Upper Embeddability of 3-Edge-Connected Simple Graphs with Independence-Number ≤ 5

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Abstract: Combined with the edge-connectivity, this paper investigates the relationship between the independence-number and the upper-embeddability of a 3-edge-connected simple graph and obtains the following result: Let G be a 3-edge-connected simple graph with $\alpha(G) \leq 5$ (where $\alpha(G)$ is the independence-number of G), then G is upper embeddable, and two minimal examples are given in the sense that there are 3-edge-connected graphs which are not upper embeddable.

Key words: graph, maximum genus, betti deficiency, upper embeddable, independence-number

CLC number O157.5 **Document code** A **Article ID** 1001-4616(2006)01-0017-04

独立数≤5 的 3-边连通简单图的上可嵌入性

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[摘要] 结合边连通度,本文探讨了 3-边连通简单图的独立数与上可嵌入性的关系,我们得到了下列结果:设 G 是一个 3-边连通简单图,$\alpha(G)$是 G 的独立数,若 $\alpha(G) \leq 5$,则 G 是上可嵌入的,同时我们又得到了两个在 3-边连通意义下最小的非上可嵌入图例.

[关键词] 图,最大亏格,Betti 亏数,上可嵌入的,独立数

0 Introduction

Graphs considered here are all connected, undirected, finite and furthermore simple. Terminology and notation without explanation in this paper will generally conform to that in [1].

A surface, always denoted by S, will mean a compact, connected and orientable 2-manifold. Such a manifold may be thought of a sphere with several handles. The number of handles on a surface S is called the genus of the surface S and is denoted by $g(S)$. By embedding a graph in a surface, we mean placing the vertices and the edges of the graphs in the surface such that edges may meet only at mutually incident vertices. A 2-cell embedding, or in other words, cellular embedding, of a graph G is one in which each of the components of the complement of G in the surface is topologically homeomorphic to an open disk. The components of complement of G are called faces or regions. Note that a disconnected graph does not admit a 2-cell embedding in any surface.

The genus, denoted by $\gamma(G)$, of a connected graph G is defined as the minimum integer $g(S)$, where S is a surface in which G has a 2-cell embedding. The maximum genus, denoted by $\gamma_M(G)$, of a connected graph G is defined to be the maximum integer $g(S)$ such that there exists a cellular embedding of G into the orientable

Received date: 2005-03-30.

Foundation item: Supported by the National Natural Science Foundation of China (10271048) and Shanghai Priority Academic Discipline and Science and Technology Commission of Shanghai Municipality (04JC14031).

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surface of genus $g(S)$. The definitions mentioned above are given with more topological rigor in the survey article by S. Stahl [2].

Since any 2-cell embedding of a connected graph G must have at least one face, the Euler polyhedral equation implies an upper bound on the maximum genus $\gamma_M(G) \leqslant \lfloor \frac{\beta(G)}{2} \rfloor$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is known as the Betti number (or cycle rank) of G (for any real number x, $\lfloor x \rfloor$ denotes the maximum integer no greater than x). A connected graph G is called upper embeddable if $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$. It is easily known from Euler's formula that a graph G is upper embeddable if and only if G has one or two face embedding on the orientable surface of the maximum genus according to $\beta(G)$ is even or odd.

Since the introductory investigation of maximum genus by E. Nordhaus, B. Stewart and A. T. White [3], the upper embeddability of graphs has been studied extensively. In particular, N. Xuong [4] has shown that a graph G is upper embeddable if and only if there is a spanning tree T of G such that at most one of the connected components of G/T consists of an odd number of edges. Moreover, Kundu [5] proved that every 4-edge-connected graph contains two spanning trees. Combining these two results shows that every 4-edge-connected graph is upper embeddable. Since 4-vertex-connectivity implies 4-edge-connectivity, it follows that every 4-vertex-connected graph is also upper embeddable. However, there are examples of 3-edge-connected graphs that are not upper embeddable [6].

1 Some Basic Lemmas

Let G be a graph and T be a spanning tree of G, $\xi(G,T)$ denotes the number of the components of $G/E(T)$ with odd number of edges, we define $\xi(G) = \min_T \xi(G,T)$ to be the Betti deficiency of G, where the minimum is taken over all the spanning trees of G. Note that $\xi(G) = \beta(G) \pmod 2$.

For the upper embeddability of graphs, in 1979 N. H. Xuong has given a sufficient and necessary condition in [4,7].

Lemma A (Xuong): Let G be a connected graph, then

(1) G is upper embeddable if and only if $\xi(G) \leqslant 1$;

(2) $\gamma_M(G) = \frac{\beta(G) - \xi(G)}{2}$.

From Lemma A above, the maximum genus of a graph G is mainly determined by the Betti deficiency $\xi(G)$ (since the Betti number $\beta(G)$ can be easily computed).

Again, for a graph G and $A \subseteq E(G)$, denoted by $c(G/A)$ the number of the components of G/A and by $b(G/A)$ the number of the components of G/A with odd Betti number, the following lemma was proved in 1981 by L. Nebesky [8].

Lemma B (Nebesky): Let G be a connected graph, then

(1) G is upper embeddable if and only if $c(G/A) + b(G/A) - 2 \leqslant |A|$ for every $A \subseteq E(G)$;

(2) $\xi(G) = \max_{A \subseteq E(G)} \{c(G/A) + b(G/A) - |A| - 1\}$.

Let $F_1, F_2, \cdots, F_l$ $(l \geqslant 2)$ be l distinct subgraphs of G, then denoted by $E_G(F_1, \cdots, F_l)$ the set of those edges of G whose two ends are, respectively, in a pair of subgraphs F_i and F_j for $1 \leqslant i, j \leqslant l$ and $i \neq j$, and let $E(F,G)$ denote the edges one of whose end vertices are in $V(F)$ and the others not in $V(F)$. The following lemma proved in [9] (or [10]) plays a fundamental role throughout this paper.

Lemma C (Huang): Let G be a connected graph. If $\xi(G) \geqslant 2$, namely G is not upper embeddable, then there exists a subset $A \subseteq E(G)$ such that the following properties hold:

(i) $c(G/A) \geqslant 2$ and $\beta(F) = 1 \pmod 2$ for any component F of G/A;

(ii) $3 \leqslant 2c(G/A) - |A| \leqslant 4$;

(iii) F is an induced subgraph of G for each connected component F of G/A;

(iv) $|E_G(F_1,\cdots,F_l)|\leq 2l-3$ for any $l\geq 2$ distinct components $F_1,\cdots,F_l$ of G/A, especially, $|E_G(F_1,F_2)|\leq 1$ for $l=2$;

(v) $\xi(G)=2c(G/A)-|A|-1$.

In the above Lemma C, we notice the following facts:

Lemma D: Under the conditions and conclusions of Lemma C, we have

(1) For any component F of G/A, if G is k-edge-connected $(k\geq 1)$, then $|E(F,G)|\geq k$.

(2) $|A|=\frac{1}{2}\sum_F |E(F,G)|$, where F is taken over all the connected components of G/A.

(3) If G is 3-edge-connected, then $c(G/A)\geq 4$.

Proof According to Lemma C, we construct a graph $G'=G/(G/A)$ as follows. The vertices of G' are the components of G/A. For each edge in A joining a pair of components in G/A, we make an edge in G' joining the corresponding vertices. It is easy to see that G' is connected because of the connectivity of G. From the definition of G', we see that the degree of each vertex of G', corresponding to a component F of G/A, equals $|E(F,G)|$, so it follows that if G is k-edge-connected $(k\geq 1)$, then $|E(F,G)|\geq k$ for any component F of G/A. Again, we have that $2|A|=2|E(G')|=\sum_{x\in V(G')} d_{G'}(x)=\sum_F |E(F,G)|$.

Furthermore, Lemma C show that A must be a set of edge-cut and $|A|\geq 3$ (Otherwise, A cannot be a set of edge-cut of a 3-edge-connected graph). If $|A|=3$, because a 3-edge-connected graph has at most 2 components after removing any 3 edges, this implies that $c(G/A)\leq 2$, which contradicts to the conclusion (ii) of Lemma C. Thus, $|A|>3$, we then get that $l=c(G/A)\geq 4$ by property (ii) of Lemma C.

2 The Main Results

The independence-number, denoted by $\alpha(G)$, of a graph G is the number of vertices in a maximum independent set of G. Since the reason mentioned above, we consider here only such graphs with edge-connectivity = 3, we have the following theorem.

Theorem 1 Let G be a 3-edge-connected simple graph with $\alpha(G)\leq 5$ (where $\alpha(G)$ is the independence-number of G), then G is upper embeddable.

Proof By contradiction that assumes G is not upper embeddable, from Lemma C, there exists $A\subseteq E(G)$ such that all the components $F_1,F_2,\cdots,F_l\,(l=c(G/A))$ of G/A satisfy the properties (i)—(v) of Lemma C. Because G is 3-edge-connected, $|E(F_i,G)|\geq 3$ for all $i=1,2,\cdots,l$. Again from (3) of Lemma D, we know that $l=c(G/A)\geq 4$. Now we shall handle two cases $l=4$ and $l\geq 5$.

Case 1 $l=4$.

Let F_1,F_2,F_3,F_4 be the four components of G/A. Because G is 3-edge-connected, we get that $|E_G(F_i,F_j)|=1$ $(i\neq j;\ i,j=1,2,3,4)$ by (iv) of Lemma C, thus $|E_G(F_1,F_2,F_3,F_4)|=6$.

Again, by (i) of Lemma C, we then have that $|E_G(F_1,F_2,F_3,F_4)|\leq 2\times 4-3=5$.

This is a contradiction!

Case 2 $l\geq 5$.

Now let x be the number of F_i with $|E(F_i,G)|=3$ for $i=1,2,\cdots,l$, we then have that $|A|=\frac{1}{2}\sum_{i=1}^{l}|E(F_i,G)|\geq\frac{1}{2}[3x+4(l-x)]=2l-\frac{x}{2}$.

From (ii) of Lemma C: $3\leq 2c(G/A)-|A|$, we have that $3\leq 2l-(2l-\frac{x}{2})$, i.e. $x\geq 6$. Without loss of generality, let F_1,F_2,F_3,F_4,F_5,F_6 be the components with $|E(F_i,G)|=3$ $(1\leq i\leq 6)$. By (i) of Lemma C: $\beta(F_i)=1\pmod 2\,(1\leq i\leq 6)$ and because G is simple, we get that $|V(F_i)|\geq 3\,(1\leq i\leq 6)$. Since $|E(F_i,G)|=3$, that there exist $x_i\in V(F_i)$ $(1\leq i\leq 6)$ such that x_1,x_2,x_3,x_4,x_5 and x_6 are not adjacent with each other, thus, we get that $\alpha(G)=6$ which contradicts to the $\alpha(G)\leqslant 5$!

Thereby, we complete the proof the theorem.

Remark If G is a 3-edge-connected simple graph with diameter three, by Claim 2 in [10], then for any $F_i(i=1,2,3,4,5,6)$. F_i is a complete graph K_3. From the Case 2 of the proof, G must be the graphs G_1 or G_2 shown in Fig. 1. Take $A=\{e_1,e_2,e_3,e_4,e_5,e_6,e_7,e_8,e_9\}$, it is easily known that $c(G_i/A)=b(G_i/A)=6$ $(i=1,2)$ and $|A|=9$, we then get that

$$c(G_i/A)+b(G_i/A)-2=6+6-2=10\geqslant 9=|A|.$$

By the Lemma B, we know that neither G_1 nor G_2 is not upper embeddable. Thus, two types of minimal examples are given in the sense that there are 3-edge-connected graphs which are not embeddable.

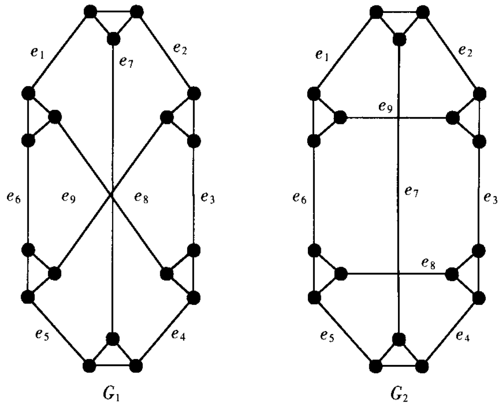

Fig.1 Two 3-edge-connected graphs with $\alpha(G)=6$ wh0ich are not upper embeddable

Acknowledgements The authors are grateful to the reviewers for thorough and constructive reviews.

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