Quenching for Degenerate Semilinear Parabolic Equations with Time Delay

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Abstract: This paper deals with the quenching problem for degenerate semilinear parabolic equations with time delay. By using regularization method and upper and lower solutions′technique, we obtain the existence of a unique classical solution to the above problem and prove that there exists a critical length a^* such that the solution u of the above problem exists globally for $a < a^*$ and quenches in finite time for $a > a^*$. Furthermore, we also get a simple estimate on the critical length a^*.

Key words: quenching problem, degenerate semilinear parabolic equation, time delay, critical length, a simple estimate

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带时滞的退化半线性抛物方程的熄灭

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[摘要] 考虑带时滞的退化半线性抛物方程的熄灭问题.利用正则化方法和上下解技巧,我们得到了上述问题经典解的存在惟一性,同时还证明了存在一个临界长度 a^* 使得上述问题的解 u 当 $a < a^*$ 时整体存在,而当 $a > a^*$ 时在有限时间内熄灭.进而我们还得到关于临界长度 a^* 的一个简单估计.

[关键词] 熄灭问题, 退化半线性抛物方程, 时滞, 临界长度, 简单估计

0 Introduction

Quenching phenomena play important roles in both steady and unsteady combustion processes. They are also important to the theory of ecology and related environmental research(see [1,2] and references therein). Since the appearance of the work [3] by karawada concerning a one-dimensional heat equation with the singular reaction function $f(u) = (1 - u)^{-1}$, quenching problem has attracted considerable attention, and extensions to various types of parabolic initial boundary value problems have been investigated by many mathematicians(e. g., [4—7]). Recently, papers [2] and [8] studied quenching for degenerate problem, and papers [1] and [9] studied quenching for uniformly parabolic equations with time delay. In this paper, we would generalize the results of [8] and [9] to degenerate parabolic equation with time delay case. Our aim in this paper is to provide sufficient conditions for the global existence and quenching in finite time of solutions to the degenerate semilinear

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parabolic equations with time delay. The problem under consideration is given by

$$\begin{aligned} &x^q u_t - u_{xx} = f(u(x,t-\tau)), && (x,t)\in(0,a)\times(0,T),\\ &u(0,t)=0,\ u(a,t)=0, && t\in(0,T),\\ &u(x,t)=\eta(x,t), && (x,t)\in(0,a)\times[-\tau,0], \end{aligned} \tag{1}$$

where $q\neq 0$ and $0<T\leqslant\infty$ are real constants, τ and a are positive constants representing the time delay and the length of the interval $(0,a)$, respectively. We assume the singular reaction function $f(u)$ and the initial function $\eta(x,t)$ satisfy the following conditions:

(H_1) $f\in C^2[0,d)$, $f(0)>0$, $f'(0)>0$, $f''(s)\geqslant 0$ for $s\in[0,d)$, $\lim\limits_{s\to d^-} f(s)=\infty$, $\int_0^d f(s)\mathrm{d}s = M<\infty$ and $d/\sqrt{M}<\pi/\sqrt{f'(0)}$, where $d>0$ is a constant.

(H_2) $\eta(x,t)\in C^\alpha([0,a]\times[-\tau,0])$, $\eta(x,t)\geqslant 0$ for $(x,t)\in[0,a]\times[-\tau,0]$, $\eta(0,t)=\eta(a,t)=0$ for $t\in[-\tau,0]$, $\eta_x(0,t)$ and $\eta_x(a,t)$ exist for $t\in[-\tau,0]$, $u_0(x)=\eta(x,0)\in C^{2+\alpha}(0,a)$, where $\alpha\in(0,1)$ is a constant.

For example, we can easily verify that $f(s)=(d-s)^{-\beta}$ $(d\geqslant 1,\ 0<\beta<1)$ and $\eta(x,t)=0$ satisfy the above conditions.

As in the case without time delay, the quenching problem of Eq. (1) is closely related to the existence and nonexistence of positive solutions of the steady state problem

$$-U''=f(U),\ x\in(0,a),\ U(0)=U(a)=0, \tag{2}$$

where $U''=d^2U/dx^2$. It is well known that under Hypothesis (H_1) there exists a constant $a^*>0$ such that (2) has a positive maximal solution $\overline{U}_s(x)$ and a positive minimal solution $\underline{U}_s(x)$ satisfying $0<\underline{U}_s(x)\leqslant\underline{U}_s(x)$ for $x\in[0,a]$ if $a<a^*$ and it has no positive solution if a^* is finite and $a>a^*$ (see [1,6]). It has been shown in [6] that

$$a^*=\sup\{a>0,\ \text{a positive solution to (2) exists}\} \tag{3}$$

and a^* is well defined. We say that the solution $u(x,t)$ of Eq. (1) quenches if there exists a finite $T_a>0$ such that

$$\limsup_{t\to T_a}\max_{0\leqslant x\leqslant a} u(x,t-\tau)=d. \tag{4}$$

Let us state our main results.

Theorem 1.1 Let (H_1) and (H_2) hold, and let $a<a^*$, where a^* is given by (3). Then

(i) a unique global solution $u(x,t)$ to problem (1) exists and satisfies the relation $0\leqslant u(x,t)\leqslant\overline{U}_s(x)$ in $(0,a)\times(0,\infty)$ if $0\leqslant\eta(x,t)\leqslant\overline{U}_s(x)$.

(ii) the solution $u(x,t)$ of problem (1) converges to $\underline{U}_s(x)$ as $t\to\infty$ if $0\leqslant\eta(x,t)\leqslant\underline{U}_s(x)$.

Theorem 1.2 Let (H_1) and (H_2) hold, and let $a>a^*$. Then for any $\eta(x,t)\geqslant 0$, the corresponding solution $u(x,t)$ of problem (1) quenches in finite time.

This paper is organized as follows. In section 2, the local existence of the classical solution of (1) is established. Sufficient conditions such that (1) has a global solution or its solution quenches in finite time are given in section 3.

1 Local Existence

To get the existence of the classical solution of problem (1), we need the following comparison principle.

Lemma 2.1 Let $w\in C^{2,1}((0,a)\times(0,T))\cap C([0,a]\times[-\tau,T))$ satisfy the relation

$$\begin{aligned} &x^q w_t - w_{xx}\geqslant c(x,t)w(x,t-\tau), && (x,t)\in(0,a)\times(0,T),\\ &w(0,t)\geqslant 0,\ w(a,t)\geqslant 0, && t\in(0,T),\\ &w(x,t)\geqslant 0, && (x,t)\in(0,a)\times[-\tau,0], \end{aligned} \tag{5}$$

if $c\in C([0,a]\times[0,T))$ and $c(x,t)\geqslant 0$ in $(0,a)\times(0,T)$. Then $w(x,t)\geqslant 0$ on $[0,a]\times[0,T)$.

Proof We first consider (5) in $D_1=(0,a)\times(0,\tau]$. By (5), we have

$$\begin{aligned} &x^q w_t - w_{xx} \geqslant 0, && (x,t) \in D_1, \\ &w(0,t) \geqslant 0,\ w(a,t) \geqslant 0, && t \in (0,\tau], \\ &w(x,0) \geqslant 0, && x \in (0,a). \end{aligned} \tag{6}$$

It follows from Lemma 1 of [10] that $w(x,t) \geqslant 0$ on $\overline{D}_1$.

We next consider (5) in the domain $D_2 = (0,a) \times (\tau, 2\tau]$. By the conclusion for $w(x,t)$ on $\overline{D}_1$, $w(x,t-\tau) \geqslant 0$ in D_2. Then again by (2.1), we get

$$\begin{aligned} &x^q w_t - w_{xx} \geqslant 0, && (x,t) \in D_2, \\ &w(0,t) \geqslant 0,\ w(a,t) \geqslant 0, && t \in (\tau,2\tau], \\ &w(x,\tau) \geqslant 0, && x \in (0,a). \end{aligned}$$

And therefore Lemma 1 of [10] implies that $w(x,t) \geqslant 0$ on $\overline{D}_2$. A continuation of the above argument leads to $w(x,t) \geqslant 0$ on $[0,a] \times [0,T)$.

From Lemma 2.1 and the strong maximum principle in [7, chapter 2], we can easily obtain the following uniqueness and positivity results.

Lemma 2.2 Let (H_1) and (H_2) hold, then (1) has at most one solution $u: u > 0$ in $(0,a) \times (0,T)$.

By (H_1) and (H_2), we know that $\hat{u} = 0$ is a lower solution of (1). In the following Lemma, we prove that (1) has an upper solution $h(x,t) \geqslant 0$ on $[0,a] \times [-\tau, t_0]$.

Lemma 2.3 Let (H_1) and (H_2) hold, then there exist positive constants $t_0 \leqslant \min(\tau, T)$ and $\bar{d} \in (0,d)$ such that (1) has an upper solution $h \in C^{\alpha}([0,a] \times [-\tau, t_0]) \cap C^{2,1}((0,a) \times (0,t_0])$, $h \in (0, \bar{d}]$ and h depends on f, a, q and τ.

Proof The proof of this lemma is similar to that of Lemma 2 in [6] and that of Lemma 2.1 in [11], and therefore is omitted here.

Now, we consider the following regular parabolic problem

$$\begin{aligned} &x^q u_{\varepsilon t} - u_{\varepsilon xx} = f(u_\varepsilon(x,t-\tau)),\ (x,t) \in (\varepsilon,a) \times (0,T), \\ &u_\varepsilon(\varepsilon,t) = u_\varepsilon(a,t) = 0,\ t \in (0,T), \\ &u_\varepsilon(x,t) = \eta(x,t),\ (x,t) \in (\varepsilon,a) \times [-\tau,0], \end{aligned} \tag{7}$$

where $\varepsilon \in (0,a)$ and $\eta(x,t): [\varepsilon,a] \times [-\tau,0] \to \mathbf{R}$ is simply the truncation of the initial data in (1). Obviously, $\tilde{u} = h(x,t)$ and $\hat{u} = 0$ are a couple of ordered upper and lower solutions of (7) in $(\varepsilon,a) \times (0,t_0]$. Even though the compatibility condition does not hold, it is well known that (7) with $\tau = 0$ has a $C^{2,1}$ solution (see [12 ~ 14]). Then by the same method of the proof of Theorem 2.8.1 of [1], we can prove that (7) has a $C^{2,1}$ solution u_ε.

In the same way as that of Lemma 2.1, we can prove

Lemma 2.4 Let $w \in C^{2,1}((\varepsilon,a) \times (0,T)) \cap C([\varepsilon,a] \times [-\tau,T))$ satisfy the relation

$$\begin{aligned} &x^q w_t - w_{xx} \geqslant c(x,t) w(x,t-\tau), && (x,t) \in (\varepsilon,a) \times (0,T), \\ &w(\varepsilon,t) \geqslant 0,\ w(a,t) \geqslant 0, && t \in (0,T), \\ &w(x,t) \geqslant 0, && (x,t) \in (\varepsilon,a) \times [-\tau,0], \end{aligned} \tag{8}$$

if $c \in C([\varepsilon,a] \times [0,T))$ and $c(x,t) \geqslant 0$ in $(\varepsilon,a) \times (0,T)$, then $w(x,t) \geqslant 0$ on $[\varepsilon,a] \times [0,T)$.

By using Lemma 2.2.1 in [1] and Lemma 2.4, we can easily get the following monotone result.

Lemma 2.5 Let $0 < \varepsilon_1 < \varepsilon_2 < a$ and suppose that u_{ε_1} and u_{ε_2} are solutions of (2.3) in $(0,t_0]$, then $u_{\varepsilon_1}(x,t) > u_{\varepsilon_2}(x,t)$ in $(\varepsilon_2,a) \times (0,t_0]$.

We set

$$u(x,t) = \begin{cases} \lim\limits_{\varepsilon \to 0} u_\varepsilon(x,t), & (x,t) \in (0,a] \times [0,t_0], \\ 0, & x = 0, t \in [0,t_0]. \end{cases} \tag{9}$$

Then by standard arguments as those of Theorems 2.3 and 2.5 of [12], we obtain

Theorem 2.6 Let (H_1) and (H_2) hold, then (1) admits a unique positive classical solution $u(x,t)$ in

$(0,a)\times(0,t_0]$. Let T be the supremum over t_0 for which there is a unique positive classical solution $u(x,t)$ to problem (1) in $(0,a)\times(0,t_0]$, then problem (1) exists a unique positive classical solution $u(x,t)$ in $(0,a)\times(0,T)$. Moreover, if $T<\infty$ then $\limsup_{t\to T}\max_{x\in[0,a]} u(x,t-\tau)=d$.

2 Global Existence and Quenching in Finite Time

To prove the main theorems, we need some lemmas.

Lemma 3.1 Let (H_1) and (H_2) hold, and let $u(x,t)$ and $\underline{u}(x,t)$ be the positive solutions of (1) corresponding to $\eta(x,t)\geqslant 0$ and $\eta(x,t)\equiv 0$, respectively. Let also $v(x,t)$ be the solution of (1) with $\tau=0$ and with the initial function $v(x,0)=v_0(x)\geqslant 0$. Assume that $\eta(x,t)\leqslant v_0(x)$ on $[0,a]\times[-\tau,0]$ and that $v_0(x)$ is a lower solution of (2), then

$$\underline{u}(x,t)\leqslant u(x,t)\leqslant v(x,t)\ \text{on}\ [0,a]\times[0,T). \tag{10}$$

Proof We first show the relation $v(x,t)\geqslant u(x,t)$. Let $w(x,t)=v(x,t)-u(x,t)$, then

$$\begin{aligned}
&x^q w_t-w_{xx}=f(v(x,t))-f(u(x,t-\tau)), && (x,t)\in(0,a)\times(0,T),\\
&w(0,t)=w(a,t)=0, && t\in(0,T),\\
&w(x,t)=v_0(x)-\eta(x,t)\geqslant 0, && (x,t)\in(0,a)\times[-\tau,0].
\end{aligned} \tag{11}$$

Since $v_0(x)$ is a lower solution of (2), similar to the proof of Lemma 5.4.1 in [1], we can prove that $v(x,t)$ is nondecreasing in t. By considering $v(x,t)=v_0(x)$ for $(x,t)\in(0,a)\times[-\tau,0]$ and using the condition $f'>0$ in (H_1), we have $f(v(x,t))\geqslant f(v(x,t-\tau))$ in $(0,a)\times(0,T)$. This and (11) lead to

$$\begin{aligned}
&x^q w_t-w_{xx}\geqslant f'(\xi)w(x,t-\tau), && (x,t)\in(0,a)\times(0,T),\\
&w(0,t)=w(a,t)=0, && t\in(0,T),\\
&w(x,t)\geqslant 0, && (x,t)\in(0,a)\times -\tau,0].
\end{aligned}$$

Since $f'(\xi)>0$, Lemma 2.1 implies that $w(x,t)\geqslant 0$ on $[0,a]\times[0,T)$, that is, $v(x,t)\geqslant u(x,t)$ on $[0,a]\times[0,T)$. The proof for $u(x,t)\geqslant\underline{u}(x,t)$ on $[0,a]\times[0,T)$ is similar.

Lemma 3.2 Let (H_1) hold, and let $\underline{u}(x,t)$ be the positive solution of (1) corresponding to $\eta(x,t)\equiv 0$, $(x,t)\in[0,a]\times[-\tau,0]$, then $\underline{u}_t(x,t)>0$ in $(0,a)\times(0,T)$.

Proof The proof is similar to that of Lemma 2.8 in [10], and hence we omit it here.

Proof of Theorem 1 (i) We take $\tilde{u}(x,t)=\overline{U}_s(x)$ and $\hat{u}(x,t)=0$ as a pair of upper and lower solutions of (1) when $\eta(x,t)\leqslant\overline{U}_s(x)$, then Theorem 2.6 implies that (1) admits a unique global solution $u(x,t)$ and it satisfies the relation $0\leqslant u(x,t)\leqslant\overline{U}_s(x)$ in $(0,a)\times(0,\infty)$.

(ii) We take $\tilde{u}(x,t)=\underline{U}_s(x)$ and $\hat{u}(x,t)=0$ as a pair of upper and lower solutions of (1) when $\eta(x,t)\leqslant\underline{U}_s(x)$, then Theorem 2.6 implies that a unique global solution $u(x,t)$ to (1) exists and satisfies the relation $u(x,t)\leqslant\underline{U}_s(x)$ in $(0,a)\times(0,\infty)$. Let $\underline{u}(x,t)$ be the solution of (1) corresponding to $\eta(x,t)\equiv 0$, then by Lemma 3.1, we have

$$\underline{u}(x,t)\leqslant u(x,t)\leqslant\underline{U}_s(x)\ \text{in}\ (0,a)\times(0,\infty). \tag{12}$$

From Lemma 3.2, we have $\underline{u}_t(x,t)>0$ in $(0,a)\times(0,\infty)$. And then we can show in the same way as that of Theorem 3.1 of [8] that $\underline{u}(x,t)$ converges to $\underline{U}_s(x)$ as $t\to\infty$. Therefore it follows from the relation (12) that $\lim_{t\to\infty}u(x,t)=\underline{U}_s(x)$.

Lemma 3.3 Let (H_1) hold, then the positive constant a^* given by (3) is well defined and $a^*\leqslant\pi/\sqrt{f'(0)}$.

Proof It follows from [6] that a^* is well defined, therefore we only need to prove that $a^*\leqslant\pi/\sqrt{f'(0)}$. Let $\lambda_1=\lambda_1(a)$ and $\phi_1=\phi_1(x)$ be the smallest eigenvalue and the corresponding eigenfunction of the following eigenvalue problem

$$\phi''(x)+\lambda\phi(x)=0,\ 0<x<a,\ \phi(0)=\phi(a)=0. \tag{13}$$

It is well known that $\lambda_1 = \lambda_1(a) = \left(\frac{\pi}{a}\right)^2$ and $\phi_1 = \phi_1(x) = \sin\frac{\pi}{a}x$, and it is obvious that $\lambda_1 = \lambda_1(a) \to 0$ as $a \to \infty$ and $\phi_1(x) > 0$ in $(0,a)$. Hence $\lambda_1 = \lambda_1(a) < f'(0)$ for all $a > \pi/\sqrt{f'(0)}$. Assume that (2) has a positive solution U_s for some $a > \pi/\sqrt{f'(0)}$, then multiplication of Equation (2) by $\phi_1(x)$ and integration over $(0,a)$ yield

$$-\int_0^a U''_s(x)\phi_1(x)\,dx = \int_0^a f(U_s(x))\phi_1(x)\,dx.$$

Upon integration by parts and application of the equation (13), we obtain

$$\lambda_1(a)\int_0^a U_s(x)\phi_1(x)\,dx = \int_0^a f(U_s(x))\phi_1(x)\,dx. \tag{14}$$

By the mean value theorem and (H_1), we have

$$f(U_s) = f(0) + f'(\xi)U_s > f'(0)U_s.$$

Hence we have

$$\int_0^a f(U_s(x))\phi_1(x)\,dx \geqslant f'(0)\int_0^a U_s(x)\phi_1(x)\,dx.$$

It follows from the above inequality and (14) that $\lambda_1(a) \geqslant f'(0)$, that is, $a \leqslant \pi/\sqrt{f'(0)}$, which contradicts our assumption. Hence (2) has no positive solution when $a > \pi/\sqrt{f'(0)}$, and this shows that $a^* \leqslant \pi/\sqrt{f'(0)}$.

By using the comparison principle Lemma 2.1 and the strong maximum principle in [13, Chapter 2], we can easily obtain the following monotonicity respect to a.

Lemma 3.4 Let (H_1) hold and denote by $\underline{u}(x,t;a)$ the positive solution of (1) in $(0,a)\times(0,T)$ corresponding to $\eta(x,t)\equiv 0$, then for any positive constant α,

$$\begin{aligned} &\underline{u}(x,t;a+\alpha) > \underline{u}(x,t;a), \\ &\underline{u}_t(x,t;a+\alpha) > \underline{u}_t(x,t;a), \end{aligned} \qquad (x,t)\in(0,a)\times(0,T). \tag{15}$$

Lemma 3.5 Let (H_1) and (H_2) hold, and let $a > a^*$, then for any $\eta(x,t)\geqslant 0$ there exists a positive constant $T_a \leqslant \infty$ such that (1) admits a unique positive solution $u(x,t)$ on $[0,a]\times[0,T_a)$ and

$$\limsup_{t\to T_a}\max_{0\leqslant x\leqslant a} u(x,t-\tau) = d. \tag{16}$$

Proof By Lemma 3.1, $u(x,t)\geqslant \underline{u}(x,t)$ on $[0,a]\times[0,T_a)$, where $\underline{u}(x,t)$ is the solution of (1) corresponding to $\eta(x,t)\equiv 0$. It suffices to show that $\underline{u}(x,t)$ satisfies (16). Since $\underline{u}(x,t)$ is monotone nondecreasing in t, it either satisfies (16) or converges to a positive solution of (2). And the latter is impossible for (2) has no positive solution when $a > a^*$. Hence (16) holds.

Lemma 3.6 Let (H_1) hold, and let $\underline{u}(x,t)$ be the positive solution of (1) corresponding to $\eta(x,t)\equiv 0$, then the set of quenching points of $\underline{u}(x,t)$ lies in $[\delta_1, a-\delta_1]$, where $\delta_1 = d^2/(2Ma)$.

Proof Similarly as the argument of Theorem 4.1 in [15], by using the property $\underline{u}_t > 0$ in $(0,a)\times(0,T_a)$, we can also show the conclusion of this lemma. Hence we omit the proof.

Proof of Theorem 2 In view of Lemma 3.1, it suffices to show that the solution $\underline{u}(x,t;a)$ quenches at finite time T_a. By Lemma 3.5, there exists $T_a \leqslant \infty$ such that $\lim_{t\to T_a}\max_{x\in[0,a]}\underline{u}(x,t-\tau;a) = d$. Now if $T_a < \infty$, then the theorem is proved. Assume by contradiction that $T_a = \infty$, then for arbitrary $\varepsilon > 0$ sufficiently small, there exist $x_1 = x_1(\varepsilon)\in(0,a)$ and $t_1 = t_1(\varepsilon) > \tau$ sufficiently large such that $\underline{u}(x_1,t_1-\tau;a) = \max_{x\in[0,a]}\underline{u}(x,t_1-\tau;a) = d-\varepsilon$. For any given positive constant α, we will prove that $T_{a+\alpha} < \infty$. Assume by contradiction that $T_{a+\alpha} = \infty$. We define

$$\beta = \beta(\varepsilon) = \inf\{x\in(x_1,a+\alpha)\mid \underline{u}(x,t_1-\tau;a+\alpha) = d-\varepsilon\} - x_1.$$

It follows from Lemma 3.4 that $\underline{u}(x_1,t_1-\tau;a+\alpha) > \underline{u}(x_1,t_1-\tau;a) = d-\varepsilon$, then $\beta = \beta(\varepsilon) > 0$. We can further deduce that $\beta(\varepsilon)\geqslant\beta_0 > 0$, where β_0 is a positive constant independent of ε. In fact, without loss of generality, we can assume that the quenching point of $\underline{u}(x,t;a)$ is unique. By Lemma 3.6, we have

$$\lim_{\varepsilon\to 0^+} x_1(\varepsilon)=x_0,\quad \lim_{(x,t)\to(x_0,\infty)^-}\underline{u}(x,t-\tau;a)=d,\ x_0\in[\delta_1,a-\delta_1]\subset(0,a).$$

Then $\bar{x}=\inf\limits_{0<\varepsilon<\frac{d}{2}} x_1(\varepsilon)\in(0,a)$. Similarly, $x_1(\varepsilon)+\beta(\varepsilon)\in(0,a+\alpha)$ and

$$\lim_{\varepsilon\to 0^+}(x_1(\varepsilon)+\beta(\varepsilon))=x'_0,\quad \lim_{(x,t)\to(x'_0,\infty)^-}\underline{u}(x,t-\tau;a+\alpha)=d,\ x'_0\in(0,a+\alpha).$$

Then $\bar{x}<\bar{\bar{x}}=\sup\limits_{0<\varepsilon<\frac{d}{2}}(x_1(\varepsilon)+\beta(\varepsilon))\in(0,\ a+\alpha)$. By Lemma 3.4, we get

$$\underline{u}_t(x,t-\tau;a+\alpha)>\underline{u}_t(x,t-\tau;a).$$

Then

$$\underline{u}(x,t_1-\tau;a+\alpha)-\underline{u}(x,t_1-\tau;a)>\underline{u}(x,t_0-\tau;a+\alpha)-\underline{u}(x,t_0-\tau;a)\geqslant\delta_2>0,\ x\in[\bar{x},\ \bar{\bar{x}}],$$

where $t_0\in(\tau,t_1(\varepsilon))$ and δ_2 are positive constants independent of ε. Therefore there exists $\xi\in(x_1,x_1+\beta)$ such that

$$\begin{aligned}\delta_2<\underline{u}(x_1,t_1-\tau;a+\alpha)-\underline{u}(x_1,t_1-\tau;a)&=\underline{u}(x_1,t_1-\tau;a+\alpha)-\underline{u}(x_1+\beta;t_1-\tau;a+\alpha)\\&=-\underline{u}_x(\xi,t_1-\tau;a+\alpha)\beta.\end{aligned}$$

From the proof of Lemma 3.6, we know that

$$\int_0^{a+\alpha}\underline{u}_x^2(x,t;a+\alpha)\mathrm{d}x\leqslant 2M(a+\alpha),\ t\in[0,T_{a+\alpha}).$$

And from Theorem 2.6, we have $\underline{u}_x(x,t;a+\alpha)\in C^{1,1}((0,a+\alpha)\times(0,T_{a+\alpha}))$. Hence

$$\underline{u}_x(x,t;a+\alpha)\in C([\bar{x},\ \bar{\bar{x}}]\times[t_0-\tau,T_{a+\alpha})).$$

It follows from $\xi\in(x_1,x_1+\beta)\subset[\bar{x},\ \bar{\bar{x}}]$ that there exists a constant K independent of ε such that $|\underline{u}_x(\xi,t_1-\tau;a+\alpha)|\leqslant K$, hence $\beta>\dfrac{\delta_2}{K}=\beta_0>0$.

By condition $\lim\limits_{u\to d^-}f(u)=\infty$ in (H_1), we can choose $\varepsilon>0$ sufficiently small and $t_1=t_1(\varepsilon)$ sufficiently large such that

$$f(v)\geqslant(x_1^*)^q\times\frac{\varepsilon}{\tau}+\frac{16\varepsilon}{\lambda^2}\quad\text{for } v\in[d-2\varepsilon,d),\tag{17}$$

where $x_1^*\in[\bar{x},\ \bar{\bar{x}}]$ equals to $x_1+\lambda$ for $q>0$ and x_1 for $q<0$ with $x_1=x_1(\varepsilon)$ being a point at which $\underline{u}(x,t_1-\tau;a)$ attains its global maximum $d-\varepsilon$ and $\lambda=\min\{\alpha,\beta\}$ with

$$\beta=\beta(\varepsilon)=\inf\{x\in(x_1,a+\alpha)\,|\,\underline{u}(x,t_1-\tau;a+\alpha)=d-\varepsilon\}-x_1>\beta_0.$$

Let $Q\equiv(x_1,x_1+\lambda)\times(t_1,t_1+\tau]$, and consider (1) in domain Q, where the boundary and initial conditions are replaced by

$$\begin{aligned}&u(x_1,t)=d-\varepsilon,\ u(x_1+\lambda,t)=d-\varepsilon,\ t\in(t_1,t_1+\tau],\\&u(x,t)=d-\varepsilon,\ (x,t)\in(x_1,x_1+\lambda)\times[t_1-\tau,t_1].\end{aligned}\tag{18}$$

By the definition of λ and Lemma 3.4, we know that $\tilde{u}(x,t)=\underline{u}(x,t;a+\alpha)$ satisfies the differential equation (1) and

$$\begin{aligned}&\tilde{u}(x_1,t)=\underline{u}(x_1,t;a+\alpha)>\underline{u}(x_1,t_1-\tau;a+\alpha)\geqslant d-\varepsilon,\ t\in(t_1,t_1+\tau],\\&\tilde{u}(x_1+\lambda,t)=\underline{u}(x_1+\lambda,t;a+\alpha)>\underline{u}(x_1+\lambda,t_1-\tau;a+\alpha)\geqslant d-\varepsilon,\ t\in(t_1,t_1+\tau],\\&\tilde{u}(x,t)=\underline{u}(x,t;a+\alpha)\geqslant d-\varepsilon,\ (x,t)\in(x_1,x_1+\lambda)\times[t_1-\tau,t_1].\end{aligned}$$

Therefore $\tilde{u}(x,t)$ is an upper solution of (1) and (18). We next construct a lower solution of (1) and (18) in the form

$$\hat{u}(x,t)=d-2\varepsilon+\delta(x-x_1)(x_1+\lambda-x)(t+\tau-t_1),\tag{19}$$

where $\delta=\dfrac{4\varepsilon}{\tau\lambda^2}$. Indeed, it is easy to verify that $\hat{u}(x,t)$ satisfies the relation

$$\begin{aligned}&\hat{u}(x_1,t)=d-2\varepsilon<d-\varepsilon,\ \hat{u}(x_1+\lambda,t)=d-2\varepsilon<d-\varepsilon,\ t\in(t_1,\ t_1+\tau],\\&\hat{u}(x,t)=d-2\varepsilon+\delta(x-x_1)(x_1+\lambda-x)(t+\tau-t_1)\\&\qquad\leqslant d-2\varepsilon+\frac{4\varepsilon}{\tau\lambda^2}\times\frac{\lambda^2}{4}\times\tau=d-\varepsilon,\ (x,t)\in(x_1,x_1+\lambda)\times[t_1-\tau,t_1].\end{aligned}$$

Moreover, a simple computation shows that

$$x^q\hat{u}_t - \hat{u}_{xx} = x^q\delta(x - x_1)(x_1 + \lambda - x) + 2\delta(t + \tau - t_1) \leqslant \frac{(x_1^*)^q\varepsilon}{\tau} + \frac{16\varepsilon}{\lambda^2},\ (x,t) \in Q.$$

Since $\hat{u}(x,t-\tau) \geqslant d - 2\varepsilon$ for all $(x,t) \in Q$, condition (17) ensures that

$$x^q\hat{u}_t - \hat{u}_{xx} \leqslant \frac{(x_1^*)^q\varepsilon}{\tau} + \frac{16\varepsilon}{\lambda^2} \leqslant f(\hat{u}(x,t-\tau)),\ (x,t) \in Q.$$

This shows that $\hat{u}(x,t)$ is a lower solution. It follows from the properties of upper and lower solutions that the function $w(x,t) = \tilde{u}(x,t) - \hat{u}(x,t)$ satisfies the equalities (5) with $c(x,t) = f'(\xi) > 0$ and $(0,a) \times (0,T)$ replaced by Q. Then Lemma 2.1 implies $\tilde{u}(x,t) \geqslant \hat{u}(x,t)$ in Q, in particular,

$$\underline{u}(x_1 + \frac{\lambda}{2}, t_1 + 2\tau - \tau; a + \alpha) = \tilde{u}(x_1 + \frac{\lambda}{2}, t_1 + \tau)$$

$$\geqslant \hat{u}(x_1 + \frac{\lambda}{2}, t_1 + \tau) = d - 2\varepsilon + \delta(t_1 + \tau + \tau - t_1)\frac{\lambda^2}{4} = d.$$

This shows that $\underline{u}(x,t;a+\alpha)$ must quench in finite time $T_{a+\alpha} < \infty$, for the arbitrariness of $\alpha > 0$ and $a > a^*$. Therefore $T_a = +\infty$ is impossible and T_a must be finite, and this completes our proof.

From Lemma 3.6 and Theorem 2, we know that if $a > a^*$ then $a - 2\delta_1 \geqslant 0$, i.e., $a - d^2/(Ma) \geqslant 0$, $a \geqslant d/\sqrt{M}$, hence $a^* \geqslant d/\sqrt{M}$. Combining this with Lemma 3.3, we obtain a simple estimate of the critical length a^*:

$$\frac{d}{\sqrt{M}} \leqslant a^* \leqslant \frac{\pi}{\sqrt{f'(0)}}.$$

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