带非局部源的双退化半线性抛物型方程解的爆破

李梅,李玲

(南京财经大学应用数学系,210003,江苏,南京)

[摘要]　该文研究双退化的半线性抛物型方程:$x^r u_t - x^\alpha u_{xx} = \int_0^a f(u)\,\mathrm{d}x$ 初边值问题,证明了局部解的存在唯一性并且得到当初值充分大时解在有限时刻爆破,得到了解的爆破点集是整个区间[0,a].

[关键词]　双退化半线性抛物型方程,非局部源,爆破,爆破点集

[中图分类号] O175.26,　[文献标识码] A,　[文章编号] 1001-4616(2005)02-0028-05

Blow-up for Double Degenerate Semilinear Parabolic Equations with Nonlocal Source

Li Mei,Li Ling

(Department of Applied Mathematics, Nanjing University of Finance and Economics, 210003, Nanjing, China)

Abstract:This paper deals with double degenerate semilinear parabolic equations $x^r u_t - x^\alpha u_{xx} = \int_0^a f(u)\,\mathrm{d}x$ with initial and boundary problem. The existence and uniqueness of local solution are given and we show that the solution blows up is a finite time provided the initial date $u_0(x)$ is large enough. Moreover, we show that the blow-up set is the entire interval $[0,a]$.

Key words:double degenerate semilinear parabolic equation, nonlocal source, blow-up, blow-up set

0　引言

本文考虑下面带非局部源的半线性退化问题

$$\begin{cases} x^r u_t - x^\alpha u_{xx} = \int_\Omega f(u)\,\mathrm{d}x, & x \in \Omega \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in \bar{\Omega}, \end{cases} \tag{1}$$

其中 $\Omega = (0,a)$,$a > 0$,$r > 0$,$0 \leqslant \alpha < 1$,$u_0(x) \geqslant 0$, $u_0(x)|_{\partial\Omega} = 0$ 且 $u_0 \in C^{2+\beta}([\bar{\Omega}])$,$\beta \in (0,1)$ 为一常数.f 满足条件(A):$f(s) \geqslant 0$, $f'(s) > 0$,$f''(s) \geqslant 0$,$s > 0$ 且 $\int_0^\infty \frac{1}{f(s)}\mathrm{d}s < +\infty$. 当 $r = 0$,$\alpha = 0$ 时,许多作者研究过问题(1),如文[1]、[2]. 当把反应源换成 u^p 时文[3]、[4] 中研究了当 $\alpha = 0$ 时,解的爆破性质. [3] 中证明了当 $p > r + 1$ 时爆破点集位于$(0,a)$ 的某个紧子集内,[4] 中证明了当 $1 < p \leqslant r + 1$ 时,若解爆破则只在 $x = 0$ 处发生. 而本文的目的是研究双退化性及非局部源对解的影响,得到了若解在有限时刻爆破则在整个区间的每一点都爆破. 这与[3]、[4] 的结果不同. 当 $r = 0$,$\alpha = 0$ 时,[2] 中的结果是本文的特殊情形.

1　局部解

记 $L = x^r \partial/\partial t - x^\alpha \partial^2/\partial x^2$,$Q_T = (0,a) \times (0,T]$,$Q_t = (0,a) \times (0,t]$,$0 < t < T$ 且 $\bar{Q}_T$,$\bar{Q}_t$ 分别表示它们的闭包.

首先根据[5] 中的思想,建立如下比较定理.

收稿日期:2004-06-28.

基金项目:南京财经大学校级课题资助项目(B0449).

作者简介:李梅,女,1966—,副教授,主要从事偏微分方程理论的教学与研究. E-mail:limei6606@tom.com

引理 1　设 $w(x,t)\in C^{2,1}((0,a]\times[0,T])\cap C(\overline{Q}_T)$，并且满足

$$\begin{cases} x^r w_t - x^\alpha \Delta w \geqslant \int_\Omega c(x,t)w(x,t)\,\mathrm{d}x, & (x,t)\in Q_T, \\ w(0,t)\geqslant 0, w(a,t)\geqslant 0, & t\in(0,T), \\ w(x,0)\geqslant 0, & x\in\overline{\Omega}, \end{cases}$$

其中 $c(x,t)\geqslant 0$ 是 Q_T 上的有界函数，则 $w(x,t)\geqslant 0$，$(x,t)\in\overline{Q}_T$.

引理 2　存在 $T_0>0$ 及有界函数 $h(x,t)$，使得 $h(x,t)$ 是问题(1)在 $\overline{Q_{T_0}}$ 上的一个上解.

证明　令(i) $\psi(x)=\dfrac{x(a^{1-\alpha}-x^{1-\alpha})}{2a^2}$，$\psi_m=\psi_{\max}=\dfrac{\frac{1-\alpha}{2-\alpha}\left(\frac{1}{2-\alpha}\right)^{\frac{1}{1-\alpha}}}{2a^\alpha}$；

(ii) $\delta<\min\left\{\left(\dfrac{1}{2-\alpha}\right)^{\frac{1}{1-\alpha}}, \dfrac{1}{2a}, \dfrac{1}{2}, \dfrac{2a^\alpha}{\xi_0}, \dfrac{\eta_0}{a}\right\}$；

(iii) $\xi(t)$ 满足

$$\begin{cases} \xi'(t)=\dfrac{2a^\alpha f'\left(f^{-1}\left(\left(\psi_m+\frac{1}{2a^\alpha}\right)\xi(t)+f(0)\right)\right)\left(\frac{5-\alpha}{4(3-\alpha)}a^{1-\alpha}\xi(t)+a f(0)\right)}{\left(\delta(1-\delta^{1-\alpha})+\delta^{\frac{1-\alpha}{2}}\right)a^r\delta^r}, \\ \xi(0)=\xi_0； \end{cases}$$

(iv) $\xi_0>\dfrac{2a^3(f(0)+1)}{(2-\alpha)(1-\alpha)}f'(f^{-1}(1+f(0)))$，$\xi_0\psi(x)+f(0)\geqslant f(u_0(x))$，$\eta_0$ 满足

$$\frac{1}{8a\eta_0^{3/2}}>\frac{5-\alpha}{4(3-\alpha)}a^{\frac{1-\alpha}{2}}f'(f^{-1}(1+f(0)))；$$

(v) $h(x,t)=f^{-1}\left(\left(\psi(x)+\dfrac{1}{2a^{\frac{1+\alpha}{2}}}x^{\frac{1-\alpha}{2}}\right)\xi(t)+f(0)\right)$，易知 $\xi(t)$ 是 t 的严格增的函数.

构造

$$J=f'(h)\left(Lh-\int_0^a f(h(x,t))\,\mathrm{d}x\right),$$

直接计算得

$$\begin{aligned} J &= f'(h)x^r\frac{\partial h}{\partial t}-f'(h)x^\alpha\frac{\partial^2 h}{\partial x^2}-f'(h)\int_0^a f(h(x,t))\,\mathrm{d}x \\ &= x^r(f(h))_t-x^\alpha(f(h))_{xx}-f'(h)\int_0^a f(h(x,t))\,\mathrm{d}x+x^\alpha f''(h)h_x^2 \\ &\geqslant x^r(f(h))_t-x^\alpha(f(h))_{xx}-f'(h)\int_0^a f(h(x,t))\,\mathrm{d}x \\ &= x^r\psi(x)\xi'(t)+\frac{1}{2a^{\frac{1+\alpha}{2}}}x^{\frac{2r+1-\alpha}{2}}\xi'(t)-x^\alpha\psi''(x)\xi(t)+\frac{1-\alpha^2}{8a^{\frac{1+\alpha}{2}}x^{3/2}}\xi(t) \\ &\quad -f'\left(f^{-1}\left(\left(\psi(x)+\frac{1}{2a^{\frac{1+\alpha}{2}}}x^{\frac{1-\alpha}{2}}\right)\xi(t)+f(0)\right)\right)\left(\frac{5-\alpha}{4(3-\alpha)}a^{1-\alpha}\xi(t)+af(0)\right), \end{aligned}$$

因为

$$\psi''(x)=-\frac{(2-\alpha)(1-\alpha)}{2a^2}x^{-\alpha}. \tag{2}$$

对于 $(x,t)\in(0,\delta a]\times(0,T_0]$，有

$$\begin{aligned} J &\geqslant \frac{(2-\alpha)(1-\alpha)}{2a^2}\xi(t)+\frac{1-\alpha^2}{8a^{\frac{1+\alpha}{2}}\eta_0^{3/2}}\xi(t) \\ &\quad -f'\left(f^{-1}\left(\psi(x)+\frac{1}{2a^{\frac{1+\alpha}{2}}}x^{\frac{1-\alpha}{2}}\right)\xi(t)+f(0)\right)\left(\frac{5-\alpha}{4(3-\alpha)}a^{1-\alpha}\xi(t)+a f(0)\right) \end{aligned}$$

$$\geqslant \xi(t)\left(\frac{(2-\alpha)(1-\alpha)}{2a^2}-\frac{a(f(0)+1)}{\xi(t)}f'\left(f^{-1}\left(\frac{\delta(1-\delta^{1-\alpha}}{2a^\alpha}+\frac{\delta^{\frac{1-\alpha}{2}}}{2a^\alpha}\right)\xi(t)+f(0)\right)\right)$$

$$+\xi(t)\left(\frac{1-\alpha^2}{8a^{\frac{1+\alpha}{2}}\eta_0^{3/2}}-\frac{5-\alpha}{4(3-\alpha)}a^{1-\alpha}f'\left(f^{-1}\left(\frac{\delta(1-\delta^{1-\alpha})+\delta^{\frac{1-\alpha}{2}}}{2a^\alpha}\xi(t)+f(0)\right)\right)\right).$$

由(iv)可知
$$\xi_0\geqslant\frac{2a^3(f(0)+1)}{(2-\alpha)(1-\alpha)}f'\left(f^{-1}\left(\frac{\delta(1-\delta^{1-\alpha})+\delta^{\frac{1-\alpha}{2}}}{2a^\alpha}\xi_0+f(0)\right)\right),\tag{3}$$

$$\frac{1-\alpha^2}{8a^{\frac{1+\alpha}{2}}\eta_0^{3/2}}>\frac{5-\alpha}{4(3-\alpha)}a^{1-\alpha}f'\left(f^{-1}\left(\frac{\delta(1-\delta^{1-\alpha})+\delta^{\frac{1-\alpha}{2}}}{2a^\alpha}\xi_0+f(0)\right)\right),\tag{4}$$

因为$\xi(t)$单调增. 故存在一个小的常数$T_0\in(0,T)$满足

$$\frac{(2-\alpha)(1-\alpha)}{2a^2}\geqslant\frac{a(f(0)+1)}{\xi_0}f'\left(f^{-1}\left(\frac{\delta(1-\delta^{1-\alpha})+\delta^{\frac{1-\alpha}{2}}}{2a^\alpha}\xi(T_0)+f(0)\right)\right),\tag{5}$$

$$\frac{1-\alpha^2}{8a^{\frac{1+\alpha}{2}}\eta_0^{3/2}}>\frac{5-\alpha}{4(3-\alpha)}a^{1-\alpha}f'\left(f^{-1}\left(\frac{\delta(1-\delta^{1-\alpha})+\delta^{\frac{1-\alpha}{2}}}{2a^\alpha}\xi(T_0)+f(0)\right)\right),\tag{6}$$

故由(5),(6)式可知,$J\geqslant 0$. 对于$(x,t)\in(a\delta,a)\times(0,T_0]$有

$$J\geqslant\left((a\delta)^r\frac{\delta(1-\delta^{1-\alpha})+\delta^{\frac{1-\alpha}{2}}}{2a^\alpha}\right)\xi'(t)+\frac{(2-\alpha)(1-\alpha)}{2a^2}\xi(t)$$

$$-f'\left(f^{-1}\left(\left(\psi_m+\frac{1}{2a^\alpha}\right)\xi(t)+f(0)\right)\right)\left(\frac{5-\alpha}{4(3-\alpha)}a^{1-\alpha}\xi(t)+af(0)\right)\geqslant 0\quad(\text{根据(v)})$$

因此$J\geqslant 0,(x,t)\in Q_{T_0}$. 因为$f'(h)>0,(x,t)\in Q_{T_0}$,所以

$$Lh-\int_0^a f(h(x,t))\,\mathrm{d}x\geqslant 0,\qquad (x,t)\in Q_{T_0},$$

$$h(0,t)=f^{-1}(f(0))=0=h(a,t),\quad t\in(0,T_0),$$

$$h(x,0)=f^{-1}\left(\xi_0\psi(x)+\xi_0\frac{1}{2a^{\frac{1+\alpha}{2}}}x^{\frac{1-\alpha}{2}}+f(0)\right)$$

$$\geqslant f^{-1}(\xi_0\psi(x)+f(0))\geqslant f^{-1}(f(u_0(x)))=u_0(x),\qquad x\in[0,a].$$

故$h(x,t)$是问题(1)在$\overline{Q}_{T_0}$上的一个上解.

由引理1采用与[9]类似的方法证.

定理1　存在$t_0<T_0$,使得问题(1)存在解$u\in C^{2,1}((0,a]\times[0,t_0])\cap C(\overline{Q}_{t_0})$.

定理2　问题(1)的解是唯一的.

证明　假设有两个不同的古典解$u\geqslant 0,v\geqslant 0$,令$w=u-v$,则

$$x^rw_t-x^\alpha w_{xx}=\int_\Omega f(u)\,\mathrm{d}x-\int_\Omega f(v)\,\mathrm{d}x=\int_\Omega f'(\xi)w\,\mathrm{d}x,\qquad (x,t)\in Q_{T_0},$$

$$w(0,t)=w(a,t)=0,\qquad t\in(0,t_0),$$

$$w(x,0)=0,\qquad x\in[0,a],$$

其中ξ位于u与v之间,因为$f'(\xi)\geqslant 0$,由引理1可得$w\geqslant 0$. 故$u\geqslant v$同理$u\leqslant v$. 故$u\equiv v$.

定理3　令T是使得问题(1)的解u在$(0,t_0)$存在t_0的上确界,则解在$(0,T)$上存在. 进而若$T<\infty$,则u在Q_T上无界.

2　解的爆破

考虑特征值问题

$$\begin{cases}-(x^\alpha\varphi)_{xx}=\lambda x^r\varphi,\quad x\in(0,a),\\ \varphi(0)=\varphi(a)=0.\end{cases}$$

记$\psi(x)=x^\alpha\varphi(x)$,上述问题化为下列问题

$$\begin{cases}-(\psi(x))_{xx}=\lambda x^{r-\alpha}\psi(x), & x\in(0,a),\\ \psi(0)=\psi(a)=0,\end{cases} \tag{7}$$

作变换 $\psi(x)=x^{\frac{1}{2}}y(x)$,则(7)可化为

$$x^2y''(x)+xy'(x)+\left(\lambda x^{r+2-\alpha}-\frac{1}{4}\right)y(x)=0.$$

令 $x=z^{\frac{2}{r+2-\alpha}}$,上式可化为

$$z^2y''(z)+zy'(z)+\frac{-1+4\lambda z^2}{(r+2-\alpha)^2}y(z)=0.$$

它的一般解是

$$Y(z)=AJ_{\frac{1}{r+2-\alpha}}\left(\frac{2\sqrt{\lambda}}{r+2-\alpha}z\right)+BJ_{\frac{-1}{r+2-\alpha}}\left(\frac{2\sqrt{\lambda}}{r+2-\alpha}z\right).$$

A、B 是任意常数,$J_{\frac{1}{r+2-\alpha}}$,$J_{\frac{-1}{r+2-\alpha}}$ 分别为第一类的阶为 $\frac{1}{r+2-\alpha}$,$-\frac{1}{r+2-\alpha}$ 的 Bessel 函数. 令 μ 为 $J_{\frac{1}{r+2-\alpha}}\left(\frac{2\sqrt{\lambda}}{r+2-\alpha}z\right)$的第一个零点. 则$\mu>0$,且其对应的特征函数为$\psi(x)=kx^{\frac{1}{2}-\alpha}\cdot J_{\frac{1}{r+2-\alpha}}\left(\frac{2\sqrt{\lambda}}{r+2-\alpha}x^{\frac{r+2-\alpha}{2}}\right)>0$,$k>0$,满足$\int_0^a\varphi(x)\mathrm{d}x=1$.

定理4 假设条件(A)成立. 则只要初值 $u_0(x)$ 充分大,问题(1)的解必在有限时刻爆破.

证明 令 $J(t)=\int_0^a x^r\varphi u\mathrm{d}x$,则有

$$J'(t)=\int_0^a x^r\varphi u_t\mathrm{d}x=\int_0^a\varphi(x)\left(x^\alpha u_{xx}+\int_0^a f(u)\mathrm{d}\xi\right)\mathrm{d}x=\int_0^a u(x^\alpha\varphi)_{xx}\mathrm{d}x+\int_0^a f(u)\mathrm{d}\xi.$$

又根据

$$(x^\alpha\varphi)_{xx}=-\lambda x^r\varphi,\quad J'(t)=-\lambda_1\int_0^a ux^r\varphi\mathrm{d}x+\int_0^a f(u)\mathrm{d}\xi=-\lambda_1J(t)+\int_0^a f(u)\mathrm{d}\xi.$$

若 C 为 $x^r\varphi$ 在$[0,a]$上的最大值,则 $C>0$,因此

$$J(t)\leqslant C\int_0^a u\mathrm{d}x.$$

由 Jensen 不等式,可得

$$\int_0^a f(u)\mathrm{d}x\geqslant af\left(\frac{1}{a}\int_0^a u\mathrm{d}x\right).$$

又$f(s)$关于 s 是单调不减的. 得到 $J'(t)\geqslant-\lambda_1J(t)+a\,f\left(\frac{1}{aC}J(t)\right)$,$t\in[0,T)$. 由$\int_0^\infty\frac{1}{f(s)}\mathrm{d}s<+\infty$ 知,只要 $J(0)$ 充分大,u 必在有限时刻爆破.

令 $G(x,\xi,t-\tau)$ 是算子 L 带初边值条件的 Green 函数. 根据[7]和[9]可得下列引理.

引理3 对任意有限时刻 $T>0$,$0\leqslant t\leqslant T$ 及 $x,\xi\in(0,a)$,$G(x,\xi;t-\tau)$ 是正的且存在正常数 C_1(不依赖于 T)及 C_2(依赖于 T),使得

$$C_1\leqslant\int_0^a G(x,\xi;t-\tau)\mathrm{d}\xi\leqslant C_2(T),\qquad 0\leqslant t\leqslant T. \tag{8}$$

定理5 假设f满足且解 u 在有限时刻爆破,则解的爆破点集为整个区间$[0,a]$.

证明 利用 Green 第二恒等式,把问题(1)的解表示成下面的形式

$$u(x,t)=\int_0^a\xi^rG(x,\xi,t)u_0(\xi)\mathrm{d}\xi+\int_0^t\int_0^a G(x,\xi;t-\tau)\left(\int_0^a f(u(x,\tau))\mathrm{d}x\right)\mathrm{d}\xi\mathrm{d}\tau, \tag{9}$$

由(8),(9)式可得,

$$u(x,t)\leqslant C_2a^{r+1}\max_{x\in[0,a]}u_0(x)+C_2\int_0^t\int_0^a f(u(x,\tau))\mathrm{d}x\mathrm{d}\tau.$$

假设 $u(x,t)$ 在 $x_0 \in [0,a]$ 处爆破并记 T^* 为爆破时间,则存在一个序列 $\{(x_m,t_m)\}$,$x_m \in [0,a]$,$t_m < T^*$ 且当 $m \to \infty$ 时,$(x_m,t_m) \to (x_0,T^*)$,使得

$$\lim_{m\to\infty}\int_0^{t_m}\int_0^a f(u(x,\tau))\,\mathrm{d}x\mathrm{d}\tau = \int_0^{T^*}\int_0^a f(u(x,\tau))\,\mathrm{d}x\mathrm{d}\tau = +\infty, \tag{10}$$

对 $[0,a]$ 中的任一点 x',由引理3及(9)式可得

$$u(x',t) \geqslant C_1\int_0^t\int_0^a f(u(x,\tau))\,\mathrm{d}x\mathrm{d}\tau. \tag{11}$$

由(10),(11)式,可得 $\lim\limits_{t\to T^*}u(x',t) = \infty$,这表明 x' 是一个爆破点. 由 x' 的任意性,可知问题(1)的解的爆破点集为整个区间 $[0,a]$.

[参考文献]

[1] Chadam J M, Yin H M. An iteration procedure for a class of integrodifferential equations of parabolic type[J]. J Integral Equations Appl, 1989, 2(1): 31—47.

[2] Chadam J M, Peirce A, Yin H M. The blow-up property of the solutions to some diffusion equations with localized nonlinear reactions[J]. J Math Anal Appl, 1992, 69(2): 313—328.

[3] Chan C Y, Liu H T. Global existence of solutions for degenerate semilinear parabolic problems[J]. Nonlinear Anal, 1998, 34(4): 617—628.

[4] Floater M S. Blow-up at the boundary for degenerate semilinear parabolic equations[J]. Arch Rat Mech Anal, 1991, 114(1): 57—77.

[5] Chan C Y, Yuen S I. Impulsive effects on global existence of solutions for degenerate semilinear parabolic equations[J]. Appl Math Comput, 1998, 90(2): 97—116.

[6] Chan C Y, Kong P C. Quenching for degenerate semilinear parabolic equations[J]. Appl Anal, 1994, 54(1): 17—25.

[7] Chan C Y, Chan W Y. Existence of classic solutions for degenerate semilinear parabolic problems[J]. Appl Math Comput, 1999, 101(2): 125—149.

[8] Mclachlan N W. Bessels Functions for Engineers[M]. 2nd ed., London: Oxford at the Carendon Press, 1955.

[9] Chan C Y, Yang J. Complete blow-up for degenerate semilinear parabolic equations[J]. J Comput Appl Math, 2000, 113(2): 353—364.

[10] Galaktionov V, Vazquez J L. Necessary and sufficient conditions for complete blow-up and extinction for one dimensional quasilinear heat equations[J]. Arch Rat Mech Anal, 1995, 129: 225—244.

[11] Anderson J R. Local existence and uniqueness of solutions of degenerate parabolic equations[J]. Comm Partial Differential Equations, 1991, 16: 105—143.

[责任编辑:陆炳新]