Some Improvement to One Normal Criteria

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Abstract: This paper obtain some normality of families of meromorphic functions allowing the functions to have zeroes but to give additional conditions, which generalize and improve some results of Lin Weichuan and Xu Yan's work. We obtain: if $f(z)f''(z)-a(f'(z))^2\neq 0$ $(a\neq 1, 1\pm\frac{1}{n})$ and $f(z)f''(z)-a(f'(z))^2=0$ implies $f'(z)=0$ then f has forms: $f(z)=\exp(\alpha z+\beta)$ or $f(z)=(\alpha z+\beta)^{\pm n}(\alpha\neq 0)$. And $\mathscr{F}$ be a family of meromorphic functions in domain D, if each $f\in\mathscr{F}$ has only zeroes of multiplicity at least $k\geqslant 3$ and satisfies: $f^{(k)}(z)=a(z)$ $(a(z)\neq 0)$, implies $|f(z)|\geqslant A$ and $f(z)=0$ implies $0<|f^{(k)}(z)|\leqslant K$. Then $\mathscr{F}$ is normal in D. Here A, K are positive constants.

Key words: transcendental meromorphic function, normal family, reside

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一类正规定则的改进

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[摘要]　采用改函数不取零值为可取零值加限制的方法改进了林伟川, 徐焱等人的结果. 得到: 若$f(z)f''(z)-a(f'(z))^2\neq 0(a\neq 1,1\pm\frac{1}{n})$及$f(z)f''(z)-a(f'(z))^2=0$蕴含$f'(z)=0$, 则$f$有形式$f(z)=\exp(\alpha z+\beta)$或$f(z)=(\alpha z+\beta)^{\pm n}(\alpha\neq 0)$. $\mathscr{F}$是区域D上的亚纯族, 若每个$f\in\mathscr{F}$的零点重数至少是$k(k\geqslant 3)$并满足: $f^{(k)}(z)=a(z)(a(z)\neq 0)$蕴含$|f(z)|\geqslant A$和$f(z)=0$蕴含$0<|f^{(k)}(z)|\leqslant K$. 则$\mathscr{F}$在区域$D$上正规. 其中$A, K$为正常数.

[关键词]　超越亚纯函数, 正规族, 留数

0　Introduction

In 1995, Bergweiler[1] obtained the following result:

Theorem A　Let f be a transcendental meromorphic function of finite order, and $a\neq 1, 1+\frac{1}{n}$. if $f(z)f''(z)-a(f'(z))^2\neq 0$, then $f(z)=\exp(\alpha z+\beta)$, where $\alpha, \beta\in\mathbf{C}$. Recently, Lin Wei Chuan [2] and Yi Hong Xun excluded the additional order restriction as follows:

Theorem B　Let f be a meromorphic function in the complex plane and let $a\neq 1, 1\pm\frac{1}{n}$, where $n\in\mathbf{N}$, and if $f(z)f''(z)-a(f'(z))^2\neq 0$, then f has one of the following forms:

i) $f(z)=\exp(\alpha z+\beta)$

ii) $f(z)=(\alpha z+\beta)$

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iii) $f=\frac{1}{(\alpha z+\beta)^{n}}$

where $n\in\mathbf{N},\alpha\neq 0,\ \beta\in\mathbf{C}$.

More over they obtain a normal family analogue of Theorem B:

Theorem C Let $\mathscr{F}$ be a family of meromorphic functions in the unit disk Δ and $a\neq 1, 1\pm\frac{1}{n}$, where $n\in\mathbf{N}$, if for every $f\in\mathscr{F}$, and $f(z)f''(z)-a(f'(z))^2\neq 0$ in Δ, then $\left\{\frac{f'}{f}: f\in\mathscr{F}\right\}$ is normal in Δ.

we note that the result ii) and iii) in Theorem B may be combined by $(\alpha z+\beta)^{\pm n}$ if we weaken the condition $f(z)f''(z)-a(f'(z))^2\neq 0$ and so we have Theorem 2.

To obtain Theorem 2, we need the following theorem which generalizes Theorem C accordingly:

Theorem 1 Let $\mathscr{F}$ be a family of meromorphic functions in the unit disk Δ and $a\neq 1, 1\pm\frac{1}{n}$, where $n\in\mathbf{N}$, if for every $f\in\mathscr{F}$, and $f(z)f''(z)-a(f'(z))^2\not\equiv 0$ and $f(z)f''(z)-a(f'(z))^2=0$ implies $f'(z)=0$, then $\left\{\frac{f'}{f}: f\in\mathscr{F}\right\}$ is normal in Δ.

And based on this Theorem, the Theorem 2 can be obtained as follows:

Theorem 2 Let f be a meromorphic function in the complex plane and let $a\neq 1, 1\pm\frac{1}{n}$, where $n\in\mathbf{N}$, $f(z)f''(z)-a(f'(z))^2\not\equiv 0$ and $f(z)f''(z)-a(f'(z))^2=0$ implies $f'(z)=0$, then f has one of the following forms:

1) $f(z)=\exp(\alpha z+\beta)$

2) $f(z)=(\alpha z+\beta)^{\pm n}$, where $n\in\mathbf{N},\alpha\neq 0,\ \beta\in\mathbf{C}$.

In 1979, Gu[3] proved a conjecture of Hayman as follows:

Theorem D Let $\mathscr{F}$ be a family of meromorphic functions in domain D, k be a positive integer, if for every $f\in\mathscr{F}$, $f\neq 0$, and $f^{(k)}\neq 1$, then $\mathscr{F}$ is normal in D.

Recently, Xu[4] improved and generalized it and obtained:

Theorem E Let k be positive integer such that $k\geqslant 3$ and K be positive number $\mathscr{F}$ be a family of meromorphic functions in domain D and $a(z)$ be non-vanishing analytic function in D suppose that for every $f\in\mathscr{F}$, and f has only zeroes of multiplicity at least k and satisfies following condition:

a) $f^{(k)}(z)\neq a(z)$. b) $f(z)=0$ implies $0<|f^{(k)}(z)|\leqslant K$. then $\mathscr{F}$ is normal in D.

The condition $f^{(k)}(z)\neq a(z)$ also may be generalized by allowing $f^{(k)}(z)=a(z)$ at some dots but restrict the values of f at these dots. And we have:

Theorem 3 Let k be positive integer such that $k\geqslant 3$ and A, K be positive number, $\mathscr{F}$ be a family of meromorphic functions in domain D and $a(z)$ be non-vanishing analytic function in D suppose that for every $f\in\mathscr{F}$, f has only zeroes of multiplicity at least k and satisfies following condition:

a) $f^{(k)}(z)=a(z)$ implies $|f(z)|\geqslant A$.

b) $f(z)=0$ implies $0<|f^{(k)}(z)|\leqslant K$.

then $\mathscr{F}$ is normal in D.

1 Some Lemmas

Lemma 1[5] Let A, B and ε be positive numbers. Let $\mathscr{F}=\{f\}$ be a family of meromorphic functions in domain D which satisify the following condition:

1) $f'(z)\neq 1$

2) if $f(z)=0$, then $o<|f'(z)|\leqslant B$

3) if Δ is a disk in D and if f has $m\geqslant 2$ zeros $z_1, z_2\cdots, z_m\in\Delta$, then $|\sum_{j=1}^{m}f'(z_j)^{-1}-1|\geqslant\varepsilon$.

Then $\mathscr{F}$ is normal in D

Lemma 2[6] Let $f(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0+\frac{p(z)}{q(z)}$, where $a_0,a_1\cdots,a_n$ are constants with $a_n\neq 0$. $p(z),q(z)$ are two co-prime polynomials with $\deg p(z)<\deg q(z)$, let k be a positive integer. if $f^{(k)}\neq 1$, then $f(z)=\frac{1}{k!}z^k+\cdots+a_0+\frac{1}{(az+b)^m}$, where $a(\neq 0)$, b are constants, m is a positive integer.

Lemma 3[2] Let f be a transcental meromorphic function with finite order. all of those zeros are of multiplicity (at least) k, and let A be positive real number. if $|f^{(k)}(z)|\leqslant A$ whenever $f(z)=0$, then for each l, $1\leqslant l\leqslant k$, $f^{(l)}(z)$ assumes any finite non-zero value infinitely often.

Lemma 4[2] Let $\{a_m\}$ be an integer sequence and $a\neq 1,1\pm\frac{1}{n}$, where $n\in\mathbf{N}$, then exists a positive number ε such that for each a_m, $|a_m(a-1)-1|\geqslant\varepsilon$.

Lemma 5[7] Let k be a positive integer and let $\mathscr{F}$ be a family of functions meromorphic in a domain D, such that each function $f\in\mathscr{F}$ has only zeros of multiplicity at least k, and suppose that there exists $A\geqslant 1$ such that $|f^{(k)}(z)|\leqslant A$ whenever $f(z)=0$. If $\mathscr{F}$ is not normal at $z_0\in D$, then, for each $0\leqslant\alpha\leqslant k$, there exist a sequence of points $z_n\in D$, $z_n\to z_0$, a sequence of positive numbers $\rho_n\to 0$, and a sequence of functions $f_n\in\mathscr{F}$ such that $g_n(\zeta)=\frac{f_n(z_n+\rho_n\zeta)}{\rho_n^{\alpha}}\to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbf{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta)\leqslant g^{\#}(0)=kA+1$. Moreover, g has order at most 2.

Lemma 6[4] Let f be meromorphic in $\mathbf{C}$ and of finite order, let $k\geqslant 3$ be a positive integer and K be positive number, suppose that f has only zeros of multiplicity at least k, $|f^{(k)}(z)|<K$ whenever $f(z)=0$, and $f^{(k)}(z)\neq 1$ Then

1) $f(z)=\alpha(z-\beta)^k,\alpha,\beta\in\mathbf{C},\alpha k!\neq 1$

2) $f(z)=\frac{1}{k!}\frac{(z-c_1)^{k+1}}{z-c}$, where c,c_1 are two distinct complex numbers.

2 Proof of Theorems

Proof of Theorem 1 Define $h(z):=\frac{f(z)}{1-a}\cdot\frac{1}{f'(z)}$, $f\in\mathscr{F}$

Then we only need to prove the family $\mathscr{H}:=\{h\}_{f\in\mathscr{F}}$ is normal in Δ.

From the definition of h, we have:

$$h'(z)=\frac{af'(z)^2-f(z)f''(z)}{(1-a)f'(z)^2}+1.$$

At first, if $h(\zeta)=0$, then $f(\zeta)=0$ or ∞. We consider two cases:

case 1.1 If ζ is a zero of f with multiplicity n, then $h'(\zeta)=\frac{1}{1-a}\frac{1}{n}$.

case 1.2 If ζ is a pole of f with multiplicity m, then $h'(\zeta)=\frac{1}{a-1}\frac{1}{m}$.

Hence, $0<|h'(\zeta)|\leqslant\left|\frac{1}{a-1}\right|$ when $h(\zeta)=0$.

Secondly we claim $h'(\zeta)\neq 1$.

If $\exists\zeta$ such that $h'(\zeta)=1$, then $h'(\zeta)-1=\frac{af'(\zeta)^2-f(\zeta)f''(\zeta)}{(1-a)f'(\zeta)^2}=0$ from case1.2, we have $f(\zeta)\neq\infty$, so $af'(\zeta)^2-f(\zeta)f''(\zeta)=0$, it implies $f'(\zeta)=0$ by condition and thus $f(\zeta)f''(\zeta)=0$.

We consider two cases:

case 2.1 If $f(\zeta)=0$, then from case 1.1, we have $f'(\zeta)=\frac{1}{1-a}\frac{1}{n}=1$, thus $a=1-\frac{1}{n}$, this is a contradiction.

case 2.2 If $f(\zeta)\neq 0$, then $f''(\zeta)=0$ and $h(\zeta)=\infty$ thus $h'(\zeta)=\infty\neq 1$, this is a contradiction.

Thirdly, suppose that $\Delta_1\subset\Delta$ is a disk and h has zeroes $z_1, z_2\cdots, z_m\in\Delta_1$. As above we have: $\left|\sum_{j=1}^{m}h'(z_j)^{-1}-1\right|=|(1-a)a_m-1|$, here a_m is an integer number. By lemma 4, there exists a positive number ε such that for each $h\in\mathcal{H}$, $\left|\sum_{j=1}^{m}h'(z_j)^{-1}-1\right|\geqslant\varepsilon$. and $\mathcal{H}$ is normal in Δ by lemma 1.

This complete the proof of Theorem 1.

Proof of Theorem 2 A meromorphic function f is called Yosida function if its spherical derivative $f^{\#}(z)=|f'(z)|/(1+|f(z)|^2)$ is uniformly bounded on $\mathbf{C}$.

As above we define $h(z):=\frac{f(z)}{1-a}\frac{1}{f'(z)}$.

We claim that h is a Yosida function.

If not, there exists a sequence z_n such that $h^{\#}(z_n)\to\infty$, write $h_n(z)=h(z_n+z)$ then $\{h_n\}$ is not normal at $z_0=0$ by marty criterion. However theorem 1 implies $\{h_n\}$ is normal at z_0, which is a contradiction.

Thus h is a Yosida function, and $\lambda(h)\leqslant 2$.

As to the proof of Theorem 1, we have that h is not a transcental function by lemma 3. Thus h is a rational function and $h'(z)\neq 1$. Then h has form $h(z)=a_0z+b$ or $h(z)=z+\beta+\frac{b}{(z+c)^l}$ by lemma 2, where $a_0\neq 1$, β, b, c are constants.

Suppose that $h(z)=z+\beta+\frac{b}{(z+c)^l}$, then $\frac{1}{h(z)}=\frac{1}{z}+O(\frac{1}{z^2})$, as $z\to\infty$, and so $\mathrm{Res}(\frac{1}{h},\infty)=-1$.

let $z_1, z_2\cdots, z_m$ be the zeroes of $h(z)$, here $m=l+1$. We note that $h(z)$ has only simple zeroes and so we have:

$$\left|\sum_{j=1}^{m}h'(z_j)^{-1}\right|=\sum_{z\in h^{-1}(0)}\mathrm{Res}\left(\frac{1}{h},z\right)=-\mathrm{Res}\left(\frac{1}{h},\infty\right)=1.$$

But on other hand $h(z)=\frac{f(z)}{1-a}\frac{1}{f'(z)}$, from the proof of Theorem 1 there exists a positive number ε such that $\left|\sum_{j=1}^{m}h'(z_j)^{-1}-1\right|\geqslant\varepsilon$. This is a contradiction.

Thus $h(z)=a_0z+b$. We consider two cases as follows:

If $a_0=0$ then $\frac{f(z)}{f'(z)}$ is non-zero constant since $f(z)f''(z)-a(f'(z))^2\neq 0$. Hence $f(z)=\exp(\alpha z+\beta)$, with $\alpha\neq 0$.

If $a_0\neq 0$, then $\frac{f(z)}{f'(z)}=\gamma(z+c)$, we note that $z=-c$ is the only zero or pole of f, then γ must be $\pm 1/n$, thus $\frac{f'(z)}{f(z)}=\pm n\frac{1}{z+c}$ and then $f=(\alpha z+\beta)^{\pm n}$ with $\alpha\neq 0$.

This complete the proof of Theorem 2.

Proof of Theorem 3 If $\mathcal{F}$ is not normal at $z_0\in D$, then by lemma 5 take $\alpha=k$ and there exist a sequence of points $z_n\in D$, $z_n\to z_0$, a sequence of positive numbers $\rho_n\to 0$, and a sequence of functions $f_n\in\mathcal{F}$ such that: $g_n(\zeta)=f_n(z_n+\rho_n\zeta)/\rho_n{}^k\to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbf{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta)\leqslant g^{\#}(0)=k(K+1)+1$. Moreover, g has order at most 2.

Let ζ_1 be a zero of $g(\zeta)$ and then by Hurwitz' Theorem there exists a sequence ζ_n, $\zeta_n\to\zeta_1$. such that

$g_n(\zeta_n)=f_n(z_n+\rho_n\zeta_n)/\rho_n^k=0$ for sufficiently large n. Thus $f_n(z_n+\rho_n\zeta_n)=0$, hence $|f_n^{(k)}(z_n+\rho_n\zeta_n)|\leq K$ by condition b). Since $g_n^{(k)}(\zeta_n)=f_n^{(k)}(z_n+\rho_n\zeta_n)\to g^{(k)}(\zeta_1)$, we deduce that $|g^{(k)}(\zeta_1)|\leq K$.

Obviously $a(z_0)\neq 0,\infty$.

We distinguish two cases:

case 3.1 If there exists ζ_0 such that $g^{(k)}(\zeta_0)=a(z_0)$, it is obvious that $g(\zeta_0)\neq\infty$.

First we claim $g^{(k)}(\zeta)-a(z_0)\not\equiv 0$.

if $g^{(k)}(\zeta)-a(z_0)\equiv 0$, and since g has only zeroes with multiplicity at least k, we have

$$g(\zeta)=\frac{a(z_0)}{k!}(z-\alpha_0)^k$$

and $|a(z_0)|=|g^{(k)}(\alpha_0)|\leq K$ from above discussion.

A simple calculation shows that: $g^{\#}(0)\leq k/2$ if $|\alpha_0|\geq 1$ and $g^{\#}(0)\leq|a(z_0)|$ if $|\alpha_0|<1$ both contradicts $g^{\#}(0)=k(K+1)+1$. So $g^{(k)}(\zeta)-a(z_0)\not\equiv 0$.

Near ζ_0, we have

$$g_n^k(\zeta)-a(z_n+\rho_n\zeta)\to g^{(k)}(\zeta)-a(z_0)$$

by Hurwitz' Theorem again there exists a sequence ζ_n, $\zeta_n\to\zeta_0$, such that for sufficiently large n:

$$g_n^{(k)}(\zeta_n)-a(z_n+\rho_n\zeta_n)=0.$$

Hence $f^{(k)}(z_n+\rho_n\zeta_n)-a(z_n+\rho_n\zeta_n)=0$.

Form condition a) we have that $|f_n(z_n+\rho_n\zeta_n)|\geq A$.

Hence $g_n(\zeta_n)=f_n(z_n+\rho_n\zeta_n)/\rho_n^k\to\infty$, and $g(\zeta_0)=\infty$, which is a contradiction.

case 3.2 If $g^{(k)}(\zeta)\neq a(z_0)$, we may assume $a(z_0)=1$, then by lemma 6 g has form:

$$\alpha(z-\beta)^k,\alpha k!\neq 1 \text{ or } \frac{1}{k!}\frac{(z-c_1)^{k+1}}{z-c},\ c\neq c_1.$$

we can exclude the former similarly as in case 3.1. We just consider the latter case. Since g has only zero c_1 with multiplicity $k+1$ this contradicts g_n has only zeroes of multiplicity k.

This complete the proof of Theorem 3.

In Theorem 3, it is easy to find out that the condition a). $f^{(k)}(z)\doteq a(z)$ implies $|f(z)|\geq A$ can be replaced by

$$f^{(k)}(z)+a_1(z)f^{(k-1)}(z)+\cdots+a_k(z)f(z)=a(z) \text{ implies } |f(z)|\geq A.$$

here $a_1(z),\cdots,a_k(z)$ are analytic functions in D.

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