

# $Q_K$ Spaces on the Unit Ball of $\mathbf{C}^n$

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**Abstract** In this paper , we introduce a class of Möbius-invariant Banach spaces  $Q_K$  of analytic functions on the unit ball of  $\mathbf{C}^n$  , where  $K : ( 0 , \infty ) \rightarrow [ 0 , \infty )$  are non-decreasing functions , and develop the general theories on  $Q_K$  including an equivalent condition for  $Q_K$  to be non-trivial , the nesting property of  $Q_K$  , and the sufficient condition for  $Q_K = \mathcal{A}(\mathbf{B})$  and so on. Let  $K(t) = t^p$  , then we get the known results on  $Q_p$  .

**Key words** Möbius invariant ,  $Q_K$  space , Bloch space

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## $\mathbf{C}^n$ 中单位球上的 $Q_K$ 空间

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[ 摘要 ] 本文中 , 我们引入了  $\mathbf{C}^n$  中单位球上 Möbius 不变的解析函数巴拿赫空间  $Q_K$  , 这里  $K : ( 0 , \infty ) \rightarrow [ 0 , \infty )$  是非减的函数 , 并发展了  $Q_K$  上一般理论 , 包括  $Q_K$  非平凡的等价条件 ,  $Q_K$  间包含关系 ,  $Q_K = \mathcal{A}(\mathbf{B})$  的充分条件等等. 令  $K(t) = t^p$  , 则得到已知的  $Q_p$  上结论.

[ 关键词 ] Möbius 不变  $Q_K$  空间 , Bloch 空间

## 0 Introduction

The notion of the spaces  $Q_p$  was first considered for holomorphic functions defined on the unit disk  $\Delta$  of the complex plane ( cf. [ 13 , 14 , 2 , 5 ] and [ 4 ] ) and , later , generalized to hyperbolic Riemann surfaces  $R$  and the unit ball of  $\mathbf{C}^n$  ( cf. [ 1 , 3 ] ). In [ 6 ] , more general spaces  $Q_K$  was introduced and investigated. In this paper we study  $Q_K$  spaces on the unit ball  $\mathbf{B}$  of  $\mathbf{C}^n$  . Let  $K : ( 0 , \infty ) \rightarrow [ 0 , \infty )$  is a right- continuous , non-decreasing function and is not equal to zero identically.  $Q_K$  space , denoted by  $Q_K(\mathbf{B})$  , is defined as the class of all holomorphic functions  $f$  on  $\mathbf{B}$  , for which  $\sup_{a \in \mathbf{B}} \int_{\mathbf{B}} |\tilde{\nabla} f(z)|^2 K(\alpha(z,a)) d\lambda(z) < \infty$  , where  $\tilde{\nabla} f$  and  $d\lambda$  denote the invariant gradient , Green 's function and the invariant volume measure respectively. When  $K(t) = t^p$  ,  $Q_K(\mathbf{B}) = Q_p(\mathbf{B})$  . It is proved in [ 8 ] that  $Q_p(\mathbf{B})$  is not trivial if and only if  $(n-1)/n < p < n/(n-1)$  , and that  $Q_p(\mathbf{B}) \subset Q_q(\mathbf{B}) \subset \mathcal{A}(\mathbf{B})$  if  $(n-1)/n < p < q < n/(n-1)$  . The purpose of this paper is to generalize these results to  $Q_K(\mathbf{B})$  .

The paper is organized as follows : In Section 2 , we explain some notations , concepts and known results which can be found in [ 9 , 10 , 12 ] . In Section 3 , we prove that  $Q_K$  are Banach spaces and contained in the Bloch space  $\mathcal{A}(\mathbf{B})$  . A sufficient and necessary condition for  $Q_K$  to be non-trivial is given in Section 4. In Section 5 ,

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we discuss the nesting property of  $Q_K$  and prove a sufficient condition for  $Q_K = \mathcal{A}(\mathbf{B})$ .

## 1 Preliminaries

Let  $\mathbf{B}$  denote the unit ball of  $\mathbf{C}^n$ ,  $\mathbf{S}$  the boundary of  $\mathbf{B}$ , and  $\text{Aut}(\mathbf{B})$  the group of biholomorphic automorphisms of  $\mathbf{B}$ . For  $a \in \mathbf{B}$ , let  $\phi_a \in \text{Aut}(\mathbf{B})$  be the Möbius transformation of  $\mathbf{B}$  which satisfies  $\phi_a(0) = a$  and  $\phi_a(a) = 0$ . By  $H(\mathbf{B})$  we denote the collection of all holomorphic functions in  $\mathbf{B}$ . For  $f \in H(\mathbf{B})$  and  $z = (z_1, \dots, z_n) \in \mathbf{B}$ , let  $\nabla f(z) = \left( \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n} \right)$  denote the complex gradient of  $f$ , and  $\mathcal{R}f(z) = \sum_{j=1}^n z_j \left( \frac{\partial f}{\partial z_j} \right)$  denote the radial derivative of  $f$ .

Let  $\nu$  denote the Lebesgue measure on  $\mathbf{C}^n = \mathbf{R}^{2n}$ , so normalized that  $\nu(\mathbf{B}) = 1$  and  $\sigma$  the normalized surface measure on  $\mathbf{S}$  so that  $\sigma(\mathbf{S}) = 1$ , the measures  $\nu$  and  $\sigma$  are related by the following formula

$$\int_{\mathbf{B}_n} f d\nu = 2n \int_0^1 r^{2n-1} dr \int_{\mathbf{S}} f(r\zeta) d\sigma(\zeta). \quad (1)$$

The identity

$$\int_{\mathbf{S}} f d\sigma = \int_{\mathbf{S}} d\sigma(\zeta) \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}\zeta) d\theta \quad (2)$$

is called integration by slices [9].

Let

$$d\lambda(z) = \frac{d\nu(z)}{(1 - |z|^2)^{n+1}}.$$

Then  $d\lambda(z)$  is Möbius invariant, which means that  $\int_{\mathbf{B}} f(z) d\lambda(z) = \int_{\mathbf{B}} f(\psi(z)) d\lambda(z)$  for  $f \in L^1(\lambda)$  and  $\psi \in \text{Aut}(\mathbf{B})$ . Let  $\tilde{\nabla}f(z) = \nabla(f \circ \psi_z)(0)$ , then  $\tilde{\nabla}f(z)$  is the invariant gradient of  $f$  (cf. [10]).  $\tilde{\nabla}f(z)$  and  $\nabla f(z)$  are related by ([7])

$$|\tilde{\nabla}f(z)|^2 = (1 - |z|^2) (|\nabla f(z)|^2 - |\mathcal{R}f(z)|^2). \quad (3)$$

As a consequence,

$$(1 - |z|^2) |\nabla f(z)| \leq |\tilde{\nabla}f(z)| \leq (1 - |z|^2)^{1/2} |\nabla f(z)|. \quad (4)$$

The Möbius invariant Green function is defined by  $G(z, a) = g(\phi_a(z))$ , where

$$g(z) = \frac{n+1}{2n} \int_{|t| \leq 1} (1 - t^2)^{n-1} t^{-2n+1} dt. \quad (5)$$

Let  $K: (0, \infty) \rightarrow [0, \infty)$  is a right-continuous and non-decreasing function and not equal to a constant identically. For  $f \in H(\mathbf{B})$ ,  $a \in \mathbf{B}$ , define

$$\begin{aligned} I_K(f, a) &= \int_{\mathbf{B}} |\tilde{\nabla}f(z)|^2 K(G(z, a)) d\lambda(z), \\ J_K(f, a) &= \int_{\mathbf{B}} |\tilde{\nabla}f(z)|^2 K((1 - |\phi_a(z)|^2)^n) d\lambda(z), \\ \|f\|_{Q_K}^2 &= \sup_{a \in \mathbf{B}} I_K(f, a), \\ \|f\|_{M_K}^2 &= \sup_{a \in \mathbf{B}} J_K(f, a), \\ Q_K(\mathbf{B}) &= \{f \in H(\mathbf{B}) : \|f\|_{Q_K} < \infty\}, \\ M_K(\mathbf{B}) &= \{f \in H(\mathbf{B}) : \|f\|_{M_K} < \infty\}. \end{aligned}$$

The Bloch semi-norm of a function in  $H(\mathbf{B})$  is defined by  $\|f\|_{\mathcal{B}} = \sup_{z \in \mathbf{B}} |\tilde{\nabla}f(z)|$ , and the Bloch space  $\mathcal{A}(\mathbf{B})$  consists of all functions  $f$ , for which  $\|f\|_{\mathcal{B}} < \infty$ .

## 2 $Q_K$ Spaces and Bloch Space

**Theorem 1**  $Q_K \subset \mathcal{A}(\mathbf{B})$ .

**Proof** By the definition of  $g(z)$  in Section 2, there exists a  $r_0 > 0$  such that  $g(z) \geq 1$  for  $|z| \leq r_0$ . Without loss of generality, assume that  $K(1) > 0$ . Then, for  $f \in Q_K$  and  $a \in \mathbf{B}$ , using (3), we have

$$\begin{aligned} \int_{\mathbf{B}} |\tilde{\nabla} f(z)|^2 K(\alpha(z, a)) d\lambda(z) &\geq \int_{B_{r_0}} |\tilde{\nabla}(f \circ \varphi_a)(z)|^2 K(g(z)) d\lambda(z) \\ &\geq \int_{B_{r_0}} (1 - |z|^2)^2 |\nabla(f \circ \varphi_a)(z)|^2 K(g(z)(1 - |z|^2)^{-n-1}) d\nu(z) \\ &\geq \int_{B_{r_0}} |\nabla(f \circ \varphi_a)(z)|^2 K(1) d\nu(z). \end{aligned}$$

By (2.1) and the subharmonicity of  $|\nabla(f \circ \varphi_a)(z)|^2$ ,

$$\begin{aligned} \int_{\mathbf{B}} |\tilde{\nabla} f(z)|^2 K(\alpha(z, a)) d\lambda(z) &\geq 2nK(1) \int_0^{r_0} r^{2n-1} dr \int_{\mathbf{S}} |\nabla(f \circ \varphi_a)(r\zeta)|^2 d\sigma(\zeta) \\ &\geq K(1) r_0^{2n} |\nabla(f \circ \varphi_a)(0)|^2 = r_0^{2n} K(1) |\tilde{\nabla} f(a)|^2. \end{aligned}$$

Thus, there exists a constant  $C$  such that

$$\|f\|_{Q_K}^2 \geq C \|f\|_{\mathcal{B}}^2, \quad (6)$$

This shows that  $f \in \mathcal{A}(\mathbf{B})$ . The proof is complete.

**Theorem 2**  $Q_K$  are Möbius invariant Banach spaces equipped with the norm  $|f(0)| + \|f\|_{Q_K}$ .

**Proof** Let  $K(t)$  be fixed. For  $f \in H(\mathbf{B})$ ,  $\varphi \in \text{Aut}(\mathbf{B})$  and  $a \in \mathbf{B}$ , by the invariance of  $\tilde{\nabla}$ , Green's function and  $d\lambda(z)$ ,

$$\begin{aligned} \int_{\mathbf{B}} |\tilde{\nabla} f(z)|^2 K(\alpha(z, a)) d\lambda(z) &= \int_{\mathbf{B}} |\tilde{\nabla} f(\varphi(z))|^2 K(\alpha(\varphi(z), a)) d\lambda(z) \\ &= \int_{\mathbf{B}} |\tilde{\nabla}(f \circ \varphi)(z)|^2 K(\alpha(z, \varphi^{-1}(a))) d\lambda(z). \end{aligned}$$

Thus,  $\|f\|_{Q_K} = \|f \circ \varphi\|_{Q_K}$ . This shows the Möbius invariance of  $Q_K$ .

From the definition of  $\tilde{\nabla}$ ,  $\tilde{\nabla}(f + h)(z) = \nabla[(f + h) \circ \varphi_z](0) = \nabla(f \circ \varphi_z)(0) + \nabla(h \circ \varphi_z)(0) = \tilde{\nabla} f(z) + \tilde{\nabla} h(z)$ . Therefore,

$$\begin{aligned} I_K(f + h, a) &\leq \int_{\mathbf{B}} (|\tilde{\nabla} f| + |\tilde{\nabla} h|)^2 K(\alpha(z, a)) d\lambda(z) \\ &\leq I_K(f, a) + I_K(h, a) + 2 \int_{\mathbf{B}} |\tilde{\nabla} f(z)| \cdot |\tilde{\nabla} h(z)| K(\alpha(z, a)) d\lambda(z). \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} \int_{\mathbf{B}} |\tilde{\nabla} f(z)| |\tilde{\nabla} h(z)| K(\alpha(z, a)) d\lambda(z) &\leq \left( \int_{\mathbf{B}} |\tilde{\nabla} f(z)|^2 K(\alpha(z, a)) d\lambda(z) \right)^{1/2} \left( \int_{\mathbf{B}} |\tilde{\nabla} h(z)|^2 K(\alpha(z, a)) d\lambda(z) \right)^{1/2} \\ &= [I_K(f, a)]^{1/2} \cdot [I_K(h, a)]^{1/2}. \end{aligned}$$

Thus,

$$\begin{aligned} I_K(f + h, a) &\leq I_K(f, a) + I_K(h, a) + 2[I_K(f, a)]^{1/2} \cdot [I_K(h, a)]^{1/2}, \\ (I_K(f + h, a))^{1/2} &\leq [I_K(f, a)]^{1/2} + [I_K(h, a)]^{1/2}, \|f + h\|_{Q_K} \leq \|f\|_{Q_K} + \|h\|_{Q_K}. \end{aligned}$$

It is obvious that  $\|f\|_{Q_K} = 0$  if and only if  $f \equiv C$ . We have proved that  $Q_K$  are linear normed spaces with the norm  $|f(0)| + \|f\|_{Q_K}$ .

Now we proceed to prove the completeness. Let  $\{f_l\}_{l=1}^\infty$  be a Cauchy sequence in  $Q_K$ . By (6),  $\{f_l\}$  is also a Cauchy sequence in  $\mathcal{A}(\mathbf{B})$ . By the completeness of  $\mathcal{A}(\mathbf{B})$  [11], there is a function  $f \in \mathcal{A}(\mathbf{B})$  such that  $f_l \rightarrow f$  in the Bloch space and, consequently,  $f_l \rightarrow f$  locally uniformly in  $\mathbf{B}$ . By Fatou's Lemma,  $\|f\|_{Q_K} \leq \liminf_{l \rightarrow \infty} \|f_l\|_{Q_K} < \infty$ . This shows that  $f \in Q_K$ . For  $\epsilon > 0$ , there is an  $N$  such that  $\|f_l - f_m\|_{Q_K} < \epsilon$  as  $l, m > N$ . Then, for  $l > N$ ,

$$I_K(f_l - f, a) = \int_{\mathbf{B}} |\tilde{\nabla}(f_l - f)(z)|^2 K(\alpha(z, a)) d\lambda(z) =$$

$$\begin{aligned} & \int_{\mathbf{B}} \lim_{m \rightarrow \infty} |\tilde{\nabla}(f_l - f_m)(z)|^2 K(\alpha(z, \mu)) d\lambda(z) \leq \\ & \liminf_{m \rightarrow \infty} \int_{\mathbf{B}} |\tilde{\nabla}(f_l - f_m)(z)|^2 K(\alpha(z, \mu)) d\lambda(z) \leq \\ & \lim_{m \rightarrow \infty} \int_{\mathbf{B}} \|f_l - f_m\|_{Q_K}^2 \leq \epsilon^2 \end{aligned}$$

Thus,  $\|f_l - f\|_{Q_K} \leq \epsilon$  for  $l > N$ . It is proved that  $f_l \rightarrow f$  in  $Q_K$ . The theorem is proved.

### 3 The Condition for Trivialness

**Lemma 1** Let  $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha} \in Q_K$  be a non-constant function, where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  is an  $n$ -tuple of non-negative integers and  $z^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$ . Then,  $z^{\alpha} \in Q_K$  if  $a_{\alpha} \neq 0$ .

**Proof** Let  $\alpha = (k_1, \dots, k_n)$  be such that  $a_{\alpha} \neq 0$ . Then

$$a_{\alpha} z^{\alpha} = \frac{1}{2\pi} \int_0^{2\pi} \dots \int_0^{2\pi} f(z_1 e^{i\theta_1}, \dots, z_n e^{i\theta_n}) e^{-ik_1\theta_1} \dots e^{-ik_n\theta_n} d\theta_1 \dots d\theta_n. \quad (7)$$

Let  $F(z) = a_{\alpha} z^{\alpha}$  and  $U_{\theta} f(z) = f(z_1 e^{i\theta_1}, \dots, z_n e^{i\theta_n}) = f \circ U(z_1, \dots, z_n)$ , where  $U(z) = (z_1 e^{i\theta_1}, \dots, z_n e^{i\theta_n})$ . Let

$$\tilde{\nabla}_j f(z) = \frac{\partial}{\partial w_j} (f \circ \varphi_z)(w) \Big|_{w=0}, \text{ that is } \tilde{\nabla} = (\tilde{\nabla}_1, \dots, \tilde{\nabla}_n). \text{ By (4.1),}$$

$$\begin{aligned} (F \circ \varphi_z)(w) &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} ((U_{\theta} f) \circ \varphi_z)(w) e^{-ik_1\theta_1} \dots e^{-ik_n\theta_n} d\theta_1 \dots d\theta_n, \\ \frac{\partial}{\partial w_j} (F \circ \varphi_z)(w) &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} \frac{\partial}{\partial w_j} ((U_{\theta} f) \circ \varphi_z)(w) e^{-ik_1\theta_1} \dots e^{-ik_n\theta_n} d\theta_1 \dots d\theta_n. \end{aligned}$$

Letting  $w = 0$ , we have  $\tilde{\nabla}_j F(z) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} \tilde{\nabla}_j U_{\theta} f(z) e^{-ik_1\theta_1} \dots e^{-ik_n\theta_n} d\theta_1 \dots d\theta_n$  and, by Jensen's inequality on convexity,

$$|\tilde{\nabla}_j F(z)|^2 \leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} |\tilde{\nabla}_j U_{\theta} f(z)|^2 d\theta_1 \dots d\theta_n.$$

Thus,

$$\begin{aligned} |\tilde{\nabla} F(z)|^2 &= \sum_{j=1}^n |\tilde{\nabla}_j F(z)|^2 \\ &\leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{j=1}^n |\tilde{\nabla}_j U_{\theta} f(z)|^2 d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} |\tilde{\nabla} U_{\theta} f(z)|^2 d\theta_1 \dots d\theta_n. \end{aligned}$$

Consequently,

$$\begin{aligned} I_K(F, \mu) &= \int_{\mathbf{B}} |\tilde{\nabla} F(z)|^2 K(\alpha(z, \mu)) d\lambda(z) \\ &\leq \int_{\mathbf{B}} \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} |\tilde{\nabla} U_{\theta} f(z)|^2 d\theta_1 \dots d\theta_n K(\alpha(z, \mu)) d\lambda(z) \\ &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} \int_{\mathbf{B}} |\tilde{\nabla} U_{\theta} f(z)|^2 K(\alpha(z, \mu)) d\lambda(z) d\theta_1 \dots d\theta_n \\ &\leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} \|U_{\theta} f\|_{Q_K}^2 d\theta_1 \dots d\theta_n = \|U_{\theta} f\|_{Q_K}^2. \end{aligned}$$

Because  $U \in \text{Aut}(\mathbf{B})$ , we have  $\|U_{\theta} f\|_{Q_K} = \|f\|_{Q_K}$  as shown at the beginning of the proof of Theorem 2. Therefore,  $\|a_{\alpha} z^{\alpha}\|_{Q_K}^2 = \|F\|_{Q_K}^2 = \sup_{\alpha \in B} I_K(F, \mu) \leq \|f\|_{Q_K}^2$  and  $z^{\alpha} \in Q_K$ . The lemma is proved.

**Theorem 3** If

$$\int_0^1 r^{2n-1} (1-r^2)^{-n} K(g(r)) dr < \infty, \quad (8)$$

where  $g$  is the function defined by (5), then  $Q_k$  contains all polynomials; otherwise,  $Q_k$  contains only constant functions.

**Proof** First, assume that (8) holds. Let  $f(z)$  be a polynomial. Then,

$$\begin{aligned} \int_{\mathbf{B}} |\tilde{\nabla} f(z)|^2 K(\alpha(z, \mu)) d\lambda(z) &= \int_{\mathbf{B}} |\tilde{\nabla} f(\varphi_a(z))|^2 K(g(z)) \frac{d\lambda(z)}{(1 - |z|^2)^n + 1} \\ &= 2n \int_0^1 r^{2n-1} (1 - r^2)^{-n-1} dr \int_{\mathbf{S}} |\tilde{\nabla} f(\varphi_a(r\zeta))|^2 d\sigma(\zeta). \end{aligned}$$

Since  $f(z)$  is a polynomial, there exists a  $M > 0$  such that  $|\nabla f(z)| \leq M$  for all  $z \in \overline{\mathbf{B}}$ . Thus,

$$\begin{aligned} |\tilde{\nabla} f(\varphi_a(r\zeta))|^2 &= (1 - |\varphi_a(r\zeta)|^2) |\nabla f(\varphi_a(r\zeta))|^2 - R(f(\varphi_a(r\zeta)))^2 \\ &\leq (1 - |\varphi_a(r\zeta)|^2) |\nabla f(\varphi_a(r\zeta))|^2 \\ &\leq \frac{M(1 - |a|^2)(1 - r^2)}{(1 - r\zeta\mu)^2} \\ &= M(1 - |a|^2)(1 - r^2) |1 + r\zeta\mu + r\zeta\mu^2 + \dots|^2. \end{aligned}$$

We have, for  $k = 0, 1, \dots$ ,

$$\int_{\mathbf{S}} |r\zeta\mu|^{2k} d\sigma(\zeta) \leq \int_{\mathbf{S}} (|r\zeta| \cdot |a|)^{2k} d\sigma(\zeta) = (r|a|)^{2k} \int_{\mathbf{S}} d\sigma(\zeta) = (r|a|)^{2k},$$

and, for  $m = 1, 2, \dots$ ,

$$\begin{aligned} \int_{\mathbf{S}} |r\zeta\mu|^{2k} r\zeta\mu^m d\sigma(\zeta) &= \int_{\mathbf{S}} d\sigma(\zeta) \frac{1}{2\pi} \int_0^{2\pi} |re^{i\theta}\zeta\mu|^{2k} re^{i\theta}\zeta\mu^m d\theta \\ &= \int_{\mathbf{S}} d\sigma(\zeta) \frac{1}{2\pi} \int_0^{2\pi} |r\zeta\mu|^{2k} r\zeta\mu^m \cdot e^{im\theta} d\theta = 0. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathbf{S}} |\tilde{\nabla} f(\varphi_a(r\zeta))|^2 d\sigma(\zeta) &= M(1 - |a|^2)(1 - r^2) \cdot \sum_{k=0}^{\infty} (r|a|)^{2k} \\ &= M(1 - |a|^2)(1 - r^2) \cdot \frac{1}{1 - r^2|a|^2} \leq M(1 - r^2), \end{aligned}$$

and  $\int_{\mathbf{B}} |\tilde{\nabla} f(z)|^2 K(\alpha(z, \mu)) d\lambda(z) \leq 2nM \int_0^1 r^{2n-1} (1 - r^2)^{-n} K(g(r)) dr$ . Since  $a$  is arbitrary, it follows that  $\|f\|_{Q_k}^2 \leq 2nM \int_0^1 r^{2n-1} (1 - r^2)^{-n} K(g(r)) dr < \infty$ . This shows that  $f \in Q_k$  and the first half of the theorem is proved.

Now, we assume that the integral in (8) is divergent. Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  is an  $n$ -tuple of non-negative integers,  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n \geq 1$ ,  $f(z) = z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$ . Then,

$$\begin{aligned} \nabla f(z) &= (\alpha_1 z_1^{\alpha_1-1} z_2^{\alpha_2} \dots z_n^{\alpha_n}, \dots, \alpha_n z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n-1}), \\ \mathcal{R}f(z) &= \alpha_1 z_1^{\alpha_1} \dots z_n^{\alpha_n} + \dots + \alpha_n z_1^{\alpha_1} \dots z_n^{\alpha_n} = |\alpha| z^\alpha. \end{aligned}$$

Thus, by (1.3),  $|\tilde{\nabla} f(z)|^2 = (1 - |z|^2) |\alpha_1^2 |z_1^{\alpha_1-1} \dots z_n^{\alpha_n}|^2 + \dots + \alpha_n^2 |z_1^{\alpha_1} \dots z_n^{\alpha_n-1}|^2 - |\alpha|^2 z^\alpha) = (1 - |z|^2) \mathcal{K}(z)$ , where  $\mathcal{K}(z) = \alpha_1^2 |z_1^{\alpha_1-1} \dots z_n^{\alpha_n}|^2 + \dots + \alpha_n^2 |z_1^{\alpha_1} \dots z_n^{\alpha_n-1}|^2 - |\alpha|^2 z^\alpha$ .

We have

$$\begin{aligned} \|f\|_{Q_k} &\geq \int_{\mathbf{B}} |\tilde{\nabla} f(z)|^2 K(\alpha(z, \mu)) d\lambda(z) \\ &= \int_{\mathbf{B}} (1 - |z|^2) \mathcal{K}(z) K(g(z)) \frac{d\lambda(z)}{(1 - |z|^2)^{n+1}} \\ &= \int_{\mathbf{B}} (1 - |z|^2)^{-n} \mathcal{K}(z) K(g(z)) d\lambda(z) \\ &= 2n \int_0^1 (1 - r^2)^{-n} r^{2n-1} dr \int_{\mathbf{S}} \mathcal{K}(r\zeta) d\sigma(\zeta). \end{aligned}$$

Since

$$\begin{aligned} \mathcal{K}(r\zeta) &= r^{2|\alpha|-2}(\alpha_1^2|\zeta_1^{\alpha_1-1}\dots\zeta_n^{\alpha_n}|^2+\dots+\alpha_n^2|\zeta_1^{\alpha_1}\dots\zeta_n^{\alpha_n-1}|^2-|\alpha|^2r^2|\zeta^\alpha|^2) \\ &\geq r^{2|\alpha|-2}(\alpha_1^2|\zeta_1^{\alpha_1-1}\dots\zeta_n^{\alpha_n}|^2+\dots+\alpha_n^2|\zeta_1^{\alpha_1}\dots\zeta_n^{\alpha_n-1}|^2-|\alpha|^2|\zeta^\alpha|^2), \end{aligned}$$

$$\text{and } \int_S |\zeta_1^{\beta_1}\dots\zeta_n^{\beta_n}|^2 d\sigma(\zeta) = \frac{(n-1)(\beta_1!\dots\beta_n!)}{(n-1+\beta_1+\dots+\beta_n)!}, \text{ we have}$$

$$\begin{aligned} \int_S \mathcal{K}(r\zeta) d\sigma(\zeta) &\geq r^{2|\alpha|-2} \left( \alpha_1^2 \frac{(n-1)(\alpha_1-1)!\dots\alpha_n!}{(n-1+|\alpha|-1)!} + \dots + \right. \\ &\quad \left. \alpha_n^2 \frac{(n-1)(\alpha_1-1)!\dots(\alpha_n-1)!}{(n-1+|\alpha|-1)!} - |\alpha|^2 \frac{(n-1)!}{(n-1+|\alpha|-1)!} \right) \\ &= \frac{(n-1)!}{(n-1+|\alpha|-1)!} \frac{|\alpha| r^{2|\alpha|-2}}{\left(1 - \frac{|\alpha|}{n-1+|\alpha|}\right)} = Cr^{2|\alpha|-2}, \end{aligned}$$

where  $C > 0$ . Thus ,

$$\|f\|_{Q_K} \geq 2nC \int_0^1 r^{2n-1+2|\alpha|-2} (1-r^2)^{-n} \mathcal{K}(g(r)) dr \geq (1/2)^{2|\alpha|-1} nC \int_{1/2}^1 r^{2n-1} (1-r^2)^{-n} \mathcal{K}(g(r)) dr. \quad (9)$$

There exists  $a \in \mathbf{B}$  such that  $\nabla f(a) \neq 0$ . By (4) ,

$$\begin{aligned} \|f\|_{Q_K} &\geq \int_{\mathbf{B}} |\tilde{\nabla} f(z)|^2 \mathcal{K}(g(z)) d\lambda(z) \\ &= \int_{\mathbf{B}} |\tilde{\nabla}(f \circ \varphi_a)(z)|^2 \mathcal{K}(g(z)) d\lambda(z) \\ &= \int_{\mathbf{B}} (1-|z|^2)^{-n-1} |\tilde{\nabla}(f \circ \varphi_a)(z)|^2 \mathcal{K}(g(z)) d\lambda(z) \\ &\geq \int_{\mathbf{B}} (1-|z|^2)^{-n+1} |\nabla(f \circ \varphi_a)(z)|^2 \mathcal{K}(g(z)) d\lambda(z) \\ &= 2n \int_0^1 r^{2n-1} (1-r^2)^{-n+1} \mathcal{K}(g(r)) \cdot \int_S |\nabla(f \circ \varphi_a)(r\zeta)|^2 d\sigma(\zeta) dr \\ &\geq (3/2)n \int_0^{1/2} r^{2n-1} (1-r^2)^{-n} \mathcal{K}(g(r)) \cdot \int_S |\nabla(f \circ \varphi_a)(r\zeta)|^2 d\sigma(\zeta) dr \end{aligned}$$

It follows from the subharmonicity of  $|\nabla(f \circ \varphi_a)(r\zeta)|^2$  that

$$\int_S |\nabla(f \circ \varphi_a)(r\zeta)|^2 d\sigma(\zeta) dr \geq |\nabla(f \circ \varphi_a)(0)|^2 = |\tilde{\nabla} f(a)|^2.$$

Thus ,

$$\|f\|_{Q_K} \geq (3/2)n |\tilde{\nabla} f(a)|^2 \int_0^{1/2} r^{2n-1} (1-r^2)^{-n} \mathcal{K}(g(r)) dr. \quad (10)$$

Combining (9) and (10) , we see that (8) implies that  $\|f\|_{Q_K} = \infty$ . It is proved that  $f \notin Q_K$  and , since  $\alpha$  is arbitrary , any non-constant polynomial is not contained in  $Q_K$ . Using Lemma 1 , we conclude that  $Q_K$  contains only constant functions. The theorem is proved.

## 4 The Nesting Property

**Theorem 4** Let  $K_1$  and  $K_2$  satisfy (8). If there exist constants  $C, t_0 > 0$  such that  $K_2(t) \leq CK_1(t)$  for  $0 < t < t_0$  , then  $Q_{K_1} \subseteq Q_{K_2}$ . As a consequence ,  $Q_{K_1} = Q_{K_2}$  if  $C_1 K_1(t) \leq K_2(t) \leq C_2 K_1(t)$  for  $0 < t < t_0$  , where  $C_1, C_2, t_0$  are non-negative constants.

**Proof** Let  $f \in Q_{K_1}$ . From the property of Green's function , there exists a  $\delta > 0$  , such that  $g(z) < t_0$  if  $|z| > \delta$ . Then , for  $a \in \mathbf{B}$  ,

$$\begin{aligned} \int_{\mathbf{B}} |\tilde{\nabla} f(z)|^2 K_2(g(z)) d\lambda(z) &= \int_{\mathbf{B}} |\tilde{\nabla}(f \circ \varphi_a)(z)|^2 K_2(g(z)) d\lambda(z) \\ &= \int_{|z| \leq \delta} |\tilde{\nabla}(f \circ \varphi_a)(z)|^2 K_2(g(z)) d\lambda(z) + \int_{|z| \geq \delta} |\tilde{\nabla}(f \circ \varphi_a)(z)|^2 K_2(g(z)) d\lambda(z) \\ &\leq \|f\|_{\mathcal{B}}^2 \int_{|z| \leq \delta} K_2(g(z)) d\lambda(z) + C \int_{|z| \geq \delta} |\tilde{\nabla}(f \circ \varphi_a)(z)|^2 K_1(g(z)) d\lambda(z) \end{aligned}$$

$$\begin{aligned} &\leq 2n \|f\|_{\mathcal{B}}^2 \int_0^\delta r^{2n-1} (1-r^2)^{-n-1} K_2(g(r)) dr + C \int_{\mathbb{B}} |\tilde{\nabla}(f \circ \varphi_a)(z)|^2 K_1(g(z)) d\lambda(z) \\ &\leq \frac{2n \|f\|_{\mathcal{B}}^2}{1-\delta^2} \int_0^\delta r^{2n-1} (1-r^2)^{-n} K_2(g(r)) dr + C \|f\|_{Q_{K_1}}^2 < \infty. \end{aligned}$$

This shows that  $\|f\|_{Q_{K_2}} < \infty$  and, consequently,  $f \in Q_{K_2}$ . The theorem is proved.

The above theorem shows that in order to compare two function spaces  $Q_{K_1}$  and  $Q_{K_2}$ , it suffices to compare the functions  $K_1(t)$  and  $K_2(t)$  in a neighborhood of the origin. Theorem 4 is still true if we only assume that the integral in (8) is convergent in a neighbourhood of the origin for  $K_1$  and  $K_2$ .

**Theorem 5** If

$$\int_0^\delta K(g(r)) r^{2n-1} dr < \infty, \quad (11)$$

then  $M_K \subset Q_K$ . Furthermore, if we assume additionally that there exists a constant  $C$  such that  $K(4t) < CK(t)$  for small  $t$ , then  $M_K = Q_K$ .

**Proof** It is easy to verify that

$$\frac{n+1}{4n} (1-|z|^2)^n \leq g(z) \leq (1-|z|^2)^n \quad (12)$$

holds if  $1-\epsilon < |z| < 1$ , for a small  $\epsilon$ . Assume that (11) holds and  $f \in M_K$ . Using the same method as in the proof of Theorem 1, we can prove that

$$\|f\|_{\mathcal{B}} \leq C \|f\|_{M_K}. \quad (13)$$

For  $a \in \mathbb{B}$ , by the second inequality of (12), we have

$$\begin{aligned} \int_{| \phi_a(z) | \geq 1-\epsilon} |\tilde{\nabla} f(z)|^2 K(g(z)) d\lambda(z) &= \int_{|z| > 1-\epsilon} |\tilde{\nabla}(f \circ \phi_a)(z)|^2 K(g(z)) d\lambda(z) \\ &\leq \int_{|z| > 1-\epsilon} |\tilde{\nabla}(f \circ \phi_a)(z)|^2 K((1-|z|^2)^n) d\lambda(z) \leq \|f\|_{M_K}^2. \end{aligned}$$

On the other hand, by (13)

$$\begin{aligned} \int_{| \phi_a(z) | < 1-\epsilon} |\tilde{\nabla} f(z)|^2 K(g(z)) d\lambda(z) &= \int_{|z| < 1-\epsilon} |\tilde{\nabla}(f \circ \phi_a)(z)|^2 K(g(z)) d\lambda(z) \\ &\leq C^2 \|f\|_{M_K}^2 \int_{|z| < 1-\epsilon} K(g(z)) d\lambda(z) \\ &= 2nC_1 \|f\|_{M_K}^2 \int_0^{1-\epsilon} r^{2n-1} K(g(r)) dr. \end{aligned}$$

This shows that  $f \in Q_K$ .

We write  $(1-t^2)^n = h(g(t))$  and  $K((1-t^2)^n) = K(h(g(t))) = K_1(g(t))$  for  $0 < t < 1$ , where  $h$  is a strict increasing function. Under the additional assumption, by the first inequality of (12), we have  $K_1(g(t)) = K((1-t^2)^n) \leq K(4g(t)) \leq CK(g(t))$  for  $t > 1-\epsilon$ . By the Theorem 4,  $Q_K \subset Q_{K_1} = M_K$ . Theorem is proved.

As a consequence of the above theorem, letting  $K(t) = t^p$ , we have  $Q_p = M_p$  for  $0 < p < n/(n-1)$ . This is proved in [8].

**Theorem 6** If  $\int_0^1 K(g(r)) r^{2n-1} (1-r^2)^{-n-1} dr < \infty$ , then  $Q_K = \mathcal{B}$

**Proof** Let  $f \in \mathcal{B}$ , then,

$$\begin{aligned} \int_{\mathbb{B}} |\tilde{\nabla} f(z)|^2 K(g(z)) d\lambda(z) &= \int_{\mathbb{B}} |\tilde{\nabla}(f \circ \varphi_a)(z)|^2 K(g(z)) (1-|z|^2)^{-n-1} d\mu(z) \\ &\leq 2n \|f\|_{\mathcal{B}}^2 \int_0^1 r^{2n-1} (1-r^2)^{-n-1} dr \int_{\mathbb{S}} d\sigma(\zeta) \\ &= 2n \|f\|_{\mathcal{B}}^2 \int_0^1 r^{2n-1} (1-r^2)^{-n-1} dr < \infty. \end{aligned}$$

Thus,  $\|f\|_{Q_K}^2 < \infty$  and  $f \in Q_K$ . This shows that  $\mathcal{B} \subset Q_K$ . We have proved  $Q_K \subset \mathcal{B}$  in Section 3. The proof of theorem is complete.

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