{B}_g = \{\{0,4,5\},\{0,10,11\},\{3,4,2\},\{3,10,8\},\{6,1,8\},\{6,7,2\},\{9,1,11\},\{9,7,5\}\}$.

(iv) $\mathscr{A}_g = \{\{0,1,2\}, \{0,7,8\}, \{3,4,2\}, \{3,10,8\}, \{6,4,11\}, \{6,10,5\}, \{9,1,11\}, \{9,7,5\}\}$,
$\mathscr{B}_g = \{\{0,4,5\}, \{0,10,11\}, \{3,1,5\}, \{3,7,11\}, \{6,1,8\}, \{6,7,2\}, \{9,4,8\}, \{9,10,2\}\}$.

(v) $\mathscr{A}_g = \{\{0,1,2\}, \{0,10,11\}, \{3,1,5\}, \{3,10,8\}, \{6,4,11\}, \{6,7,2\}, \{9,4,8\}, \{9,7,5\}\}$,
$\mathscr{B}_g = \{\{0,4,5\}, \{0,7,8\}, \{3,4,2\}, \{3,7,11\}, \{6,1,8\}, \{6,10,5\}, \{9,1,11\}, \{9,10,2\}\}$.

(vi) $\mathscr{A}_g = \{\{0,1,2\}, \{0,10,11\}, \{3,4,2\}, \{3,7,11\}, \{6,1,8\}, \{6,10,5\}, \{9,4,8\}, \{9,7,5\}\}$,
$\mathscr{B}_g = \{\{0,4,5\}, \{0,7,8\}, \{3,1,5\}, \{3,10,8\}, \{6,4,11\}, \{6,7,2\}, \{9,1,11\}, \{9,10,2\}\}$.

Lemma 5.2 In the sense of isomorphism, there is a unique way to partition $\mathscr{T}_2$ into two parts $\mathscr{A}_g$ and $\mathscr{B}_g$ such that they satisfy the condition (ii) of Lemma 2.2.

Proof Let $\tau_1 = (1,2)(4,5)(7,8)(10,11)$, $\tau_3 = \tau_2 = \tau_1$, $\tau_4 = (3,6)(4,7)(5,8)$ and $\tau_5 = (6,9)(7,10)(8,11)$. It is easily checked that there are isomorphic mappings τ_1 from (i) to (ii), τ_2 from (iii) to (iv), τ_3 from (v) to (vi), τ_4 from (i) to (iii), and τ_5 from (iii) to (v).

6 OGDDs Based on $\mathscr{T}_2$

From Lemma 5.2 we only need to consider the case (i) of Lemma 5.1. In the following we always let $\{1,2\} \in \mathscr{P}_{\mathscr{B},0,0}$.

Lemma 6.1 There are 3 ways to arrange $\mathscr{P}_{\mathscr{B},0,0}$ and $\mathscr{P}_{\mathscr{A},0,0}$ such that they satisfy Lemma 2.5.

Proof From Lemma 2.4 we have $\mathscr{P}_{\mathscr{B},0,0} \subset \{\{1,2\}, \{4,5\}, \{7,11\}, \{10,8\}, \{1,8\}, \{4,11\}, \{7,5\}, \{10,2\}\}$ and $\mathscr{P}_{\mathscr{A},0,0} \subset \{\{7,8\}, \{10,11\}, \{1,5\}, \{4,2\}, \{7,2\}, \{10,5\}, \{1,11\}, \{4,8\}\}$.

Since $\{1,2\} \in \mathscr{P}_{\mathscr{B},0,0}$, $\{4,5\} \notin \mathscr{P}_{\mathscr{B},0,0}$ by Lemma 2.8. Hence there are only three arrangements:

(i) $\mathscr{P}_{\mathscr{B},0,0} = \{\{1,2\}, \{7,11\}\}$, $\mathscr{P}_{\mathscr{A},0,0} = \{\{10,5\}, \{4,8\}\}$;

(ii) $\mathscr{P}_{\mathscr{B},0,0} = \{\{1,2\}, \{4,11\}\}$, $\mathscr{P}_{\mathscr{A},0,0} = \{\{10,5\}, \{7,8\}\}$;

(iii) $\mathscr{P}_{\mathscr{B},0,0} = \{\{1,2\}, \{7,5\}\}$, $\mathscr{P}_{\mathscr{A},0,0} = \{\{10,11\}, \{4,8\}\}$.

Based on Lemma 6.1 we have

Lemma 6.2 There are 7 ways to arrange $\mathscr{P}_{\mathscr{B},i,0}$ and $\mathscr{P}_{\mathscr{A},i,0}$, for $i = 0,1,2,3$, such that they satisfy Lemma 2.4, Lemma 2.5 and Lemma 2.8.

Proof The 7 arrangements are given in the following table.

	$\mathscr{P}_{\mathscr{A},0,0}$	$\mathscr{P}_{\mathscr{A},1,0}$	$\mathscr{P}_{\mathscr{A},2,0}$	$\mathscr{P}_{\mathscr{A},3,0}$
(i)	{10,5}, {4,8}	{7,2}, {1,11}	{10,11}, {4,2}	{7,8}, {1,5}
(ii)	{10,5}, {4,8}	{10,11}, {7,2})	{1,11}, {4,2}	{7,8}, {1,5}
(iii)	{10,5}, {4,8}	{7,8}, {1,11}	{1,5}, {7,2}	{10,11}, {4,2}
(iv)	{10,5}, {7,8}	{7,2}, {1,11}	{1,5}, {4,8}	{10,11}, {4,2}
(v)	{10,5}, {7,8}	{7,2}, {10,11}	{1,5}, {4,8}	{1,11}, {4,2}
(vi)	{10,11}, {4,8}	{7,2}, {1,11}	{10,5}, {4,2}	{7,8}, {1,5}
(vii)	{10,11}, {4,8}	{7,8}, {1,11}	{10,5}, {4,2}	{7,2}, {1,5}
	$\mathscr{P}_{\mathscr{B},0,0}$	$\mathscr{P}_{\mathscr{B},1,0}$	$\mathscr{P}_{\mathscr{B},2,0}$	$\mathscr{P}_{\mathscr{B},3,0}$
(i)	{1,2}, {7,11}	{4,5}, {10,8}	{1,8}, {7,5}	{4,11}, {10,2}
(ii)	{1,2}, {7,11}	{4,5}, {1,8}	{10,8}, {7,5}	{4,11}, {10,2}
(iii)	{1,2}, {7,11}	{4,5}, {10,2}	{10,8}, {4,11}	{1,8}, {7,5}
(iv)	{1,2}, {4,11}	{4,5}, {10,8}	{10,2}, {7,11}	{1,8}, {7,5}
(v)	{1,2}, {4,11}	{4,5}, {1,8}	{10,2}, {7,11}	{1,8}, {7,5}
(vi)	{1,2}, {7,5}	{4,5}, {10,8}	{1,8}, {7,11}	{4,11}, {10,2}
(vii)	{1,2}, {7,5}	{4,5}, {10,2}	{1,8}, {7,11}	{4,11}, {10,8}

Lemma 6.3 There are isomorphic mappings $\tau_1 = (1,10)(4,7)(2,8)(5,11)(0,3)(6,9)$ from (ii) to (iv), $\tau_2 = \tau_1$ from (iii) to (vi), $\tau_3 = (1,5)(4,2)(7,8)(10,11)(6,9)$ from (ii) to (vi) and $\tau_4 = \tau_3$ from (v) to (vii).

Lemma 6.4 There is no OGDD of type 4^4 for the cases (ii) and (v).

Proof For (v), it is clear that for $i, j = 0,1,2,3$,

$$b_{0,j} \in \{5,7,8,10\},\ b_{1,j} \in \{2,7,10,11\},\ b_{2,j} \in \{1,5,4,8\},\ b_{3,j} \in \{1,2,4,11\},$$
$$b_{i,0} \in \{1,2,4,5\},\ b_{i,1} \in \{7,8,10,11\},\ b_{i,2} \in \{1,4,8,11\},\ b_{i,3} \in \{2,5,7,10\}.$$

This forces that

$$b_{0,0} = b_{2,3} = 5,\ b_{0,2} = b_{2,1} = 8,\ b_{1,0} = b_{3,3} = 2,\ b_{1,2} = b_{3,1} = 11,$$

Similarly we have

$$a_{0,0} = a_{2,3} = 11,\ a_{0,2} = a_{2,1} = 2,\ a_{1,0} = a_{3,3} = 8,\ a_{1,2} = a_{3,1} = 5.$$

By Lemma 2.6, (v) can not be arranged into an OGDD of type 4^4.

For (ii), similar to (v), we have

$$b_{1,0} = b_{2,3} = 2,\ b_{1,2} = b_{2,1} = 11,\ a_{1,0} = a_{2,3} = 8,\ a_{1,2} = a_{2,1} = 5.$$

By Lemma 2.6, (ii) can not be arranged into an OGDD of type 4^4.

Lemma 6.5 There are at most two isomorphism classes of OGDDs of type 4^4.

Proof We only need to consider the case (i) of Lemma 6.2. Silimar to the proof of Lemma 6.4, we have $M_{\mathscr{B}} = M_1$ or M_2 and $M_{\mathscr{A}} = N_1$ or N_2, where

$$M_1 = \begin{array}{|c|c|c|c|}\hline 4 & 10 & 8 & 5 \\ \hline 1 & 7 & 11 & 2 \\ \hline 2 & 11 & 4 & 10 \\ \hline 5 & 8 & 1 & 7 \\ \hline \end{array} \quad M_2 = \begin{array}{|c|c|c|c|}\hline 5 & 8 & 4 & 10 \\ \hline 2 & 11 & 1 & 7 \\ \hline 4 & 10 & 11 & 2 \\ \hline 1 & 7 & 8 & 5 \\ \hline \end{array} \quad N_1 = \begin{array}{|c|c|c|c|}\hline 11 & 2 & 7 & 1 \\ \hline 8 & 5 & 10 & 4 \\ \hline 7 & 1 & 5 & 8 \\ \hline 10 & 4 & 2 & 11 \\ \hline \end{array} \quad N_2 = \begin{array}{|c|c|c|c|}\hline 7 & 1 & 2 & 11 \\ \hline 10 & 4 & 5 & 8 \\ \hline 8 & 5 & 7 & 1 \\ \hline 11 & 2 & 10 & 4 \\ \hline \end{array}$$

It follows from Lemma 2.7 that it is not an OGDD for $M_{\mathscr{B}} = M_1$ and $M_{\mathscr{A}} = N_2$ or $M_{\mathscr{B}} = M_2$ and $M_{\mathscr{A}} = N_1$. It is easily checked that it is an OGDD for $M_{\mathscr{B}} = M_1$ and $M_{\mathscr{A}} = N_1$ or $M_{\mathscr{B}} = M_2$ and $M_{\mathscr{A}} = N_2$.

Theorem 6.6 There is a unique isomorphism class of OGDDs of type 4^4.

Proof The two OGDDs of Lemma 6.5 are isomorphic by the permutation $\tau = (1,2)(4,5)(7,11)(10,8)(6,9)(\alpha_2,\alpha_3)$. Therefore, there is a unique isomorphism class of OGDDs of type 4^4, namely, $(X,\mathscr{G},\mathscr{A},\mathscr{B})$, where

$$\begin{aligned}\mathscr{A} = \{&\{0,1,2\},\{0,4,5\},\{3,7,11\},\{3,10,8\},\{6,1,8\},\{6,4,11\},\{9,7,5\},\{9,10,2\},\\ &\{\alpha_0,10,5\},\{\alpha_0,4,8\},\{\alpha_0,0,11\},\{\alpha_0,3,2\},\{\alpha_0,6,7\},\{\alpha_0,9,1\},\\ &\{\alpha_1,1,11\},\{\alpha_1,7,2\},\{\alpha_1,0,8\},\{\alpha_1,3,5\},\{\alpha_1,6,10\},\{\alpha_1,9,4\},\\ &\{\alpha_2,4,2\},\{\alpha_2,10,11\},\{\alpha_2,0,7\},\{\alpha_2,3,1\},\{\alpha_2,6,5\},\{\alpha_2,9,8\},\\ &\{\alpha_3,1,5\},\{\alpha_3,7,8\},\{\alpha_3,0,10\},\{\alpha_3,3,4\},\{\alpha_3,6,2\},\{\alpha_3,9,11\},\\ \mathscr{B} = \{&\{0,7,8\},\{0,10,11\},\{3,1,5\},\{3,4,2\},\{6,7,2\},\{6,10,5\},\{9,1,11\},\{9,4,8\},\\ &\{\alpha_0,1,2\},\{\alpha_0,7,11\},\{\alpha_0,0,4\},\{\alpha_0,3,10\},\{\alpha_0,6,8\},\{\alpha_0,9,5\},\\ &\{\alpha_1,4,5\},\{\alpha_1,10,8\},\{\alpha_1,0,1\},\{\alpha_1,3,7\},\{\alpha_1,6,11\},\{\alpha_1,9,2\},\\ &\{\alpha_2,1,8\},\{\alpha_2,7,5\},\{\alpha_2,0,2\},\{\alpha_2,3,11\},\{\alpha_2,6,4\},\{\alpha_2,9,10\},\\ &\{\alpha_3,4,11\},\{\alpha_3,10,2\},\{\alpha_3,0,5\},\{\alpha_3,3,8\},\{\alpha_3,6,1\},\{\alpha_3,9,7\}.\end{aligned}$$

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