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Summation Formulas Involving Second-order Recurrent Sequences

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Abstract: Let $\{w_n\}$ be a second-order recurrence sequence and let n be integers. According to the definition and characteristics of the second-order recurrent sequences, the involving second-order recurrent sequences sums $S_{m,k}$ and $T_{m,k}$ are defined. The sums $S_{m,k}$ and $T_{m,k}$ are studied. The recursion formulas of $S_{m,k}$ and $T_{m,k}$ are determined. The conclusion extends two result from Melham R S.

Key words: second-order recurrent sequences, Lucas numbers, companion sequence

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关于二阶线性递归序列乘积的和式

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[摘要] 设 $\{w_n\}$ 是二阶线性递归序列,n 为整数. 根据二阶线性递归序列的定义和性质,给出了关于二阶线性递归序列乘积的和式 $S_{m,k}$,$T_{m,k}$ 的定义,研究了关于二阶线性递归序列的和式 $S_{m,k}$,$T_{m,k}$,得到了关于和式 $S_{m,k}$,$T_{m,k}$ 的重要结论. 本文的主要结论推广了 Melham R S 的二个结果.

[关键词] 二阶线性递归序列,Lucas 序列,伴随序列

0 Introduction

Let $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of the integers and real numbers, respectively. Let $\mathbf{R}^* = \mathbf{R}\setminus\{0\}$. Fix $p, q \in \mathbf{R}^*$, and let $\mathscr{L}(p,q)$ consist of all those second-order recurrent sequences $\{w_n\}_{n\in\mathbf{Z}}$ of complex numbers satisfying the recursion:

$$w_{n+2} = pw_{n+1} - qw_n,\ w_0 = a,\ w_1 = b \text{ for } n = 0, \pm 1, \pm 2, \cdots \tag{1}$$

Write $\Delta = p^2 - 4q$. If $\Delta \geqslant 0$, then it is known that (see [1])

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{2}$$

where $\alpha = (p + \sqrt{\Delta})/2$, $\beta = (p - \sqrt{\Delta})/2$, $A = b - a\beta$, and $B = b - a\alpha$.

In [2,3], Melham defined a sequence $\{W_n\}$ in the special case $q = -1$.

The Lucas sequences $\{u_n\}_{n\in\mathbf{Z}}$ and $\{v_n\}_{n\in\mathbf{Z}}$ in $\mathscr{L}(p,q)$ take specialvalues at $n = 0$ and 1, namely,

$$u_0 = 0, u_1 = 1, v_0 = 2, v_2 = p. \tag{3}$$

It is well known that

$$(\alpha - \beta)u_n = \alpha^n - \beta^n \text{ and } v_n = \alpha^n + \beta^n \text{ for } n \in \mathbf{Z}. \tag{4}$$

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If $q=-1$ Melham[2,3] defined $U_n=u_n$ and $V_n=v_n$.

If $p=1$ and $q=-1$, then those $F_n=u_n$ and $L_n=v_n$ are called Fibonacci numbers and Lucas numbers, respectively.

We define a companion sequence $\{\overline{w}_n\}$ of $\{w_n\}$ as

$$\overline{w}_n=A\alpha^n+B\beta^n. \tag{5}$$

Aspects of this sequence have been treated, for example, in [4] and [5].

From (2) and (5) we see that $\overline{u}_n=v_n$ and $\overline{v}_n=\Delta u_n$.

The purpose of this paper is to investigate the infinite sums

$$S_{k,m}=\sum_{n=1}^{\infty}\frac{q^{kn}\overline{w}_{k(n+m)}}{w_{kn}w_{k(n+m)}w_{k(n+2m)}}, \tag{6}$$

$$T_{k,m}=\sum_{n=1}^{\infty}\frac{q^{2kn}}{w_{kn}w_{k(n+m)}w_{k(n+2m)}w_{k(n+3m)}}, \tag{7}$$

where k and m are positive integers, and $w_{k(n+im)}\neq 0$ for $i=0,1,2,3$.

If $q=-1$ and $2\mid k$, we have

$$S_{k,m}=\sum_{n=1}^{\infty}\frac{\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}}, \tag{8}$$

$$T_{k,m}=\sum_{n=1}^{\infty}\frac{1}{W_{kn}W_{k(n+m)}W_{k(n+2m)}W_{k(n+3m)}}. \tag{9}$$

In [2,3], Melham gave the formulas for $S_{k,m}$ and $T_{k,m}$ in the special case $q=-1$ and $2\mid k$. In this paper we generalize Melham's results under the conditions that p and q are nonzero real numbers, and k is a positive integer.

1 Preliminary Results

We require the following, in which k and m are taken to be integers.

$$\frac{\beta^{kn}}{w_{kn}}-\frac{\beta^{k(n+m)}}{w_{k(n+m)}}=\frac{Aq^{kn}u_{km}}{w_{kn}w_{k(n+m)}}, \tag{10}$$

$$w_{k(n+m)}w_{k(n+2m)}-w_{kn}w_{k(n+3m)}=ABq^{kn}u_{km}u_{2km}, \tag{11}$$

$$w_{n+k}-q^k w_{n-k}=\overline{w}_n u_k, \tag{12}$$

$$B\beta^n=w_{n+1}-\alpha w_n. \tag{13}$$

Identities (10) and (11) are readily proved using (2) and (5). Identity (12) can be obtained from [4], while (13) can be obtained from [6].

We will also make use of the following lemma:

Lemma 1 Let k and m be positive integers, then

$$\sum_{n=1}^{\infty}\frac{q^{kn}}{w_{kn}w_{k(n+m)}}=\frac{1}{ABu_{km}}\left[\sum_{n=1}^{m}\frac{w_{kn+1}}{w_{kn}}-m\alpha\right]. \tag{14}$$

Proof If summing both sides of (10), we obtain

$$\sum_{n=1}^{\infty}\frac{q^{kn}}{w_{kn}w_{k(n+m)}}=\sum_{n=1}^{\infty}\frac{1}{Au_{km}}\left(\frac{\beta^{kn}}{w_{kn}}-\frac{\beta^{k(n+m)}}{w_{k(n+m)}}\right)=\sum_{n=1}^{m}\frac{1}{Au_{km}}\frac{\beta^{kn}}{w_{kn}}$$

and (14) follows from (13).

Remark 1 Theorem 3 of [7] is essentially our (14) in the special case $m=1$.

2 Main Results

Our main results can now be given in two theorems.

Theorem 1 Let k and m be positive integers, then

$$S_{k,m}=\frac{1}{u_{km}}\sum_{n=1}^{m}\frac{q^{kn}}{w_{kn}w_{k(n+m)}}. \tag{15}$$

Proof Let k and m be positive integers. Setting

$$I=\frac{\beta^{kn}}{w_{kn}}-\frac{\beta^{k(n+m)}}{w_{k(n+m)}}+\frac{\beta^{k(n+2m)}}{w_{k(n+2m)}}, \tag{16}$$

and using (10) we get

$$I=\frac{Aq^{kn}u_{km}}{w_{kn}w_{k(n+m)}}+\frac{\beta^{k(n+2m)}}{w_{k(n+2m)}}, \tag{17}$$

and

$$I=\frac{\beta^{kn}}{w_{kn}}-\left[\frac{\beta^{k(n+m)}}{w_{k(n+m)}}-\frac{\beta^{k(n+2m)}}{w_{k(n+2m)}}\right]=\frac{\beta^{kn}}{w_{kn}}-\frac{Aq^{k(n+m)}u_{km}}{w_{k(n+m)}w_{k(n+2m)}}. \tag{18}$$

From (17) and (18), we have

$$2I=\frac{\beta^{kn}}{w_{kn}}+\frac{\beta^{k(n+2m)}}{w_{k(n+2m)}}+\frac{Aq^{kn}u_{km}}{w_{kn}w_{k(n+m)}}-\frac{Aq^{k(n+m)}u_{km}}{w_{k(n+m)}w_{k(n+2m)}}. \tag{19}$$

Now, by (12), we have

$$\frac{Aq^{kn}u_{km}}{w_{kn}w_{k(n+m)}}-\frac{Aq^{k(n+m)}u_{km}}{w_{k(n+m)}w_{k(n+2m)}}=\frac{Aq^{kn}u_{km}(w_{k(n+2m)}-q^{km}w_{kn})}{w_{kn}w_{k(n+m)}w_{k(n+2m)}}=\frac{Aq^{kn}u_{km}^2\bar{w}_{k(n+m)}}{w_{kn}w_{k(n+m)}w_{k(n+2m)}}. \tag{20}$$

Thus

$$2I=\frac{\beta^{kn}}{w_{kn}}+\frac{\beta^{k(n+2m)}}{w_{k(n+2m)}}+\frac{Aq^{kn}u_{km}^2\bar{w}_{k(n+m)}}{w_{kn}w_{k(n+m)}w_{k(n+2m)}},$$

so

$$\frac{Aq^{kn}u_{km}^2\bar{w}_{k(n+m)}}{w_{kn}w_{k(n+m)}w_{k(n+2m)}}=\left[\frac{\beta^{kn}}{w_{kn}}-\frac{\beta^{k(n+m)}}{w_{k(n+m)}}\right]-\left[\frac{\beta^{k(n+m)}}{w_{k(n+m)}}-\frac{\beta^{k(n+2m)}}{w_{k(n+2m)}}\right]. \tag{21}$$

Finally, summing both sides of (21), we obtain

$$Au_{km}^2S_{k,m}=\sum_{n=1}^{m}\frac{\beta^{kn}}{w_{kn}}-\sum_{n=1}^{m}\frac{\beta^{k(n+m)}}{w_{k(n+m)}}$$

and (15) follows from (10).

If we take $m=1$ and put $w_n=u_n$ and $w_n=v_n$, respectively, then (15) becomes

Corollary 1 Let k be a positive integer, then

$$\sum_{n=1}^{\infty}\frac{q^{kn}v_{k(n+1)}}{u_{kn}u_{k(n+1)}u_{k(n+2)}}=\frac{q^k}{u_k^3v_k}, \tag{22}$$

and

$$\sum_{n=1}^{\infty}\frac{q^{kn}u_{k(n+1)}}{v_{kn}v_{k(n+1)}v_{k(n+2)}}=\frac{q^k}{\Delta\ u_{4k}}. \tag{23}$$

If we put $k=1$, (22) and (23) become

Corollary 2

$$\sum_{n=1}^{\infty}\frac{q^nv_{n+1}}{u_nu_{n+1}u_{n+2}}=\frac{q}{p}, \tag{24}$$

and

$$\sum_{n=1}^{\infty}\frac{q^nu_{n+1}}{v_nv_{n+1}v_{n+2}}=\frac{q}{(p^2-4q)(p^3-2pq)}. \tag{25}$$

If we put $u_n=F_n$ and $v_n=L_n$, (24) and (25) become

Corollary 3

$$\sum_{n=1}^{\infty}\frac{(-1)^nL_{n+1}}{F_nF_{n+1}F_{n+2}}=-1, \tag{26}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^nF_{n+1}}{L_nL_{n+1}L_{n+2}}=-\frac{1}{15}. \tag{27}$$

Remark 2 (3.7) and (3.8) of Melham[3] are essentially our (22) and (23) in the special case $u_n=F_n$, $v_n=L_n$ and $k=2$.

Theorem 2 Let k and m be positive integers, then

$$ABu_{km}u_{2km}T_{k,m} = \frac{1}{AB}\Big[\frac{1}{u_{3km}}\sum_{n=1}^{3m}\frac{w_{kn+1}}{w_{kn}} - \frac{1}{q^{km}u_{km}}\sum_{n=1}^{m}\frac{w_{kn+1}}{w_{kn}}\Big] + \sum_{n=1}^{m}\frac{q^{k(n-m)}}{w_{kn}w_{k(n+m)}} + \frac{m\alpha}{AB}\Big[\frac{1}{q^{km}u_{km}} - \frac{1}{u_{3km}}\Big]. \tag{28}$$

Proof From (11), we see that

$$\frac{ABq^{2kn}u_{km}u_{2km}}{w_{kn}w_{k(n+m)}w_{k(n+2m)}w_{k(n+3m)}} = \frac{q^{kn}}{w_{kn}w_{k(n+3m)}} - \frac{q^{kn}}{w_{k(n+m)}w_{k(n+2m)}}.$$

Summing both sides we obtain with the aid of (14),

$$\begin{aligned}ABu_{km}u_{2km}T_{k,m} &= \sum_{n=1}^{\infty}\frac{q^{kn}}{w_{kn}w_{k(n+3m)}} - \frac{1}{q^{km}}\sum_{n=1}^{\infty}\frac{q^{k(n+m)}}{w_{k(n+m)}w_{k(n+2m)}} \\ &= \frac{1}{ABu_{3km}}\Big[\sum_{n=1}^{3m}\frac{w_{kn+1}}{w_{kn}} - 3m\alpha\Big] - \frac{1}{ABq^{km}u_{km}}\Big[\sum_{n=1}^{m}\frac{w_{kn+1}}{w_{kn}} - m\alpha\Big] + \sum_{n=1}^{m}\frac{q^{k(n-m)}}{w_{kn}w_{k(n+m)}}\end{aligned}$$

which is (28).

If we take $k=1$ and $m=1$, and put $w_n = F_n$ and $w_n = L_n$, respectively, then (28) becomes

Corollary 4
$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}F_{n+2}F_{n+3}} = \frac{12-5\sqrt{5}}{4}, \tag{29}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{L_nL_{n+1}L_{n+2}L_{n+3}} = \frac{5-2\sqrt{5}}{40}. \tag{30}$$

Remark 3 (3.10) and (3.11) of Melham [3] are essentially our (28) in the special case $u_n = F_n$, $v_n = L_n$, $m=1$ and $k=2$.

Remark 4 Theorems 1 and 2 of Melham [3] are essentially our (15) and (28) in the special case $q = -1$ and k is taken to be even integers.

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