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A Theorem on 3-Colorable Plane Graphs

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Abstract: Borodin and Raspaud proposed a conjecture which claims that every plane graph without 5-circuits and adjacent triangles is 3-colorable. This strengthens a conjecture of Steinberg. In this paper, we study the structure of plane graphs without 5-, 6- and 9-circuits and adjacent triangles. As a corollary, we prove that such graphs are 3-colorable, this improves a result by Borodin, and independently by Sanders and Zhao, and also provides a positive support to Borodin and Raspaud's conjecture.

Key words: plane graph, circuit, coloring

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关于平面图 3-可着色的一个定理

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[摘要]　Borodin 和 Raspaud 提出一个猜想:任何既没有 5-圈也没有相邻三角形的平面图是 3-可着色. 这个猜想强化了 Steinberg 提出的猜想. 在本文中，我们研究了没有 5-,6-,9-圈并且没有相邻三角形的平面图的结构. 利用这个结构，证明了这类图是 3-可着色的. 它加强了由 Borodin 及 Sanders 和 Zhao 的结果，并且又是对 Borodin 和 Raspaud 猜想的一个正面的支持.

[关键词]　平面图，圈，着色

0　Introduction

In 1976, Steinberg (see [1] or [2]) conjectured that every plane graph without 4- and 5-circuits is 3-colorable. This conjecture was relaxed by Erdös (see [1]) in 1990 to the following question: Is there an integer $k \geqslant 5$ such that every plane graph without i-circuits for $4 \leqslant i \leqslant k$ is 3-colorable? The first positive result appeared in [3] in which the authors showed that $k = 11$ is acceptable. Sanders and Zhao [4], and Borodin [2] independently, improved that to $k = 9$. Recently, the parameter k was again improved to 7 by Borodin et al [5].

The distance between triangles in a graph is defined as the length of the shortest path between vertices of different triangles. Two triangles are said to be adjacent if they have an edge in common. In [6], Borodin and Raspaud proved that if G is a plane graph without 5-circuits and triangles of distance less than four, then G is 3-colorable, and they conjectured that every plane graph without 5-circuits and adjacent triangles is 3-colorable. It is proved that every plane graph without 5- and 7-circuits and adjacent triangles is 3-colorable, every plane graph without 5-circuits and triangles of distances less than 3 is 3-colorable, and every plane graph without 5- and 6-circuits and triangles of distances less than 2 is 3-colorable. It seems that Borodin and Raspaud's conjecture is far

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away from being solved entirely.

We consider a similar problem, the 3-colorability of plane graph without adjacent triangles. Motivated by a result of [7], in which the authors consider the 3-choosability of plane graphs without circuits of certain length, we consider in this paper the 3-colorability of plane graphs, and prove the following.

Theorem 1 Let G be a 2-connected plane graph that contains no adjacent triangles and circuits of length $i \in \{5, 6, 9\}$. Then, one of the following holds

(1) $\delta(G) < 3$;

(2) G contains a 4-face;

(3) G contains a 10-face incident with ten 3-vertices and adjacent to five 3-faces.

Theorem 2 Every plane graph without adjacent triangles and circuits of length in $\{5, 6, 9\}$ is 3-colorable.

Theorem 1 strengthens a result of Borodin (Theorem 2 of [2]). Theorem 2 provides a positive support to Borodin and Raspaud's conjecture, and improves the result by Borodin (Theorem 1 of [2]), and independently by Sanders and Zhao [4].

Let G be a plane graph. A k-vertex is a vertex of degree k. Let f be a face of G. The degree of f, denoted by $\lambda(f)$, is the number of edges incident with it, where every cut-edge is counted twice. A k-face is a face of degree k. A k-circuit is a circuit of length k. A set S of vertices is deleted from a graph G if S together with all the edges with at least one end in S is removed from G, and the resulting graph is denoted by $G \backslash S$. A k-circuit is a circuit of length k. Terminologies and notation not defined here can be found in [8].

1 Proofs of the Theorems

The proof of Theorem 1 is similar to the proof of Theorem 1.4 of [7]. We present the sketch of the proof here for completeness (the detailed proof is presented as an appendix for referee).

Proof of Theorem 1 To prove this theorem by contradiction, we assume that there exists a plane graph G that is a counterexample of minimum order, i.e., G is a plane graph that contains no circuits of length in $\{5, 6, 9\}$ and adjacent triangles, but none of the three conclusions of Theorem 1 holds.

By Euler's formula $|V(G)| + |F(G)| - |E(G)| = 2$, we have $\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2\lambda(f) - 6) = -12$.

We define $\omega(v) = d(v) - 6$ for $v \in V(G)$ and $\omega(f) = 2\lambda(f) - 6$ for $f \in F(G)$. Then $\sum_{x \in V(G) \cup F(G)} \omega(x) = -12$.

We use $T(v)$ to denote the set of 3-faces incident with a vertex v, and use $N(f)$ to denote the set of faces adjacent to a face f. To construct a new weight ω' for $x \in V \cup F$, we discharge the weight from one element to another according to following rules. Every non-triangular face f transfers to each incident vertex v the following charge.

$\frac{3}{2}$ if $d(v) = 3$ and $|T(v)| = 1$,

1 if $d(v) = 3$ and $|T(v)| = 0$,

1 if $d(v) = 4$ and $|T(v)| = 2$ or $|T(v) \backslash N(f)| = 1$,

$\frac{1}{2}$ if $d(v) = 4$ and $|T(v)| = 0$ or $|T(v) \cap N(f)| = 1$,

$\frac{1}{3}$ if $d(v) = 5$.

It is clear that $\sum_{x \in V \cup F} \omega'(x) = -12$. But a detailed calculation shows that $\omega'(x) \geqslant 0$ for each $x \in V \cup F$ (We present the detailed proof in the appendix for refereeing, interested readers may find the detailed proof in [7]). This contradiction completes the proof.

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Proof of Theorem 2 Assume to contrary that the conclusion is not true. Then, we may choose G to be a plane graph which is a counterexample with minimum order.

Claim 1 G is a 2-connected.

Proof If it is not so, let x be a cut vertex of G, and let G_1 and G_2 be two subgraphs of G such that $V(G_1) \cap V(G_2) = \{x\}$ and $E(G_1) \cup E(G_2) = E(G)$. By the choice of G, both G_1 and G_2 are 3-colorable, and we can assign G_i a 3-coloring $\phi_i\,(i=1,2)$ with the property that $\phi_1(x) = \phi_2(x)$. Obviously, ϕ_1 and ϕ_2 yield a 3-coloring of G, a contradiction.

Claim 2 $\delta(G) \geqslant 3$.

Proof If it is not the case, let u be a vertex of degree 2 in G. Clearly, $G-u$ still contains no adjacent triangles and circuits of length in $\{5,6,9\}$. By the choice of G, $G-u$ is 3-colorable. But in any 3-coloring ϕ of $G-u$, at most two colors are used by the neighbors of u. Then, we can extend ϕ to a 3-coloring of G, a contradiction.

Claim 3 G contains no 4-faces.

Proof Assume to the contrary, let f be a 4-face with boundary $uvwxu$. Since G contains no 5-circuits, f is not adjacent to triangles. Let $G_{u,w}$, $G_{v,x}$ be the graphs obtained from G by identifying u and w, and by identifying v and x, respectively. It is evident that both $G_{u,w}$ and $G_{v,x}$ contain no adjacent triangles. If one of $G_{u,w}$ and $G_{v,x}$, say $G_{u,w}$ contains no circuits of length in $\{5,6,9\}$, then by the choice of G, $G_{u,w}$ is 3-colorable. But any 3-coloring of $G_{u,w}$ yields a 3-coloring of G also, a contradiction. So, we assume that each of $G_{u,w}$ and $G_{v,x}$ contains a circuit of length in $\{5,6,9\}$. Then, G contains a path P_1 joining u to w, and a path P_2 joining v to x, of which each has length in $\{5,6,9\}$. Since G is a plane graph, $P_1 \setminus \{u,w\}$ and $P_2 \setminus \{v,x\}$ has a vertex in common. We consider serval cases according to the length l_i of P_i, $i=1,2$. For a path P and two distinct vertices u and v on P, we use $P_{[u,v]}$ to denote the segment of P that joins u and v.

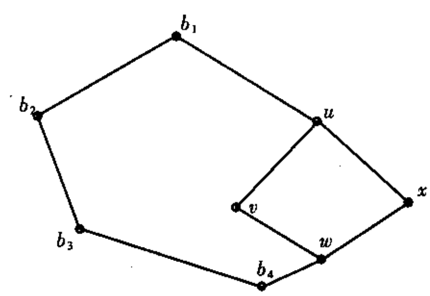

Fig 1

If one of ux and uv (see Fig. 1), say uv, is in P_2, then the length of $P_2[x,u]$ is at least 10 for otherwise G contains a circuits of length in $\{5,6,9\}$. This contradicts $l_2 \leqslant 9$. If one of u and w, say u, is on P_2, then both $P_2\{[v,u]\}$ and $P_2\{[u,x]\}$ have length at least 8, a contradiction also. Therefore, we suppose by symmetry that $\{u,w\} \cap V(P_2) = \emptyset$ and $\{v,x\} \cap V(P_1) = \emptyset$.

Let $P_1 = ub_1b_2 \cdots b_{l_1-1}w$ (See Fig. 1 for $l_1 = 5$), and assume that $b_i \in V(P_2)$ for some i is the only common vertex of P_1 and P_2. Let $P_u = P_1[u,b_i]$, $P_w = P_1[b_i,w]$, $P_v = P_2[v,b_i]$ and $P_x = P_2[b_i,x]$.

For $s \in \{u,v,w,x\}$, let l_s be the number of vertices on the path P_s. Then, the subgraph induced by $V(P_1) \cup V(P_2)$ contains eight circuits that have length in $\{l_s + l_t - 1, l_s + l_t + 1\}$ for some $s \in \{u,w\}$ and $t \in \{v,x\}$.

Since $l_1, l_2 \in \{5,6,9\}$ and G contains no circuits of length in $\{5,6,9\}$,

$$\begin{cases} l_u + l_w - 2,\ l_v + l_x - 2 \in \{5,6,9\}, \\ l_u + l_v - 1, l_u + l_v + 1 \notin \{5,6,9\}, \\ l_u + l_x - 1, l_u + l_x + 1 \notin \{5,6,9\}, \\ l_w + l_v - 1, l_w + l_v + 1 \notin \{5,6,9\}, \\ l_w + l_x - 1, l_w + l_x + 1 \notin \{5,6,9\}. \end{cases} \tag{1}$$

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By a detailed calculation, one can find that the above equation (1) has no solution. Here as an example, we discuss the case where $l_u + l_w - 2 = l_1 = 5$. By symmetry, we may suppose that $l_u = 2$ or 3, and $l_w = 5$ or 4 then. If $l_u = 2$ and $l_w = 5$, then $l_u + l_v - 1, l_u + l_v + 1 \notin \{5, 6, 9\}$ indicates that $l_v \geq 7$, and $l_u + l_x - 1, l_u + l_x + 1 \notin \{5, 6, 9\}$ indicates that $l_x \geq 7$. If $l_u = 3$ and $l_w = 4$, then $l_v \geq 8$ and $l_w \geq 8$. In both cases, $l_v + l_x - 2 \geq 12 \notin \{5, 6, 9\}$, a contradiction.

The same arguments show us also a contradiction when P_1 and P_2 have two or more inner vertices in common. We omit the detailed proof.

By Claims 1, 2 and 3, we see that G is 2-connected plane graph with $\delta(G) \geq 3$ that contains no 4-faces, no adjacent triangles and no circuits of length in $\{5, 6, 9\}$. By Theorem 1, G must contains a 10-face f that is incident with 10 3-vertices. Let C be the boundary of f that is a circuit of length 10.

By our choice of G, $G \backslash V(C)$ admits a 3-coloring ϕ. Since each vertex x of C has exactly one neighbor in $G \backslash V(C)$ that is colored in ϕ, there are still two colors that can be used for coloring x. One can easily construct a coloring of C that together with ϕ yields a 3-coloring of G. This contradiction completes the proof of Theorem 2.

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