Mountain Pass Lemma Without the P S Condition

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Abstract: In this paper, the well-known Mountain Pass Lemma is considered without the Palais – Smale (P. S.) Condition. It is obtained that the existence of asymptotical-critical points of a functional which does not satify the P. S. condition. The main result generalizes the classical Mountain Pass Lemma. This paper also provides a new proof method for classical Mountain Pass Lemma under weaker conditions.

Key words: P S condition, Mountain Pass Lemma, asymptotical-critical points

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没有 P S 条件的山路引理

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[摘要] 本文研究了没有 Palais-Smale 条件的山路引理. 对于不满足 Palais-Smale 条件的泛函,得到了渐近临界点的存在性,推广了古典的山路引理. 本文还提供了更弱条件下的山路引理的新的证明.

[关键词] P S 条件, 山路引理, 渐近临界点

0 Introduction

Minimax theorem is one of basic theorems in critical point theory. In 1973, Ambrosetti and Rabinowitz proposed the well-known Mountain Pass Lemma[1], which was stated as follows:

Theorem A Let E be a real Banach space. Assume that $f\in C1[E, R]$ satisfies the Palais-Smale (P. S. in short) condition, Ω is an open neighborhood of x_0, $x_0, x_1\in E$, $x_1\notin\bar{\Omega}$,

$$\max\{f(x_0), f(x_1)\} < \inf_{x\in\partial\Omega} f(x). \tag{1}$$

Set $c=\inf_{h\in\Phi}\max_{t\in[0,1]} f(h(t))$, where $\Phi=\{h\mid h:[0,1]\to E \text{ is continuous and } h(0)=x_0,\ h(1)=x_1\}$. Then c must be a critical value of f, i. e., there exists $x^*\in E$ such that $f'(x^*)=\theta$ and $f(x^*)=c$.

Mountain Pass Lemma was extensively used in many disciplines of pure and applied mathematics including ordinary and partial differential equations, mathematical physics, geometrical analysis, etc. [2][3]. Here, we state the P. S. condition as follows:

Definition 1[4] Let E be a real Banach space, $f\in C^1[E, R]$. The functional f is called satisfying the P. S. condition if any sequence $\{x_n\}\subset E$ with

$$f(x_n) \text{ being bounded and } f'(x_n)\to\theta$$

contains a convergent subsequence.

In 1987, Qi Guijie[5] obtained the existence of a critical point with the weaker condition

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$$\max\{f(x_0), f(x_1)\} \leqslant \inf_{x\in\partial\Omega} f(x) \tag{2}$$

instead of (1) without any additional condition.

The P. S. condition is very important in the variational methods. We know that the P. S. condition make an important rule in establishing Deformation Lemma. But for some functional, it is very difficult to verify the compact condtion, so it is significant to study the problem of the existance of critical point without the P. S. condition. In this paper, we study Mountain Pass Lemma and get a new result of determining the asymptotical critical points.

1 Main Result

In order to state the main result of this paper, we first introduce the following basic lemma:

Lemma 1[6, Lemma 2.3] Let X be a Banach space. Assume $\varphi\in C^1(X, R)$, $S\subset X$, $c\in R$, $\varepsilon,\delta > 0$ such that

$$\|\varphi'(u)\| \geqslant 8\varepsilon/\delta,\ \forall u\in\varphi^{-1}([c-2\varepsilon, c+2\varepsilon])\cap S_{2\delta}$$

where $S_{2\delta}=\{u\in X \mid \operatorname{dist}(u,S)<2\delta\}$. Then there exists $\eta\in C([0,1]\times X, X)$ such that

(i) $\eta(t,u)=u$, $\forall u\notin\varphi^{-1}[c-2\varepsilon, c+2\varepsilon]\cap S_{2\delta}$ or $t=0$;

(ii) $\eta(1, \varphi^{c+\varepsilon}\cap S)\subset\varphi^{c-\varepsilon}$;

(iii) $\varphi(\eta(\cdot, u))$ is nonincreasing, $\forall u\in X$.

Now we give the main result of this paper.

Theorem 1 Let X be a Banach space. Assume $\varphi\in C^1(X, R)$ and Ω is an open neighborhood of θ, $e\notin \bar{\Omega}$. Set

$$c_1=\max\{\varphi(\theta), \varphi(e)\},\ c_0=\inf_{x\in\partial\Omega}\varphi(x),\ c=\inf_{\gamma\in\Gamma}\ \max_{t\in[0,1]}\varphi(\gamma(t)),$$

where $\Gamma=\{\gamma \mid \gamma:[0,1]\to E$ is continuous and $\gamma(0)=\theta$, $\gamma(1)=e\}$.

If $c_0\geqslant c_1$, then for every $\varepsilon>0$ there exists $x\in X$ such that

(a) $c-2\varepsilon\leqslant\varphi(x)\leqslant c+2\varepsilon$;

(b) $\|\varphi'(x)\|<2\varepsilon$.

Proof From the assumptions, we have $c_1\leqslant c_0\leqslant c$. If $c_0<c$, then $c_1(\leqslant c_0)<c$. Thus in this case, the theorem is proved by Theorem A. So it is only needed to prove the theorem in the case of $c_0=c$. In the following, we use a proof by contradiction. Assume there exists $\varepsilon_1>0$ such that

$$\|\varphi'(x)\|\geqslant 2\varepsilon_1,\ \forall x\in\varphi^{-1}([c-2\varepsilon_1, c+2\varepsilon_1]).$$

Set $S=X$. Then the assumption of Lemma 1 is satisfied. So there exists $\eta\in C([0,1]\times X, X)$ such that

(i) $\eta(t,u)=u$, $\forall x\notin\varphi^{-1}([c-2\varepsilon_1, c+2\varepsilon_1])$ or $t=0$;

(ii) $\eta(1,\varphi^{c+\varepsilon_1})\subset\varphi^{c-\varepsilon_1}$;

(iii) $\varphi(\eta(\cdot,))$ is nonincreasing, $\forall u\in X$.

From the proof of Lemma 1, $\sigma(t,u)$ is the unique solution of

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\sigma(t,u)=f(\sigma(t,u)),\\ \sigma(0,u)=u\end{cases}$$

for every $u\in X$ and $\eta(t,u)=\sigma(2\varepsilon_1 t,u)$, where

$$f(u)=\begin{cases}-\psi(u)\|g(u)\|^{-2}g(u), & u\in A,\\ \theta, & u\in X\backslash A,\end{cases}$$

$$A=\varphi^{-1}([c-2\varepsilon_1, c+2\varepsilon_1]),\ B=\varphi^{-1}([c-\varepsilon_1, c+\varepsilon_1]),$$

$$\psi(u)=\frac{\operatorname{dist}(u,X\backslash A)}{\operatorname{dist}(u,X\backslash A)+\operatorname{dist}(u,X\backslash B)},$$

g is a pseudogradient vector field of φ'. Since $c=\inf_{\gamma\in\Gamma}\ \max_{t\in[0,1]}\varphi(\gamma(t))$, there exists $\gamma_0\in\Gamma$ such that

$$\max_{t\in[0,1]}\varphi(\gamma(t))<c+\frac{\varepsilon_1}{2}.$$

Let $\beta(t)=\eta(1,\gamma_0(t))$, $0\leqslant t\leqslant 1$. Then β is continuous and

$$\beta(0)=\eta(1,\theta)\equiv x_0,\beta(1)=\eta(1,e)\equiv x_1.$$

We claim $x_0\in\Omega$, $x_1\notin\bar{\Omega}$. Indeed, if $x_0\notin\Omega$, then from the continuity of $\eta:[0,1]\times X\to X$, we can see that there exists $s_0\in[0,1]$ such that $\eta(s_0,\theta)\in\partial\Omega$. Thus

$$c_0\leqslant\varphi(\eta(s_0,\theta))\leqslant\varphi(\eta(0,\theta))=\varphi(\theta)\leqslant c_1\leqslant c_0,$$

which means $\varphi(\eta(s_0,\theta))=\varphi(\theta)=c_0=c$. From the definition of B, we have $\eta(s_0,\theta)\in B$. It is easy to see

$$\frac{\mathrm{d}}{\mathrm{d}s}\varphi(\eta(s,\theta))=\left(\varphi'(\eta(s,\theta)),\frac{\mathrm{d}}{\mathrm{d}s}\eta(s,\theta)\right)=\left(\varphi'(\eta(s,\theta)),2\varepsilon_1\frac{\mathrm{d}}{\mathrm{d}(2\varepsilon_1 s)}\sigma(2\varepsilon_1 s,\theta)\right)$$
$$=(\varphi'(\eta(s,\theta)),2\varepsilon_1 f(\sigma(2\varepsilon_1 s,\theta)))=(\varphi'(\eta(s,\theta)),2\varepsilon_1 f(\eta(s,\theta)))$$
$$\leqslant-2\varepsilon_1\varphi(\eta(s,\theta)),$$

thus

$$\int_0^{s_0}2\varepsilon_1\psi(\eta(s,\theta))\,\mathrm{d}s\leqslant\varphi(\eta(0,\theta))-\varphi(\eta(s_0,\theta))=c_0-c_0=0.$$

Since ψ is locally Lipschitz continuous and $0\leqslant\psi(u)\leqslant 1$, $\forall u\in X$, we have $\psi(\eta(s,\theta))\equiv 0$, $\forall 0\leqslant s\leqslant s_0$, which implies $\psi(\eta(s_0,\theta))=0$. But from the definition of ψ and $\eta(s_0,\theta)\in B$, we have $\psi(\eta(s_0,\theta))\neq 0$, which leads a contradiction. Similarly, we can prove $x_1\notin\bar{\Omega}$.

Since $\gamma_0(t)\in\varphi^{c+\varepsilon_1}$, $0\leqslant t\leqslant 1$ and $\eta(1,\varphi^{c+\varepsilon_1})\subset\varphi^{c-\varepsilon_1}$, it follows that

$$\varphi(\beta(t))=\varphi(\eta(1,\gamma_0(t))\leqslant c-\varepsilon_1,0\leqslant t\leqslant 1.$$

However, since $x_0\in\Omega$, $x_1\notin\bar{\Omega}$ and β is a path connecting x_0 and x_1, there exists $\tilde{t}\in(0,1)$ such that $\beta(\tilde{t})\in\partial\Omega$. Thus

$$c-\varepsilon_1\geqslant\varphi(\beta(\tilde{t}))\geqslant\inf_{x\in\partial\Omega}\varphi(x)=c_0=c,$$

which is a contradition. Therefore Theorem 1 is proved.

Remark 1 Let $S=\partial\Omega$, then by Lemma 1 the result of Theorem 1 can be strengthened as: $\forall\varepsilon>0,\delta>0$, $\exists x\in X$ such that

(a) $c-2\varepsilon\leqslant\varphi(x)\leqslant c+2\varepsilon$;

(b) $\|\varphi'(x)\|<8\varepsilon/\delta$;

(c) $\mathrm{dist}(x,\partial\Omega)\leqslant 2\delta$.

From Theorem 1, it is easy to prove the following theorem:

Theorem 2 Assume that all the conditions of Theorem 1 are valid and φ satisfies the P. S. condition, then c is a critical value of φ and

(1) $K_c-\{\theta,e\}\neq\emptyset$;

(2) $K_c\cap\partial\Omega\neq\emptyset$, when $c_0=c$, where $K_c=\{u\in X\mid\varphi'(u)=\theta,\varphi(u)=c\}$.

Remark 2 Theorem 2 is just the main result of [3]. Acturally, Theorem 1 also provides a new proof method for classical Mountain Pass Lemma.

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