Limit Cycle Problem for a Non-Liénard Type Cubic System

Ali M, Luo Dingjun

(School of Mathematics and Computer Science, Nanjing Normal University, 210097, Nanjing, China)

Abstract: In this paper we discuss the limit cycle problem for a non-Liénard type cubic system. Firstly for quadratic case, we find an example to show the sufficiency about the center conditions. Secondly we prove the uniqueness of limit cycles by translating the system to Liénard type and by using the related uniqueness theorem of limit cycles. Lastly we show that the cubic system may have at least two limit cycles by using the generalized Hopf bifurcation theorem.

Key words: cubic system, quadratic system, uniqueness of limit cycles, bifurcation

CLC number O175.12,　**Document code** A,　**Article ID** 1001-4616(2005)04-0008-05

一类非 Liénard 型三次系统的极限环问题

阿里,罗定军

(南京师范大学数学与计算机科学学院,210097,江苏,南京)

[摘要]　本文讨论了一类非 Liénard 型三次系统的极限环问题. 首先对二次系统情况我们给出了补充中心充要条件的一个例子. 其次将此系统化成 Liénard 系统后设法证明了极限环的惟一性. 最后利用广义 Hopf 分支定理说明此系统可具有两个极限环.

[关键词]　三次系统,二次系统,极限环的惟一性,分支

0 Introduction

In paper [1] and [2], we studied a non-Liénard type cubic system

$$\frac{dx}{dt} = -y + cx + ax^2 + bx^3,\ \frac{dy}{dt} = x + cy, \tag{1}$$

and proved the uniqueness of limit cycles. For system (1), the second equation has no nonlinear term. We now consider the case with a quadratic term dy^2 to instead linear term cy, or equivalently consider the system by exchanging x and y

$$\frac{dx}{dt} = y + ax^2,\ \frac{dy}{dt} = -x + cy + by^2 + dy^3. \tag{2}$$

First we consider the simple case that $c = d = 0$, i.e.,

$$\frac{dx}{dt} = y + ax^2,\ \frac{dy}{dt} = -x + by^2. \tag{3}$$

Change the sign of y, then system (3) becomes

$$\frac{dx}{dt} = -y + ax^2,\ \frac{dy}{dt} = x - by^2. \tag{4}$$

Received date: 2005-05-28.

Foundation item: Supported by the National Natural Science Foundations of China(10471066).

Biography: Ali M, born in 1969, doctor, came from Alzaiem Alazhari University(Sudan), majored in qualitative theory of polynomial differential system.

By comparing (4) with the system (12.16) in [3] we get $a_{20}=a$, $b_{02}=-b$ and other quadratic terms there being all zeroes. Then it is easy to get

$$W_1=0,\ W_2=6ab(a^2+b^2)(a^2-b^2),\ W_3=0. \tag{5}$$

Without loss of generality we may consider $a\geqslant 0$ (otherwise change (y,t) to $(-y,-t)$) and $b\geqslant 0$ (otherwise change (x,t) to $(-x,-t)$).

The system (4) can be divided into two cases (a, b both not zero):

Case 1 $a=b$. It is easy to see the system now is symmetry with respect to the line $y=x$ and O is a center as shown in Fig. 1. There is a homoclinic singular closed orbit Γ passing through saddle S with a family of closed orbits inside Γ. This system gives a gap of the conditions of Theorem 9.1 in [3]. We now give the following conclusion.

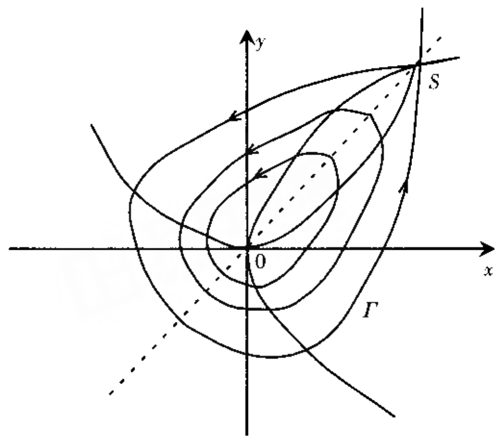

Fig.1 Center system

Theorem 1 For quadratic system in the form

$$\frac{dy}{dx}=-\frac{x+Ax^2+(2B+\alpha)xy+Cy^2}{y+Bx^2+(2C+\beta)xy+Dy^2}, \tag{6}$$

O is a center if and only if one of the following conditions is satisfied:

1) $\alpha=\beta=0$;

2) $A+C=B+D=0$;

3) $A=C=\beta=0$ (or $B=D=\alpha=0$);

4) $A+C=\beta=\alpha+5(B+D)=BD+2D^2+A^2=0$, $B+D\neq 0$ (or $B+D=\alpha=\beta+5(A+C)=AC+2A^2+D^2=0$, $A+C\neq 0$);

5) $A=D=0$, $|B|=|C|$, $2B+\alpha=2C+\beta=0$.

It gives an improvement of Theorem 9.1 in [3] with the additional condition 5). Comparing system (4) with system (6), for case 1 in [3] $B=-a$, $C=-b$, $\alpha=2a$, $\beta=2b$, $A=D=0$. The case does not belong to the conditions 1) ~4) in Theorem 9.1 there. but belongs to the condition 5). We think the reason is that Theorem 9.1 only considered the case $\alpha(a+c)=\beta(b+d)=0$, and missed the case that both sides of the above equality are not zeroes. The condition 5) is just a supplement to this case.

Case 2 $a\neq b$. From (5) we know that O is a second order weak focus. For simplicity we may assume $a=1$ (by rescaling), then consider the case $b>a=1$ and O is a stable weak focus (the case $b<a$ is similar by exchanging x and y).

Theorem 2 For $a\neq b$, the system (4) has no limit cycle.

We shall prove it at the end of next section.

1 Quadratic System Case as $d=0$

Now consider system (2) as $d=0$, i. e, the system

$$\frac{dx}{dt}=y+ax^2\equiv P(x,y),\ \frac{dy}{dt}=-x+cy+by^2\equiv Q(x,y). \tag{7}$$

Lemma 1 For system (7) there is no limit cycle around O as $a=0$.

Proof If $a=c=0$, then it is easy to see that the system has a family of closed orbits around O. From

$$P\frac{\partial Q}{\partial c}-Q\frac{\partial P}{\partial c}=y^2,$$

we know that (7) forms a rotated vector field with respect to parameter c, and then there is no limit cycle for $c\neq 0$.

Thus we only need to consider the case $a\neq 0$, and as above mentioned we may assume $a=1$ to consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y+x^2,\ \frac{\mathrm{d}y}{\mathrm{d}t}=-x+cy+by^2. \tag{8}$$

For (8) as $c=0$, O is a stable weak focus, then by Hopf bifurcation a limit cycle is created from O when c becomes positive. Except the critical point O, the x coordinates of other critical points satisfy the cubic equation

$$bx^3-cx-1=0. \tag{9}$$

When $c<\frac{3}{2}\sqrt[3]{2b}$, there is only one positive root x_0 of (9), for which $S(x_0,x_0^2)$ is a saddle as shown in Fig. 2 (i); when $c\geqslant\frac{3}{2}\sqrt[3]{2b}$, (9) has one or two negative roots, and the phase-portrait is shown in Fig. 2(ii) (S_1 and S_2 coincide together as $c=\frac{3}{2}\sqrt[3]{2b}$). It is easy to see that there is no limit cycle around O at this time. Thus in the following we only consider the case $c<\frac{3}{2}\sqrt[3]{2b}$ and $c<2$ (if $c\geqslant 2$, then there is no limit cycle around O since O is a node).

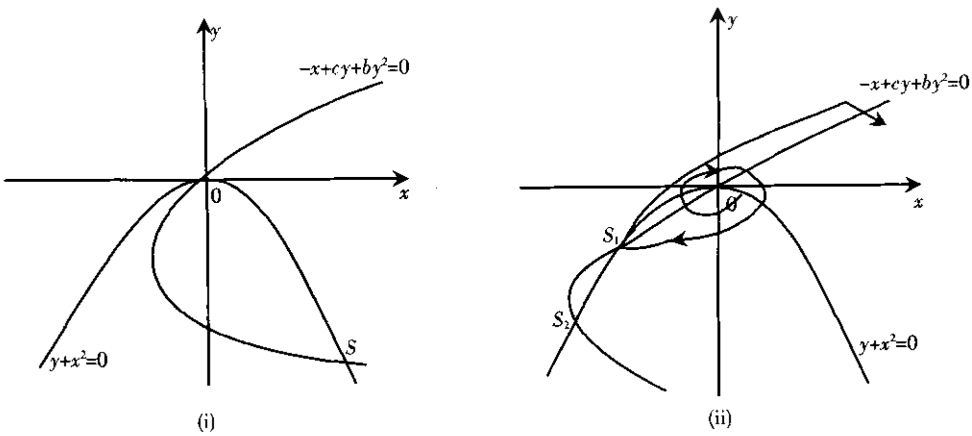

Fig.2 Phase-portraits without limit cycle

To prove the uniqueness of this limit cycle we need to translate the system (8) to Liénard type.

Lemma 2 System (8) can be reduced to a system of Liénard type

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y-F(x),\ \frac{\mathrm{d}y}{\mathrm{d}t}=-g(x) \tag{10}$$

with $F(x)=\int_0^x(2bx^2-2x-c)\mathrm{e}^{-bx}\mathrm{d}x$ and $g(x)=-x(bx^3-cx-1)\mathrm{e}^{-2bx}$.

Proof Let $x=x$, $\zeta=y+x^2$, then we get $\dot{x}=\zeta$, $\dot{\zeta}=-x-cx^2+bx^4+(-2bx^2+2x+c)\zeta+b^2\zeta^2$. Next, let $x=x$, $\zeta=u\mathrm{e}^{bx}$, then $\dot{x}=u\mathrm{e}^{bx}$ and $\dot{u}=(bx^4-cx^2-x)\mathrm{e}^{-bx}+(-2bx^2+2x+c)u$.

At last, let $x=x$, $v=u+\int_0^x(2bx^2-2x-c)\mathrm{e}^{bx}\mathrm{d}x$, $\frac{\mathrm{d}t}{\mathrm{d}\tau}=\mathrm{e}^{-bx}$, then we find the following system after changing v to y and τ to t:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y-\int_0^x(2bx^2-2x-c)\mathrm{e}^{-bx}\mathrm{d}x,\ \frac{\mathrm{d}y}{\mathrm{d}t}=x(bx^3-cx-1)\mathrm{e}^{-2bx},$$

which is in the Liénard type (10) with

$$F(x)=\int_0^x(2bx^2-2x-c)\mathrm{e}^{-bx}\mathrm{d}x=\left(-2x^2-\frac{2}{b}x+\frac{bc-2}{b^2}\right)\mathrm{e}^{-bx}-\frac{bc-2}{b^2},$$

$$g(x)=-x(bx^3-cx-1)\mathrm{e}^{-2bx}. \tag{11}$$

Denote $f(x)=F'(x)=(2bx^2-2x-c)\mathrm{e}^{-bx}$ and $G(x)=\int_0^x g(x)\mathrm{d}x$.

For Liénard system, we use the following result about the uniqueness of limit cycles. Suppose that $f(x)$,

$g(x)$ are continuous in (r_1, r_2), $r_1 < 0 < r_2$, $xg(x) > 0$ for $x \in (r_1, r_2)$, $x \neq 0$, $f(x) < 0$, $x \in (x_1, x_2)$ and $f(x) > 0$ outside of $x \in (x_1, x_2)$, where $r_1 < x_1 < 0 < x_2 < r_2$. By the Filippov transformation $z = G(x) = \int_0^x g(x)\,dx$, the inverse functions are $x_1 = x_1(z) > 0$ and $x_2 = x_2(z) < 0$, $z > 0$. $F_i(z) = F(x_i(z))$, $i = 1, 2$, $F(x) = \int_0^x f(x)\,dx$.

Lemma 3[4] For above $f(x)$, $g(x)$ if they also satisfy the following conditions:

(i) there exists $\tilde{z} > 0$, such that $F_1(z) < F_2(z)$, $0 < z < \tilde{z}$, $F_1(z) > F_2(z)$, $z > \tilde{z}$;

(ii) $\dfrac{f(x)}{g(x)}$ is increasing for $x \in (u, r_2)$, $u = x_1(\tilde{z})$,

then the Liénard system (10) has at most one limit cycle in the strip region $r_1 < x < r_2$, $-\infty < y < +\infty$.

Theorem 3 System (8) has at most one limit cycle.

Proof We now check that after changing system (8) to (10), the conditions above are satisfied. Now

$$f(x) = (2bx^2 - 2x - c)e^{-bx},\quad F(x) = \left(-2x^2 - \frac{2}{b}x + \frac{bc-2}{b^2}\right)e^{-bx} - \frac{bc-2}{b^2},$$

$$g(x) = -x(bx^3 - cx - 1)e^{-2bx}.$$

We omit the explicit expression of $G(x)$ since it is not used in the following. The pictures of them are shown in Fig. 3.

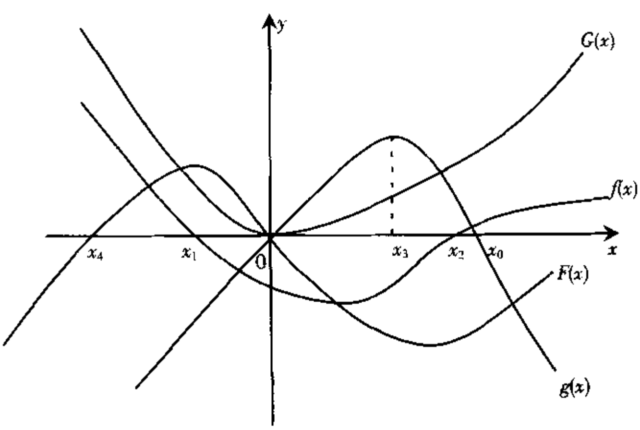

Fig.3 Figures of $f(x)$, $g(x)$

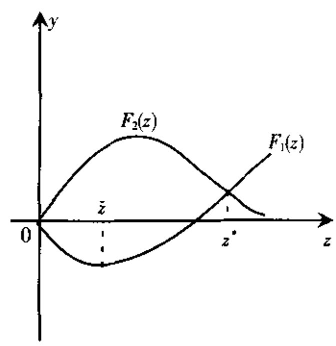

Fig.4 Figures of $F_i(z)$

$g(x)$ has a positive zero point x_0, for which there is a saddle lies on $x = x_0$. The limit cycle locates in the part of $x < x_0$, in which $f(x)$ has two zeros $x_1 < 0 < x_2$ (notice that if $x_2 > x_0$ the Lemma still can be used). $F(x)$ has no positive zero point as $bc \geqslant 2$ and has one positive zero point as $bc < 2$, and has one negative zero point x_4.

Now $F'_i(z) = \dfrac{dF_i(z)}{dx} \Big/ \dfrac{dz}{dx}$, then from Fig. 3 we see that

$$F'_1(z) = \frac{f(x)}{g(x)} \begin{matrix} <0, & 0 < x < x_2, \\ >0, & x_2 < x; \end{matrix} \quad F'_2(z) = \frac{f(x)}{g(x)} \begin{matrix} >0, & x_1 < x < 0, \\ <0, & x < x_1. \end{matrix}$$

Then the condition (i) is satisfied as in Fig. 4.① To check the condition (ii), let x_3 be the zero point of $g'(x)$. For $x \in (0, x_0)$, $g(x) > 0$ and $g'(x) > 0$, $x \in (0, x_3)$ and $g'(x) < 0$, $x \in (x_3, x_0)$, see Fig. 3.

$$\begin{aligned} g'(x) &= [-2b(-bx^4 + cx^2 + x) + (-4bx^3 + 2cx + 1)]e^{-2bx} \\ &= (2b^2x^4 - 4bx^3 - 2bcx^2 + 2(c-b)x + 1)e^{-2bx} = g_1(x)e^{-2bx}, \\ g_1(x) &= 2b^2x^4 - 4bx^3 - 2bcx^2 + 2(c-b)x + 1 \end{aligned}$$

① If $y = F_1(z)$ and $y = F_2(z)$ do not intersect each other, then $F_2(z) > F_1(z)$ and there is no limit cycle by Theorem 5.4 of [3]. The conclusion is obtained.

$$= (2bx^2 - 2x - c)\left(bx^2 - x - \frac{2+bc}{2b}\right) - \frac{2b^2+2}{2b}x + \frac{2b-(2+bc)c}{2b}.$$

$$g_1(x_2) = -\frac{2b^2+2}{b}x_2 + 1 - \frac{2+bc}{b}c, \text{ since } x_2 = \frac{1+\sqrt{1+2bc}}{2b} > \frac{1}{b}, \text{ then}$$

$$g_1(x_2) < -\frac{2b^2+2}{b}\cdot\frac{1}{b} + 1 - \frac{2+bc}{b}c = -2 - \frac{2}{b^2} + 1 - \frac{2+bc}{b}c < 0.$$

That means $x_3 < x_2$. From Fig. 4 we see that $\tilde{z} > z^*$, $u = x_1(\tilde{z}) > x_1(z^*) = x_2$. For $x \in (u, x_0) \Rightarrow x \in (x_2, x_0)$ $\Rightarrow g(x) > 0$, $f(x) > 0$, $f'(x) > 0$ and $g'(x) < 0$, then we get

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} > 0,\ x \in (u, x_0).$$

Thus Theorem 3 has been proved.

Now as a corollary, we give a proof of Theorem 2, i. e., for $c = 0$ system (7) has no limit cycle.

Proof Assume that for $c = 0$ system (7) has a limit cycle with O stable, then changing c from zero to $c > 0$, we get the second limit cycle by Hopf bifurcation, which is a contradiction with Theorem 3.

2 Quadratic System Case as $d \neq 0$

Let $d \neq 0$, then system (2) as $a = 1$ becomes

$$\frac{dx}{dt} = y + x^2,\ \frac{dy}{dt} = -x + cy + by^2 + dy^3. \tag{12}$$

We now use the formula of focal value in [5] for the system

$$\frac{dx}{dt} = -\omega y + \sum_{i+j=1}^{\infty} F_{ij}x^i y^j,\ \frac{dy}{dt} = \omega x + \sum_{i+j=1}^{\infty} G_{ij}x^i y^j. \tag{13}$$

and the first focal value of (13) is

$$P_4 = \frac{1}{8}\{3(F_{30} + G_{03}) + F_{12} + G_{21} + \frac{1}{\omega}[F_{11}(F_{20} + F_{02}) - G_{11}(G_{20} + G_{02}) + 2(F_{02}G_{02} - F_{20}G_{20})]\}.$$

For system (12), only $F_{20} = 1$, $G_{02} = b$, $G_{03} = d$ and other F_{ij} are all zeros, $\omega = 1$, then the focal value is

$$P_4 = \frac{3}{8}d.$$

Theorem 3 For certain parameters b, c, d the system (12) has at least two limit cycles.

Proof Since from (2), as $c = d = 0$, $b > 1$ O is a stable weak focus of order two. Firstly, keep $c = 0$, let d become positive, then O becomes an unstable weak focus of order one and a stable limit cycle L_1 bifurcates. Secondly, change c to negative with $|c|$ small enough, such that L_1 does not disappear but O changes its stability again and an unstable limit cycle L_2 bifurcates in the interior of L_1. The conclusion is obtained.

[References]

[1] Yuan Weili, Luo Dingjun. Qualitative study of a non-Liénard type cubic system [J]. Journal of Nanjing University, Math Biquarterly, 2002, 19(1): 8—17.

[2] Ali M, Luo Dingjun. Qualitative study of a non-Liénard type cubic system [J]. Annales of Differential Equations, 2004, 20(4): 331—336.

[3] Ye Yanqian, *et al*. Theory of Limit Cycles [M]. Providence: Amer Math Soc Translation Math Monographs 66, 1986.

[4] Luo Dingjun, Wang Xian, Zhu Deming, *et al*. Bifurcation Theory and Methods of Dynamical Systems [M]. Singapore: World Scientific Publ Co, 1997.

[5] Gobber F, Willamowski K D. Ljapunov approach to multiple Hopf-bifurcation [J]. J Math Anal Appl, 1979, 71(2): 333—350.

[责任编辑:陆炳新]