On Subsets of Star-Lindelöf Spaces

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Abstract: A subset B of a space X is strongly star-Lindelöf(star-Lindelöf) if for every cover $\mathscr{U}$ of B by open subsets of X, there exists a countable subset $F\subseteq B$ (respectively, $F\subseteq \cup\mathscr{U}$) such that $B\subseteq St(F, \mathscr{U})$. In this paper, we study the star-Lindelöfness of subsets of star-Lindelöf spaces, moreover investigate the relationship between star-Lindelöf subset and relative star-Lindelöf subset.

Key words: star-Lindelöf subset, relative star-Lindelöf subset

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星林德洛夫空间的子集

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[摘要] 一个空间 X 的子集 B 称为强星林德洛夫(星林德洛夫) 如果对于由 X 的开子集构成的 B 的任意开覆盖 $\mathscr{U}$,存在一个可数子集 $F\subseteq B(F\subseteq\cup\mathscr{U})$ 使得 $B\subseteq St(F,\mathscr{U})$. 本文研究星林德洛夫空间的子集星林德洛夫性,进而研究了星林德洛夫的子集和相对星林德洛夫的子集的关系.

[关键词] 星林德洛夫的子集, 相对星林德洛夫的子集

0 Introduction

By a space, we mean a topological space. Recall from [1,2,3] that a subspace Y of a space X is Lindelöf in X if for every open cover $\mathscr{U}$ of X, there exists a countable subfamily covering Y. A space X is star-Lindelöf (by different names, see [4,5,6]) if for every open cover $\mathscr{U}$ of X, there exists a countable subset F of X such that $St(F,\mathscr{U})=X$, where $St(F,\mathscr{U})=\cup\{U\in\mathscr{U}: U\cap F\neq\varnothing\}$. These definitions motivate us to introduce the following concepts.

Definition 0.1 ([7]) A subset B of a space X is called star-Lindelöf(strongly star-Lindelöf) in X if for every open cover $\mathscr{U}$ of X, there exists a countable subset $F\subseteq X$ (respectively, $F\subseteq B$) such that $B\subseteq St(F,\mathscr{U})$.

Definition 0.2 A subset B of a space X is called star-Lindelöf(strongly star-Lindelöf) if for every cover $\mathscr{U}$ of B by open subsets of X, there exists a countable subset $F\subseteq\cup\mathscr{U}$ (respectively, $F\subseteq B$) such that $B\subseteq St(F,\mathscr{U})$.

From the above definitions, it is clear that if B is a strongly star-Lindelöf subset of X, then B is strongly star-Lindelöf in X; if B is a star-Lindelöf subset of X, then B is star-Lindelöf in X; if B is a strongly star-Lindelöf subset of X, then B is a star-Lindelöf subset of X; if B is strongly star-Lindelöf in X, then B is star-Lindelöf in X. But the converses do not hold.

The purpose of this paper is to study the star-Lindelöfness of subsets of star-Lindelöf spaces, moreover inves-

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tigate the relationship between star-Lindelöf subset and relative star-Lindelöf subset by constructing some examples.

Moreover, the cardinality of a set A is denoted by $|A|$. Let ω denote the first infinite cardinal and c the cardinality of the continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each pair of ordinals α, β with $\alpha<\beta$, we write $(\alpha,\beta)=\{\gamma:\alpha<\gamma<\beta\}$. Other terms and symbols that we do not define will be used as in [8].

1 The Subsets of Star-Lindelöf Spaces

From the definition of relative star-Lindelöf subset, it is clear that every subset of a star-Lindelöf space is relative star-Lindelöf. The following example shows it need not be relative strongly star-Lindelöf even if a regular-closed subset. For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 1.1 There exist a Tychonoff star-Lindelöf space X and a regular-closed subset B of X such that B is a star-Lindelöf subset of X, but B is not strongly star-Lindelöf in X.

Proof Let $S_1=\omega\cup\mathscr{R}$ be the Isbell-Mrówka space [9], where $\mathscr{R}$ is maximal almost disjoint family of infinite subsets of ω with $|\mathscr{R}|=c$. Then, S_1 is star-Lindelöf, since ω is a dense subset of S_1.

Let D be the discrete space of cardinality c and let

$$S_2=(\beta D\times(\omega+1))\times((\beta D\backslash D)\times\{\omega\}).$$

We assume that $S_1\cap S_2=\varnothing$. Since $|\mathscr{R}|=c$ and $|D\times\{\omega\}|=c$, we can enumerate $\mathscr{R}$ and $D\times\{\omega\}$ as $\{r_\alpha:\alpha<c\}$ and $\{\langle d_\alpha,\omega\rangle:\alpha<c\}$ respectively.

Let

$$\varphi:\mathscr{R}\to D\times\{\omega\}$$

be a bijection by

$$\varphi(r_\alpha)=\langle d_\alpha,\omega\rangle \text{ for } \alpha<c.$$

Let X be the quotient space obtained from the discrete sum $S_1\oplus S_2$ by identifying r_α with $\varphi(r_\alpha)$ for each $\alpha<c$. Let $\pi:S_1\oplus S_2\to X$ be the quotient map and $B=\pi(S_2)$.

First, we show that X is star-Lindelöf. For this end, let $\mathscr{U}$ be an open cover of X. Since $\pi(S_1)$ is homeomorphic to S_1, then $\pi(S_1)\subseteq St(\pi(\omega),\mathscr{U})$, since ω is a dense subset of S_1. On the other hand, since $\pi(\beta D\times\{n\})$ is compact for each $n\in\omega$, then there exists a finite subset $F_n\subseteq\pi(\beta D\times\{n\})$ such that

$$\pi(\beta D\times\{n\})\subseteq St(F_n,\mathscr{U}) \text{ for each } n\in\omega.$$

If we put $F=\pi(\omega)\cup\bigcup_{n\in\omega}F_n$, then F is a countable subset of X and $X=St(F,\mathscr{U})$, which shows that X is star-Lindelöf.

Next, we show that B is a star-Lindelöf subset of X. For this end, let $\mathscr{U}$ be a cover of B by open subsets of X. Let $F'=\pi(\omega)\cap(\cup\mathscr{U})$, then F' is a countable subset of $\cup\mathscr{U}$ and $\pi(D\times\{\omega\})\subseteq St(F',\mathscr{U})$. On the other hand, similar to the above proof, for $n\in\omega$ there exists a finite subset $F_n\subseteq\pi(\beta D\times\{n\})$ such that

$$\pi(\beta D\times\{n\})\subseteq St(F_n,\mathscr{U}).$$

If we put $F=F'\bigcup_{n\in\omega}F_n$, then F is a countable subset of $\cup\mathscr{U}$ and $B\subseteq St(F,\mathscr{U})$, which shows that B is a star-Lindelöf subset of X.

Finally, we show that B is not strongly star-Lindelöf in X. Let

$$U_n=\pi(\beta D\times\{n\}) \text{ for each } n\in\omega$$

and

$$V_\alpha=\pi((\{d_\alpha\}\times(\omega+1))\cup r_\alpha) \text{ for } \alpha<c.$$

Let us consider the open cover $\mathscr{V}=\{U_n:n\in\omega\}\cup\{V_\alpha:\alpha<c\}\cup\{\pi(\omega)\}$ of X. Let F be any countable subset of B, then there exists a $\alpha<c$ such that $V_\alpha\cap F=\varnothing$, hence $\pi(\langle d_\alpha,\omega\rangle)\notin St(F,\mathscr{V})$, since V_α is the only element of $\mathscr{V}$ containing $\pi(\langle d_\alpha,\omega\rangle)$. This shows that B is not strongly star-Lindelöf in X, which completes the

proof.

Remark 1 Since every strongly star-Lindelöf subset B of X is strongly star-Lindelöf in X, the Example 1.1 shows that a regular-closed subset B of a Tychonoff star-Lindelöf space X need not be a strongly star-Lindelöf subset of X and a star-Lindelöf subset B of X need not be a strongly star-Lindelöf subset of X.

Example 1.2 There exist a compact space X and an open subset B of X such that B is strongly star-Lindelöf in X, but B is not a strongly star-Lindelöf subset of X.

Proof Let D be the discrete space of cardinality c and let $X=\beta D$ and $B=D$. Then, B is strongly star-Lindelöf in X, since X is compact and D is dense in X. But B is not a strongly star-Lindelöf subset of X, since B is a discrete open subspace of X with $|D| = c$.

Remark 2 The Example 1.2 shows there exist a compact space X and a subset B of X such that B is star-Lindelöf in X, but B is not a star-Lindelöf subset of X, and an open subset of a compact space need not be a star-Lindelöf subset of X.

Remark 3 In [7], the author constructed an example shows there exist a Tychonoff space X and a subset B of X such that B is star-Lindelöf in X, but B is not strongly star-Lindelöf in X.

We give two positive results on the relative strongly star-Lindelöfness.

Theorem 1.3 Let X be a space and B a closed subset of X. Then B is strongly star-Lindelöf in X if and only if B is strongly star-Lindelöf of X.

Proof The sufficiency is clear.

Necessity. Let *$\mathscr{U}$ be a cover of* B *by open subsets of* X. *Then* $\mathscr{V}=\mathscr{U}\cup\{X\backslash B\}$ is an open cover of X. Since B is strongly star-Lindelöf in X, there exists a countable subset $F\subseteq B$ such that $B\subseteq St(F,\mathscr{V})$. Note that $F\cap(X\backslash B)=\phi$, so $B\subseteq St(F,\mathscr{U})$, which completes the proof.

Theorem 1.4 Let X be a star-Lindelöf space and B an open F_σ-subset of X. Then B is strongly star-Lindelöf in X.

Proof Let $B=\cup\{H_n:n\in\omega\}$, where H_n is a closed subset of X for each $n\in\omega$. Let $\mathscr{U}$ be an open cover of X and $\mathscr{U}'=\{U\cap B:U\in\mathscr{U}\}$. For each $n\in\omega$, let $\mathscr{U}_n=\mathscr{U}'\cup\{X\backslash H_n\}$, then $\mathscr{U}_n$ is an open cover of X, then there exists a countable subset $B_n\subseteq X$ such that $X=St(B_n,\mathscr{U}_n)$. Let $A_n=B\cap B_n$ for each $n\in\omega$. Then,

$$H_n\subseteq St(A_n,\ \mathscr{U}')\subseteq St(A_n,\ \mathscr{U}).$$

If we put $F=\cup\{A_n:n\in\omega\}$, then F is a countable subset of B and $B\subseteq St(F,\mathscr{U})$, which completes the proof.

Since a cozero – set is open F_σ-set, thus we have the following corollary.

Corollary 1.5 Let X be a star-Lindelöf space and a cozero – set B of X. Then, B is strongly star-Lindelöf in X.

Similar to the proof of Theorem 1.3, we can prove the following results.

Theorem 1.6 Let X be a star-Lindelöf space and an open F_σ-subset B of X. Then B is a strongly star-Lindelöf subset of X.

Corollary 1.7 Let X be a star-Lindelöf space and a cozero – set B of X. Then B is a strongly star-Lindelöf subset of X.

Theorem 1.8 Let X be a star-Lindelöf space and a F_σ-subset B of X. Then B is a star-Lindelöf subset of X.

Proof Let $B=\cup\{H_n:n\in\omega\}$, where H_n is a closed subset of X for each $n\in\omega$. Let $\mathscr{U}$ be a cover of B by open subsets of X. For each $n\in\omega$, let $\mathscr{U}_n=\mathscr{U}\cup\{X\backslash H_n\}$. Then $\mathscr{U}_n$ is an open cover of X, and there exists a countable subset $B_n\subseteq X$ such that $X=St(B_n,\ \mathscr{U}_n)$. Let $A_n=B_n\cap(\cup\mathscr{U})$ for each $n\in\omega$. Then,

$$H_n\subseteq St(A_n,\ \mathscr{U}).$$

If we put $F=\cup\{A_n:n\in\omega\}$, then F is a countable subset of $\mathscr{U}$ and $B\subseteq St(F,\ \mathscr{U})$, which completes the proof.

Since a closed set is a F_σ-set, thus we have the following corollary.

Corollary 1.9 Let X be a star-Lindelöf space and a closed subset B of X. Then B is a star-Lindelöf subset of X.

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