A V-cycle Multigrid Method for Mortar-type Rotated Q_1 Element

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Abstract: In this paper, a V-cycle multigrid method is presented for the Mortar-type rotated Q_1 nonconforming element. The uniform convergence rate is proven, which is independent of mesh size and mesh level. Numerical experiments are presented to confirm our theoretical results.

Key words: multigrid method, Mortar finite element, rotated Q_1 element
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Mortar 型旋转 Q_1 元的 V 循环多重网格

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[**摘要**] 展现了一种 Mortar 类型的旋转 Q_1 有限元的多重网格方法. 通过定义一些算子证明了这种 V 循环的多重网格是一致收敛的, 它的收敛率不依赖于网格的尺寸和层数, 并通过数值实验验证了理论分析的正确性.

[**关键词**] 多重网格, Mortar 型有限元, 旋转 Q_1 元

0 Introduction

The mortar finite element method is a nonconforming domain decomposition technique tailored to handle problems posed on domains that are partitioned into independently subdomains. The meshes on different subdomains need not align across subdomains interfaces. Adequate weak continuity conditions replace the pointwise continuity at the interfaces. This offers the advantages of freely choosing highly varying mesh sizes on different subdomains and is very promising to approximate the problems with abruptly changing diffusion coefficients or local anisotropy.

The rotated Q_1 element is an important nonconforming element. It was first proposed and analyzed in [1] for numerically solving the Stokes problem. The rotated Q_1 element provides the simplest example of discretely divergence-free nonconforming element on quadrilaterals. Due to its simplicity, the rotated Q_1 element is used to simulate the deformation of martensitic crystals with microstructure in [2]. Independently, it also was derived with-

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in the framework of mixed element method (see [3]). Recently, Chen and Xu proposed a mortar element version for rotated Q_1 element in [4].

In this paper, we focus our attention on studying multigrid method for the mortar finite element method for the rotated Q_1 element. Since the Mortar-type rotated Q_1 element space can't consist of any bilinear element space, a special intergrid transfer operator is presented for the nonnested spaces. Based on this operator, we give a V-cycle multigrid algorithm and prove that the V-cycle multigrid is uniform convergence, i. e., the convergence rate is independent of mesh size and level. We use the rotated Q_1 mortar element only on the last level L, and use the conforming bilinear element spaces as the coarse-grid correction spaces on all coarse levels. It is shown that this V-cycle multigrid requires only one smoothing step on all coarse levels $l < L$, while on the last level L a sufficiently large number of smoothing steps is needed.

The outline of this paper is organized as follows. In section 2, we introduce model problem, the rotated Q_1 mortar element method, and some notations. In section 3, we construct an intergrid transfer operator and propose our multigrid algorithm. In section 4, We give the proof of uniform convergence for the V-cycle multigrid. Numerical experiments is presented in the last section.

1 Preliminaries

For simplicity, we consider the following model problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^2$ is a bounded rectangular domain, $f \in L^2(\Omega)$. There is no difficulty to extend the results in this paper to more general second elliptic problems.

The variational form of (1.1) is to find $u \in H_0{}^1(\Omega)$ such that

$$a(u,v) = (f,v), \qquad \forall v \in H_0{}^1(\Omega), \tag{1.2}$$

where the bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \qquad \forall u,v \in H^1(\Omega).$$

From [5] we know for any $f \in L^2(\Omega)$, there is a solution of (1.1): $u \in H^2(\Omega)$, such that

$$\|u\|_2 \leqslant C\|f\|_0. \tag{1.3}$$

Divide Ω into geometrically conforming rectangular substructures, i. e., $\overline{\Omega} = \bigcup_{k=1}^{N} \overline{\Omega}_k$ with $\overline{\Omega}_k \cap \overline{\Omega}_l$ being empty set or a vertex or an edge for $k \neq l$. With each Ω_k we associate a quasi-uniform triangulation $\mathscr{T}_h(\Omega_k)$ made of elements that are rectangles whose edges are parallel to x-axis or y-axis. Denote the global mesh $\cup_k \mathscr{T}_h(\Omega_k)$ by $\mathscr{T}_h$. The mesh parameter h_k is the diameter of the largest element in $\mathscr{T}_h(\Omega_k)$. Let Γ_{kl} denote the open edge that is common to Ω_k and Ω_l. Denote by Γ the set of all interfaces between the subdomains, i. e., $\Gamma = \cup\, \partial\Omega_k \setminus \partial\Omega$. Each edge inherits two triangulations made of segments that are edges of elements of the triangulations of Ω_k and Ω_l respectively. In this way each Γ_{kl} is provided with two independent and different one dimensional meshes, which are denoted by $\mathscr{T}_h^k(\Gamma_{kl})$ and $\mathscr{T}_h^l(\Gamma_{kl})$ respectively.

For each triangulation $\mathscr{T}_h(\Omega_k)$, the rotated Q_1 element space is defined by

$$X_h(\Omega_k) = \{v \in L^2(\Omega_k) \mid \quad v|_E = a_E^1 + a_E^2 x + a_E^3 y + a_E^4(x^2 - y^2), a_E^i \in \mathscr{R}, \int_{\partial E \cap \partial\Omega} v|_{\partial\Omega} \mathrm{d}s = 0,$$

$$\forall E \in \mathscr{T}_h(\Omega_k); \text{for } E_1, E_2 \in \mathscr{T}_h(\Omega_k), \text{ if } \partial E_1 \cap \partial E_2 = e, \text{ then} \int_e v|_{\partial E_1} \mathrm{d}s = \int_e v|_{\partial E_2} \mathrm{d}s\}$$

with the so called broken norm and the broken seminorm

$$\|v\|_{H_h^1(\Omega_k)} = \Big(\sum_{E \in \mathscr{T}_h(\Omega_k)} \|v\|_{H^1(E)}^2\Big)^{\frac{1}{2}}, \qquad |v|_{H_h^1(\Omega_k)} = \Big(\sum_{E \in \mathscr{T}_h(\Omega_k)} \|v\|_{H^1(E)}^2\Big)^{\frac{1}{2}}.$$

Next the global space $X_h(\Omega)$ is define as follow:

$$X_h(\Omega) = \prod_{k=1}^{N} X_h(\Omega_k),$$

with the following norm and seminorm:

$$\| v \|_{1,h} = \Big(\sum_{k=1}^{N} \|v\|^2_{H^1_h(\Omega_k)} \Big)^{\frac{1}{2}}, \qquad | v |_{1,h} = \Big(\sum_{k=1}^{N} | v |^2_{H^1_h(\Omega_k)} \Big)^{\frac{1}{2}}.$$

Define one of the sides of Γ_{kl} as mortar denoted by $\gamma_{m(k)}$ and the other as nonmortar denoted by $\delta_{m(l)}$. Assume that the mortar for $\gamma_{m(k)} = \delta_{m(l)} = \Gamma_{kl}$ is chosen by the condition $h_k \leqslant h_l$, i. e. , the fine side is chosen as mortar. Based on this assumption, the two elements of the slave triangulation $\mathscr{T}^l_h(\delta_{m(l)})$ that touch the ends of $\delta_{m(l)}$ are longer than the respective elements of the mortar triangulation $\mathscr{T}^k_h(\gamma_{m(k)})$. Define an auxiliary test space $M^{h_l}(\delta_{m(l)})$ to be a subspace of the space $L^2(\Gamma_{kl})$ such that its functions are piecewise constants on $\mathscr{T}^l_h(\delta_{m(l)})$. The dimension of $M^{h_l}(\delta_{m(l)})$ is equal to the number of elements on the $\delta_{m(l)}$. For each nonmortar $\delta_{m(l)} = \Gamma_{kl}$, we define an L^2-orthogonal projection $Q_m: L^2(\Gamma_{kl}) \to M^{h_l}(\delta_{m(l)})$ by

$$(Q_m v, w)_{L^2(\delta_{m(l)})} = (v, w)_{L^2(\delta_{m(l)})}, \ \forall w \in M^{h_l}(\delta_{m(l)}). \tag{1.4}$$

Now we define rotated Q_1 mortar element space

$$V_h = \{ v \in X_h(\Omega) \mid Q_m v_l = Q_m v_k, \forall \delta_{m(l)} = \gamma_{m(k)} \in \Gamma \},$$

where $v_k = v|_{\gamma_{m(k)}}$ and $v_l = v|_{\delta_{m(l)}}$. The condition of the equality of the L^2-orthogonal projection of traces onto the test space for each interface is called the mortar condition.

The rotated Q_1 mortar element approximation of problem (1.2) is: find $u_h \in V_h$, such that

$$a_h(u_h, v_h) = (f, v_h), \qquad \forall v_h \in V_h, \tag{1.5}$$

where

$$a_h(u_h, v_h) = \sum_{k=1}^{N} a_{h,k}(u_h, v_h), \qquad a_{h,k}(u_h, v_h) = \sum_{E \in \mathscr{T}_h(\Omega_k)} \int_E \nabla u_h \cdot \nabla v_h \mathrm{d}x.$$

Define operator $A_h: V_h \to V_h$ as follows:

$$(A_h u, v) = a_h(u, v) \qquad \forall u, v \in V_h,$$

then (1.5) can be represented as:

$$A_h u_h = f_h, \tag{1.6}$$

where $f_h \in V_h$, $(f_h, v) = (f, v)$, $\forall v \in V_h$.

2 Multigrid Algorithm

In this section an effective V-cycle multigrid algorithm is presented for the mortar-type rotated Q_1 element.

Let $\mathscr{T}_1$ be the coarsest triangulation of Ω with mesh size h_1, which is made of rectangles. We refine the triangulation $\mathscr{T}_{l-1}$ to produce $\mathscr{T}_l$ by joining the opposite mid-points of the edge of the rectangles in $\mathscr{T}_{l-1}$, with mesh size $h_l, l = 2, 3, \cdots, L-1$. Obviously we have $h_l = h_{l-1}/2 \ (2 \leqslant l \leqslant L-1)$. We use the conforming bilinear element spaces as the coarse-grid correction spaces on all coarse levels $l = 1, \cdots, L-1$. In order to construct a multigrid algorithm for (1.6), we define the conforming bilinear finite element spaces $S_l \subset H^1_0(\Omega)$ on the grid $\mathscr{T}_l$, $l < L$. It is obvious that $S_1 \subset S_2 \subset \cdots \subset S_{L-1} \not\subset V_h$.

Because $S_{L-1} \not\subset V_h$ we must define a suitable intergrid transfer operator $I_h: S_{L-1} \to V_h$. On the last level L, for each triangulation $\mathscr{T}_h(\Omega_i)$, we define S^i_L by the conforming bilinear finite spaces. Let $S_L = \prod_{i=1}^{N} S^i_L$. We take $\mathscr{T}_L = \mathscr{T}_h$ and assume $h_L = h$.

Let $G_h: S_{L-1} \to S_L$ be the usual nodal value interpolation operator (see [6. Lemma 2.1]). The operator G_h has the following property.

Lemma 2.1 (i) $\|G_h v\|_1 \leqslant C \|v\|_1$, $\forall v \in S_{L-1}$;

(ii) $\|v - G_h v\|_0 \leqslant Ch \|v\|_1$, $\forall v \in S_{L-1}$.

Proof Please refer to [6].

Let the operator $F_h^i: S_L^i \to X_h(\Omega_i)$ be the usual nodal value interpolation operator, i. e. ,

$$\int_e F_h^i v \mathrm{d}s = \int_e v \mathrm{d}s, \quad \forall v \in S_L^i,$$

where e is an edge of $K \in \mathscr{T}_h(\Omega_i)$.

Based on the operator F_h^i, we define the operator $F_h: S_L \to X_h(\Omega)$ as follows:

$$F_h(v) = (F_h^1 v, F_h^2 v, \cdots, F_h^N v), \forall v \in S_L.$$

According to the estimate of the interpolation operator and the inverse inequality, we can obtain the following lemma.

Lemma 2.2 $\|v - F_h v\|_0 \leqslant Ch|v|_1, |F_h v|_{1,h} \leqslant C|v|_1, \forall v \in S_L$.

It is necessary to define the operator $J_h: J_h = F_h \circ G_h: S_{L-1} \to X_h(\Omega)$.

Next define the operator $E_h: X_h(\Omega) \to X_h(\Omega)$ by

$$\int_e E_h(v) \mathrm{d}s = \begin{cases} \int_e Q_m(v|_{\gamma_{m(k)}} - v|_{\delta_{m(l)}}) \mathrm{d}s, & e \in \delta_{h,m(l)}, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\forall v \in X_h(\Omega)$, let

$$v^* = v + \sum_{\delta_{m(l)} \in \Gamma} E_h(v),$$

we can check that $v^* \in V_h$. In fact, for any $\omega \in M^{h_l}(\delta_{m(l)})$, we have

$$\begin{aligned} \int_{\delta_{m(l)}} v^*|_{\delta_{m(l)}} \omega \mathrm{d}s &= \int_{\delta_{m(l)}} v|_{\delta_{m(l)}} \omega \mathrm{d}s + \int_{\delta_{m(l)}} (E_h(v))|_{\delta_{m(l)}} \omega \mathrm{d}s = \int_{\delta_{m(l)}} v|_{\delta_{m(l)}} \omega \mathrm{d}s + \int_{\delta_{m(l)}} Q_m(v|_{\gamma_{m(k)}} - v|_{\delta_{m(l)}}) \omega \mathrm{d}s \\ &= \int_{\delta_{m(l)}} v|_{\delta_{m(l)}} \omega \mathrm{d}s + \int_{\delta_{m(l)}} (v|_{\gamma_{m(k)}} - v|_{\delta_{m(l)}}) \omega \mathrm{d}s = \int_{\delta_{m(l)}} v|_{\gamma_{m(k)}} \omega \mathrm{d}s \\ &= \int_{\delta_{m(l)}} v^*|_{\gamma_{m(k)}} \omega \mathrm{d}s. \end{aligned}$$

After the above preparation, we can define an intergrid transfer operator $I_h: S_{L-1} \to V_h$ as follows:

$$I_h v = J_h v + \sum_{\delta_{m(j)} \in \Gamma} E_h(J_h v), \quad \forall v \in S_{L-1}.$$

Define the operators $A_{S_l}: S_l \to S_l$ and $Q_{S_l}: S_{L-1} \to S_l, l = 1, \cdots, L-1$ by

$$(A_{S_l} u, v) = a(u,v), \ \forall u, v \in S_l, (Q_{S_l} u, v) = (u,v), \ u \in S_{L-1}, \forall v \in S_l.$$

Define the projection operators $Q_{L-1}, P_{L-1}: V_h \to S_{L-1}$ by

$$(Q_{L-1} u, v) = (u, I_h v), \ u \in V_h, \forall v \in S_{L-1}, a(P_{L-1} u, v) = a_h(u, I_h v), \ u \in V_h, \forall v \in S_{L-1}. \quad (2.1)$$

Using the similar technique in [7], we can construct certain smoothing operator $R_h: V_h \to V_h$ such that

$$C \frac{1}{\lambda_h}(v,v) \leqslant (R_h v, v), \ \forall v \in V_h, \quad (2.2)$$

$$a_h(R_h A_h v, v) \leqslant \theta a_h(v,v), \ \forall v \in V_h, \quad (2.3)$$

where λ_h is the largest eigenvalue of A_h and $\theta \in (0,2)$.

Similarly, on the coarse spaces $S_l (l = 1, \cdots, L-1)$, we can construct the smoothing operators $R_{S_l}: S_l \to S_l$ such that

(1) $C \frac{1}{\lambda_l}(v,v) \leqslant (R_{S_l} v, v), \forall v \in S_l;$ (2.4)

(2) $a(R_{S_l} A_{S_l} v, v) \leqslant \theta a(v,v), \forall v \in S_l,$ (2.5)

where λ_l is the largest eigenvalue of A_{S_l} and $\theta \in (0,2)$.

Now we introduce our V-cycle multigrid algorithm as follows.

Given $g \in V_h$ define $B_L g$ by

(1) Set $x_0 = 0$, $x^n = x^{n-1} + R_h(g - A_h x^{n-1}), n = 1, \cdots, m$;

(2) Define $x^{m+1} = x^m + I_h q$, where $q = M_{L-1} Q_{L-1}(g - A_h x^m)$;

(3) Set $y^0 = x^{m+1}$ and $y^n = y^{n-1} + R_h(g - A_h y^{n-1}), n = 1, \cdots, m$;

(4) Define $B_L g = y^m$.

The operator M_{L-1} in the above algorithm is defined as follows.

Let $M_1 = A_{S_1}^{-1}$, for a given $g_l \in S_l, M_l(l=2,\cdots,L-1)$ is defined by

(i) Set $x_1 = R_l g_l$;

(ii) Define $M_l g_l = x_1 + p$, where $p \in S_{l-1}$ is given by $p = M_{l-1}Q_{S_{l-1}}(g_l - A_{S_l}x_1)$.

We can see that on each coarse grid space S_l only one smoothing step is need. It is easy to check that

$$I - B_L A_h = K_h^m(I - I_h P_{L-1} + I_h(I - M_{L-1}A_{S_{L-1}})P_{L-1})K_h^m. \tag{2.6}$$

3 Convergence Analysis

In this section, the convergence analysis of the V-cycle multigrid method is given. Let $t_{L-1}: C(\bar{\Omega}) \to S_{L-1}$ be the bilinear interpolation operator. First, we prove some Lemmas.

Lemma 3.1 For the operators I_h, t_{L-1}, we have

(i) $\|I_h v - v\|_0 \leqslant Ch|v|_1$, $|I_h v|_{1,h} \leqslant C|v|_1$, $\forall v \in S_{L-1}$;

(ii) $\|t_{L-1}\xi - I_h t_{L-1}\xi\|_1 \leqslant Ch|\xi|_2$, $\forall \xi \in H^2(\Omega) \cap H_0^1(\Omega)$.

Proof By the definition of I_h, we get

$$\|I_h v - v\|_0 = \Big\|J_h v - v + \sum_{\delta_{m(j)} \in \Gamma} E_h(J_h v)\Big\|_0 \leqslant \|J_h v - v\|_0 + \Big\|\sum_{\delta_{m(j)} \in \Gamma} E_h(J_h v)\Big\|_0. \tag{3.1}$$

Using the definition of J_h and Lemmas 2.1, 2.2, we obtain

$$\|J_h v - v\|_0 = \|F_h G_h v - G_h v + G_h v - v\|_0 \leqslant C_1 h|G_h v|_1 + C_2 h|v|_1 \leqslant Ch|v|_1. \tag{3.2}$$

Using norm equivalence we derive

$$\begin{aligned}
\|E_h(J_h v)\|_0^2 &\leqslant C\sum_{e_i \in \delta_{m(j)}}\Big(\int_{e_i} E_h(J_h v)\,ds\Big)^2 = C\sum_{e_i \in \delta_{m(j)}}\Big(\int_{e_i} Q_m(J_h v)|_{\gamma_{m(i)}} - (J_h v)|_{\delta_{m(j)}}\,ds\Big)^2 \\
&\leqslant Ch\|Q_m((J_h v)|_{\gamma_{m(i)}} - (J_h v)|_{\delta_{m(j)}})\|_{0,\gamma_m}^2 \leqslant Ch\|(J_h v)|_{\gamma_{m(i)}} - (J_h v)|_{\delta_{m(j)}}\|_{0,\gamma_m}^2 \\
&\leqslant Ch(\|(J_h v)|_{\gamma_{m(i)}} - v|_{\delta_{m(j)}}\|_{0,\gamma_m}^2 + \|v|_{\delta_{m(j)}} - (J_h v)|_{\delta_{m(j)}}\|_{0,\gamma_m}^2) := Ch(K_1 + K_2).
\end{aligned} \tag{3.3}$$

By trace theorem and (3.2), it follows that

$$K_2 \leqslant Ch\|v\|_{1,j}^2. \tag{3.4}$$

So we only need to estimate K_1. Owing to $v \in H_0^1(\Omega)$, then

$$\|(J_h v)|_{\gamma_{m(i)}} - v|_{\delta_{m(j)}}\|_{0,\gamma_m}^2 = \|(J_h v)|_{\gamma_{m(i)}} - v|_{\gamma_{m(i)}}\|_{0,\gamma_m}^2 \leqslant Ch\|v\|_{1,i}^2, \tag{3.5}$$

(3.3) ~ (3.5) gives

$$\|E_h(J_h v)\|_0^2 \leqslant Ch^2(\|v\|_{1,i}^2 + \|v\|_{1,j}^2), \tag{3.6}$$

then

$$\Big\|\sum_{\delta_{m(j)} \in \Gamma} E_h(J_h v)\Big\|_0 \leqslant Ch|v|_1.$$

By the inverse inequality, we can see that the first inequality of Lemma 3.1 is valid.

Arguing as lemma 4.3 in [8], we can prove the second inequality.

By the definition of P_{L-1} and Lemma 3.1, we can deduce the following lemma.

Lemma 3.2 $$|P_{L-1}v|_1 \leqslant C|v|_{1,h}, \forall v \in V_h. \tag{3.7}$$

Proof $\forall v \in V_h$, we have

$$|P_{L-1}v|_1^2 = a(P_{L-1}v, P_{L-1}v) = a_h(v, I_h P_{L-1}v) \leqslant |v|_{1,h}|I_h P_{L-1}v|_{1,h} \leqslant C|v|_{1,h}|P_{L-1}v|_1,$$

and then we can obtain

$$|P_{L-1}v|_1 \leqslant C|v|_{1,h}.$$

By (2.2), (2.3) and a similar argument of Theorems 3.6 and 5.1 in [7], we have

Lemma 3.3 For any $v \in V_h$, it holds

$$c\frac{\|A_hK_h^mv\|_0^2}{\lambda_h}\leqslant a_h((I-K_h^2)K_h^mv,K_h^mv)\leqslant C\frac{1}{m}a_h(v,v),$$

where $K_h=I-R_hA_h$, and m is the number of smoothing steps.

By a similar argument in [9], we can prove

Lemma 3.4 For the operator $I-M_{L-1}A_{S_{L-1}}$, we have

$$|a_{L-1}((I-M_{L-1}A_{S_{L-1}})v,v)|\leqslant\delta_0a_{L-1}(v,v),\forall v\in S_{L-1},$$

where the constant $\delta_0\in(0,1)$ is independent of the mesh h and the level L.

Let $\{\lambda_j\}_{j=1}^{N_h}$ and $\{\varphi_j\}_{j=1}^{N_h}$ be the eigenvalues and corresponding normalized eigenfunctions of A_h, i. e. ,

$$A_h\varphi_j=\lambda_j\varphi_j,\ j=1,\cdots,N_h,$$

and

$$(\varphi_i,\varphi_j)=\delta_{ij},$$

where δ_{ij} is Kronecker symbol.

For any $v\in V_h$, we write $v=\sum_{j=1}^{N_h}c_j\varphi_j$. Let $A_h^sv=\sum_{j=1}^{N_h}\lambda_j^sc_j\varphi_j$, then we define the following discrete norm on the space V_h:

$$\|v\|_{s,h}:=(A_h^sv,v)^{\frac{1}{2}}.$$

It is easy to see that

$$\|v\|_{1,h}=a_h(v,v)^{\frac{1}{2}},\|v\|_{0,h}=\|v\|_0.\qquad(3.8)$$

Lemma 3.5 For the operator P_{L-1} defined by (2.1) we have

$$\|v-P_{L-1}v\|_0\leqslant Ch\|v\|_{1,h},\ \forall v\in V_h.$$

Proof Consider the following auxiliary problem

$$\begin{cases}-\Delta\xi=v-P_{L-1}v, & \text{in }\Omega,\\ \xi=0, & \text{on }\partial\Omega.\end{cases}\qquad(3.9)$$

By elliptic regularity property (1.3) we have

$$\|\xi\|_2\leqslant C\|v-P_{L-1}v\|_0.\qquad(3.10)$$

On the other hand,

$$\begin{aligned}\|v-P_{L-1}v\|_0^2&=(-\Delta\xi,v-P_{L-1}v)=a_h(\xi-t_{L-1}\xi,v)+a_h(t_{L-1}\xi-I_ht_{L-1}\xi,v)\\&+a(t_{L-1}\xi-\xi,P_{L-1}v)+\sum_{K\in\mathcal{T}_h}\oint_{\partial K}\sum_{i=1}^{2}\frac{\partial\xi}{\partial x_i}v\cos(n,x_i)\mathrm{d}s:=I_1+I_2+I_3+I_4\end{aligned}$$

An application of Lemma 3.1, interpolation estimate [10], and (3.7), (3.8), (3.10) yield

$$|I_i|\leqslant Ch\|v-P_{L-1}v\|_0\|v\|_{1,h}.$$

So we get Lemma 3.5.

Lemma 3.6 For all $v\in V_h$, we have

$$\|v-I_hP_{L-1}v\|_{1,h}\leqslant Ch\|v\|_{2,h}.$$

Proof By Lemma 3.1 and Lemma 3.5, we get

$$\|v-I_hP_{L-1}v\|_0\leqslant\|v-P_{L-1}v\|_0+\|(I-I_h)P_{L-1}v\|_0\leqslant Ch\|v\|_{1,h}+Ch\|P_{L-1}v\|_1\leqslant Ch\|v\|_{1,h}.$$

On the other hand,

$$\begin{aligned}\|v-I_hP_{L-1}v\|_{1,h}&=\sup_{w\in V_h,\|w\|_{1,h}=1}a_h(v-I_hP_{L-1}v,w)=\sup_{w\in V_h,\|w\|_{1,h}=1}a_h(v,w-I_hP_{L-1}w)\\&\leqslant\sup_{w\in V_h,\|w\|_{1,h}=1}\|v\|_{2,h}\|w-I_hP_{L-1}w\|_0\leqslant Ch\|v\|_{2,h}.\end{aligned}$$

The proof is completed.

Finally, we show the main result of this paper.

Theorem 3.1 For any $\delta\in(\delta_0,1)$, if the smoothing number on the last level is large enough, then

$$|a_h((I-B_LA_h)v,v)|\leqslant\delta\,a_h(v,v),\forall v\in V_h.$$

where the constant $\delta_0 \in (0,1)$ is independent of the mesh h and the level L.

Proof Let $\tilde{v} = K_h^m v$, by (2.6) and Lemma 3.4, we get

$$\begin{aligned} | a_h((I - B_L A_h)v,v) | &\leqslant | a_h((I - I_h P_{L-1})\tilde{v},\tilde{v}) | + | a((I - M_{L-1}A_{S_{L-1}})P_{L-1}\tilde{v},P_{L-1}\tilde{v}) | \\ &\leqslant | a_h((I - I_h P_{L-1})\tilde{v},\tilde{v}) | + \delta_0 | a_h(I_h P_{L-1}\tilde{v},\tilde{v}) | \\ &\leqslant (1 + \delta_0) | a_h((I - I_h P_{L-1})\tilde{v},\tilde{v}) | + \delta_0 | a_h(\tilde{v},\tilde{v}) |. \end{aligned}$$

Lemma 3.3 and Lemma 3.6 imply

$$\begin{aligned} | a_h((I - I_h P_{L-1})\tilde{v},\tilde{v}) | &\leqslant Ch\|\tilde{v}\|_{2,h}\|\tilde{v}\|_{1,h} = C(\frac{\|A_h\tilde{v}\|_0^2}{\lambda_h})^{\frac{1}{2}}\|\tilde{v}\|_{1,h} \\ &\leqslant C(a_h((I - K_h^2)K_h^m v,K_h^m v))^{\frac{1}{2}}\|\tilde{v}\|_{1,h} \leqslant C\frac{1}{\sqrt{m}}a_h(v,v). \end{aligned}$$

Then, if m is large enough, we have

$$| a_h((I - B_l A_h)v,v) | \leqslant (\frac{C(1 + \delta_0)}{\sqrt{m}} + \delta_0)a_h(v,v) \leqslant \delta a_h(v,v).$$

Remark 3.1 The uniform convergence rate is proven, which is independent of mesh size and mesh level.

4 Numerical Experiments

In this section we present the results of numerical examples to illustrate the theory developed in the earlier sections. These numerical examples deal with the poisson equation on the unit square. For the problem (1.1), let $\Omega = [0,1] \times [0,1]$, $\Omega_1 = [0,1] \times [0,0.5]$ is mortar sub-domain, $\Omega_2 = [0,1] \times [0.5,1]$ is non-mortar sub-domain. In this test we assume $f = 2y(1-y) + 2x(1-x)$. Obviously, we have the exact solution as follows:

$$u = xy(1-x)(1-y).$$

Table 1 shows the number of iterations required to achieve the error reduction 10^{-3}, where the starting vector for the iteration is zero. In the following table, h_i is the mesh size of level L in $\mathscr{T}_h(\Omega_i)$. CG is iteration steps of conjugate gradients method. $iter_1$ and $iter_2$ are the numbers of iteration steps for the V-cycle multigrid at level L, with damp-Jacobi, SOR smoothing operators, respectively.

Table 1 Iteration numbers for the V-cycle multigrid

L	h_1^{-1}	h_2^{-1}	CG	$iter_1$	$iter_2$
2	4	2	7	6	5
3	8	4	12	7	6
4	16	8	24	7	6
5	32	16	42	7	6
6	64	32	80	8	6

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