# An Equivalence Relation Characterization of Distributive Lattices

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**Abstract**: A characterization of distributive lattices, Heyting algebras and Boolean algebras was given by means of an equivalence relation defined on them. Furthermore, some interesting properties of Heyting algebras and Boolean algebras were obtained.

**Key words**: equivalence relation, distributive lattice, Heyting algebra, Boolean algebra

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# 分配格的等价关系刻画

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[摘要] 通过在格上定义等价关系,给出了分配格,Heyting 代数,Boolean 代数的一致等价刻画。由此得到了 Heyting 代数与 Boolean 代数分解定理.

[关键词] 等价关系,分配格,Heyting 代数,Boolean 代数

## 0 Introduction

Boolean algebra is an important notion of order algebra and logic, which was studied firstly by Boole in [1]. The definition of Heyting algebra was introduced by Heyting in connection with his formilization of intuitionistic propositional calculus in [2]. It is well known that the class of Heyting algebra contains the class of Boolean algebra, and is contained in the class of distributive lattice, but it is not so obvious because they are presented from different views. In this paper we give a uniform characterization of them by means of an equivalence relation defined on them. Furthermore, we get a decomposition theorem for Heyting algebras and Boolean algebras.

Recall that a lattice $L$ is a Boolean algebra if it is distributive in the sense that,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \text{ for all elements } x,\ y,\ z \in L$$

and every element $x$ has a complement $y$ in the sense that

$$x \wedge y = 0 \text{ and } x \vee y = 1.$$

A Heyting algebra is a lattice $L$ satisfying the following conditions:

for every element $a$, the function $x \mapsto a \wedge x : L \to L$ has an upper adjoint.

By an interval $[p,q]$ we mean the set $\{x \mid p \leqslant x \leqslant q\}$. For a general background on distributive lattice and set theory, we refer to [3] and [4].

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## 1 Main Results

Let $A$ be a lattice. For any element $a \in A$ , we define a two - element relation $R_a$ on $A$ as follows:

$$bR_a b', \text{ if for any } x \in A,\ a \wedge x \leqslant b \Longleftrightarrow a \wedge x \leqslant b' .$$

It is obvious that $R_a$ is an equivalence relation, and each equivalence class $[b]$ of $b$ has the following properties:

(1) $[b]$ is an ordered convex set of $A$;

(2) For any $x \in [b]$, $a \wedge x = a \wedge b$, and $a \wedge b \in [b]$;

(3) $[b] \cap \downarrow a = \{a \wedge b\}$;

(4) $[b]$ is closed under arbitrary meets in $A$ if they exist, moreover $a \wedge b$ is the minimal element in $[b]$;

(5) For each $b$, $b' \in A$, we have $bR_a b'$ iff $a \wedge b = a \wedge b'$.

**Lemma 1** If $A$ is a distributive lattice, then $R_a$ is closed under joins for any $a \in A$, i. e. for any $x_1 \in [k_1]$, $x_2 \in [k_2]$, we have $x_1 \vee x_2 \in [k_1 \vee k_2]$.

**Proof** If $x_1 \in [k_1]$, $x_2 \in [k_2]$, then we have $a \wedge k_1 = a \wedge x_1$, $a \wedge k_2 = a \wedge x_2$, moreover $a \wedge (x_1 \vee x_2) = (a \wedge x_1) \vee (a \wedge x_2) = (a \wedge k_1) \vee (a \wedge k_2) = a \wedge (k_1 \vee k_2)$, i. e. $x_1 \vee x_2 \in [k_1 \vee k_2]$.

**Lemma 2** If $A$ is a Heyting algebra, $b \in A$, then we have:

(1) $bR_a(a \to b)$, moreover, we have $bR_a b'$, for any $b$, $b' \in A$ iff $(a \to b) = (a \to b')$

(2) $(a \to b)$ is the maximal element in $[b]$, moreover, $[b] = [a \wedge b, a \to b]$.

**Proof** (1) If $a \wedge x \leqslant b$, then $a \wedge x \leqslant (a \to b)$ since $b \leqslant (a \to b)$. Conversely, if $a \wedge x \leqslant (a \to b)$, then $a \wedge x \leqslant a \wedge (a \to b)$, i. e. $a \wedge x \leqslant a \wedge b \leqslant b$, by the definition of the relation, $bR_a(a \to b)$.

(2) For any element $b_0 \in [b]$, since $a \wedge b_0 \leqslant b_0$ and $bR_a b_0$, we have $a \wedge b_0 \leqslant b$, so $b_0 \leqslant (a \to b)$. By (1), $a \to b \in [b]$, so $(a \to b)$ is the maximal element in $[b]$.

In locale theory [5], the map $a \to (-): A \to A$ is a nucleus on locale $A$, the corresponding sublocale is harder to describe. Denote $A/R_a = \{[b] \mid b \in A\}$ ordered by the inducing order in $A$, we have:

**Corollary 3** If $A$ is a Hetying algebra , then $A/R_a \cong \downarrow a \cong \{a \to b \mid b \in A\}$

**Remark** Note that the isomorphic maps between $\downarrow a$ and $\{a \to b \mid b \in A\}$ are $f: x \mapsto (a \to x)$, for any $x \in \downarrow a$ and $g: y \mapsto a \wedge y$, for any $y \in \{a \to b \mid b \in A\}$, for that $a \wedge (a \to x) = a \wedge x = x$, $a \to (a \wedge y) = a \to y = y$.

**Definition 4** We say an equivalence relation $R_a$ on $A$ with property $P_d$ in sense that, for any $b \in A$, we have:

(1) $[b]$ is a order convex set in $A$;

(2) $|[b] \cap \downarrow a| = 1$;

(3) $R_a$ is closed under joins.

If condition (1) is replaced by (1'): $[b]$ is an interval in $A$, we call that the relation satisfies property $P_h$. Furthermore, $R_a$ has property $P_b$ if $a \vee m = 1$ for the maximal element $m$ in $[b]$.

Distributive lattices, Heyting algebras and Boolean algebras have the property $P_d$, $P_h$, and $P_b$ respectively. The question whether the converse is true is raised naturally. In the following we give a positive answer for it.

**Proposition 5** Lattice $L$ is distributive iff for any $a \in A$, there exists an equivalence $R_a$ on $L$ with the property $P_d$.

**Proof** From the remarks above we only need to give the sufficiency. Let $[b]$ be an equivalence class of $R_a$ and $b' \in [b]$, by the condition $|[b] \cap \downarrow a| = 1$, we have $a \wedge b = a \wedge b' \in [b]$. For any two elements $b, c \in L$, suppose that $b \in [k_1]$, $c \in [k_2]$, we have $a \wedge b = a \wedge k_1 \in [k_1]$, $a \wedge c = a \wedge k_2 \in [k_2]$. By the condition (3) of property $P_d$, $(a \wedge b) \vee (a \wedge c) \in [k_1 \vee k_2]$ and $b \vee c \in [k_1 \vee k_2]$, then $a \wedge [(a \wedge b) \vee (a \wedge c)] = (a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$.

Similarly, for Heyting algebra we have:

**Proposition 6** Lattice $L$ is a Heyting algebra iff for each $a \in A$, there exists an equivalence $R_a$ on $L$ with the property $P_h$.

**Proof** ($\Rightarrow$): Trivially.

($\Leftarrow$): By the definition of Heyting algebra, we only need to show that there exists a maximal element in $\{x \mid a \wedge x \leqslant b\}$, for any elements $a, b \in A$. Let $[k]$ be an equivalence class of $R_a$, the maximal element denoted by $m$, the minimal element denoted by $n$. From the proof of proposition 5, we have:

$$\text{for any } x \in [k],\ a \wedge m = a \wedge x = n \in [k],$$

$L$ is distributive. Suppose that $b \in [k_1]$ and the maximal element in $[k_1]$ is $m_1$, then $a \wedge m_1 = a \wedge b \leqslant b \leqslant m_1$. If there exists an element $x \in A$ such that $a \wedge x \leqslant b$, but $x \not\leqslant m_1$, then $x \vee m_1 > m_1$, so $x \vee m_1$ is not in $[k_1]$. Suppose $x \vee m_1 \in [k_0]$, then $a \wedge (x \vee m_1) = (a \wedge x) \vee (a \wedge m_1) \leqslant b \leqslant m_1 < x \vee m_1$, so $b \in [k_0]$ and $[k_0] \cap [k_1] \neq \varnothing$, it is impossible, so $m_1$ is the maximal element in $\{x \mid a \wedge x \leqslant b\}$.

**Proposition 7** Lattice $L$ is a Boolean lattice iff for any $a \in A$, there exists an equivalence relation $R_a$ on $L$ with the property $P_b$.

**Proof** ($\Rightarrow$) Trivially.

($\Leftarrow$) By Proposition 6, $L$ is a Heyting algebra. Suppose $0 \in [b]$ and the maximal element is $m$, then $a \vee m = 1$, then we have $m = a \rightarrow 0$, i. e. $a$ has a complement element $m$.

From the equivalence relation characterization of Heyting algebra and Boolean algebra, we can get the following interesting properties, which give a lively description of them.

**Proposition 8** If lattice $L$ is a Heyting algebra, then for any element $a \in L$, there is a family of intervals $\{[p_i, q_i]\}_{i \in I}$ such that for any $i, j \in I$, $[p_i, q_i] \cap [p_j, q_j] = \varnothing$, $|[p_i, q_i] \cap \downarrow a| = 1$, and $L = \cup \{[p_i, q_i]\}_{i \in I}$.

**Proposition 9** If lattice $L$ is a Boolean algebra, then for any element $a \in L$, there is a family of intervals $\{[p_i, q_i]\}_{i \in I}$ such that for all $i, j \in I$, $[p_i, q_i] \cap [p_i, q_i] = \varnothing$, $[p_i, q_i] \cong [p_i, q_i]$, $|[p_i, q_i] \cap \downarrow a| = 1$, $q_i \vee a = 1$ and $L = \cup \{[p_i, q_i]\}_{i \in I}$.

**Proof** By Proposition 7, there exists an equivalence $R_a$ on $L$ with property $P_b$. The proof of Proposition 6 implies that for any $b \in [p_i, q_i]$, $p_i = a \wedge b$, $q_i = a \rightarrow b$, and $[a] = [a, 1]$, we only need to show that the equivalence class $[b]$ is isomorphism to the equivalence class $[a]$. Now suppose we have two elements $x$, $y$ with $x \in [a]$, $y \in [b]$, define $f: [b] \rightarrow [a]$, provided that $f(y) = a \vee y$; $g: [a] \rightarrow [b]$, provided that $g(x) = (a \rightarrow b) \wedge y$. It is obvious that $f$ and $g$ are order preserving, then we have $g(f(y)) = (a \rightarrow b) \wedge (a \vee y) = [(a \rightarrow b) \wedge a] \vee [(a \rightarrow b) \wedge y] = (a \wedge b) \vee y = y$; $f(g(x)) = a \vee ((a \rightarrow b) \wedge x) = [a \vee (a \rightarrow b)] \wedge (a \vee x) = x$, i. e. $g \cdot f = 1_{[b]}$, $f \cdot g = 1_{[a]}$, so $[a] \cong [b]$.

**Remark** If $L$ is a distributive lattice, then the condition is sufficient. We only need to define the relation $R_a$ on $L$ as follows:

$$b \sim b', \text{ if for some } i \in I,\ p \in [p_i, q_i] \text{ and } q \in [p_i, q_i].$$

It is easy to verify that $R_a$ has the property $P_b$.

Assume $L$ be a Boolean algebra, $a \in L$, then $\uparrow a$ and $\downarrow a$ are Boolean algebras, moreover, $\uparrow a \times \downarrow a$ is a Boolean algebra. By the proof of Proposition 9, we know that an element $x$ of $L$ is uniquely determined by $a \vee x \in \uparrow a$ and $a \wedge x \in \downarrow a$. Conversely, for any $(x, y) \in \uparrow a \times \downarrow a$ determined an element $(a \rightarrow y) \wedge x$ of $L$. So we have:

**Corollary 10** Let $L$ be a Boolean algeba. For any $a \in L$, $L \cong \uparrow a \times \downarrow a$, moreover, $|L| = |\downarrow a| \cdot |\uparrow a|$.

**Corollary 11** Let $L$ be a Boolean algebra. If $|L| < \aleph_0$, then for some set $X$, $L \cong 2^X$.

**Remark** In the theory of continuous lattices [6], Boolean algebra $L$ is isomorphic to the algebra of all subsets of some set if and only if it is continuous. Since a finite distributive lattice is continuous, we can obtain the corollary. However, by the Corollary 10 and finite inducing, it is obvious.

In [7] and [8], the category of Boolean algebras is isomorphic to the category of Boolean rings. The partial order on Boolean rings is defined by $a \leqslant b$ iff $a \cdot b = a$. By the Corollary 10, we have:

**Corollary 12** If $R$ is a Boolean ring, for any $a \in R$, $A = \{x \in R \mid a \cdot x = x\}$, $B = \{x \in R \mid a \cdot x = a\}$, we have $|L| = |A| \cdot |B|$.

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