New Duality Between Generalized Smash Products and Smash Coproducts

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Abstract This paper mainly proves that $_H(A\#B)^0$ is isomorphic to $_HA^0\times_H B^0$ as coalgebras, where $_H(A\#B)^0$ denotes the new duality of the generalized smash product $A\#B$, and $_HA^0\times_H B^0$ denotes the generalized smash coproduct.

Key words Hopf algebra, generalized smash product, generalized smash coproduct, dimodule algebra

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广义 Smash积与 Smash余积之间的新对偶

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[摘要] 主要证明余代数$_H(A\#B)^0$和余代数$_HA^0\times_H B^0$同构, 其中$_H(A\#B)^0$是广义 Smash积$A\#B$的新对偶, $_HA^0\times_H B^0$是广义 Smash余积.

[关键词] Hopf代数, 广义 Smash积, 广义 Smash余积, 重模代数

0 Introduction

In [1], Molnar introduced the concepts of smash product $A\#H$ for module algebras and smash coproduct $C\times H$ for comodule coalgebras.

In 2003, Zhang and Chen introduced the new dual concept of module algebras, and gave the dual relation between module algebras and comodule coalgebras, that is, there exists an isomorphism of coalgebras in [2]:

$$(A\#H)^0\cong {}_HA^0\times H^0.$$

In [3], Kan introduced the concepts of generalized smash product $A\#B$ and generalized smash coproduct $C\times D$, which extend the above smash product $A\#H$ and smash coproduct $C\times H$.

In [4], Zhang and Tong further studied the generalized smash product over H-dimodule algebras and the generalized smash coproduct over H-dimodule coalgebras.

The aim of this paper is to study the new dual relation between generalized smash products and generalized smash coproducts.

1 Preliminaries

We always work over a fixed field k and follow Sweedler's book[5] for terminologies.

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A k-coalgebra is a k-space C together with two k-linear maps, comultiplication $\Delta: C\to C\otimes C$ and counit $\varepsilon: C\to k$ such that

$$(\Delta\otimes I)\Delta=(I\otimes\Delta)\Delta,\ (\varepsilon\otimes I)\Delta=I=(I\otimes\varepsilon)\Delta$$

In the sequel, the comultiplication structure map Δ of C is written as $\Delta(c)=\sum c_1\otimes c_2$ for $c\in C$.

For a coalgebra C, a left C-comodule is a k-space M with a linear map $\rho: M\to C\otimes M$ such that

$$(I\otimes\rho)\rho=(\Delta\otimes I)\rho,\ (\varepsilon\otimes I)\rho=I,$$

where the comodule structure ρ of M is written by $\rho(m)=\sum m_{(-1)}\otimes m_{(0)}$.

If H is both an algebra and a coalgebra such that comultiplication Δ and counit ε are algebra maps, then we call H a bialgebra.

In this paper, H is always considered as a bialgebra. In the following we recall some concepts used in this paper.

Definition 1.1 (1) A left H-module algebra A is both an algebra and a left H-module with module structure "$\cdot$" such that for any $h\in H$, $a, b\in A$,

$$h\cdot(ab)=\sum(h_1\cdot a)(h_2\cdot b),\ h\cdot 1_A=\varepsilon(h)1_A.$$

(2) A left H-comodule algebra A is both an algebra and a left H-comodule with comodule structure map ρ such that for any $a, b\in A$,

$$\rho(ab)=\rho(a)\rho(b),\ \rho(1_A)=1_H\otimes 1_A.$$

(3) A left H-module coalgebra is both a coalgebra and a left H-module with module structure "$\cdot$" such that for any $h\in H$, $c\in C$,

$$\Delta(h\cdot c)=\Delta(h)\Delta(c),\ \varepsilon(h\cdot c)=\varepsilon(h)\varepsilon(c).$$

(4) A left H-comodule coalgebra C is both a coalgebra and a left H-comodule with comodule structure map ρ such that for any $h\in H$, $c\in C$,

$$\sum c_{(-1)}\otimes c_{(0)1}\otimes c_{(0)2}=\sum c_{1(-1)}c_{2(-1)}\otimes c_{1(0)}\otimes c_{2(0)},\ \sum c_{(-1)}\varepsilon(c_{(0)})=\varepsilon(c)1_H.$$

Definition 1.2 (1) A k-module M which is both a left H-module and a left H-comodule is called a left H-dimodule if for any $m\in M$,

$$\rho(h\cdot m)=\sum m_{(-1)}\otimes h\cdot m_{(0)}.$$

(2) A left H-dimodule algebra A in [3] is a left H-dimodule which is both a left H-module algebra and a left H-comodule algebra.

Definition 1.3 (1) Let A be a left H-module algebra, and B a left H-comodule algebra. A generalized smash product $A\#B$ is defined as follows:

$A\#B=A\otimes B$ as k-modules, and its multiplication is given by

$$(a\#x)(b\#y)=\sum a(x_{(-1)}\cdot b)\#x_{(0)}y,$$

for any $a, b\in A$, $x, y\in B$.

(2) Let C be a left H-comodule coalgebra, and D a left H-module coalgebra. A generalized smash coproduct $C\times D$ is defined as follows:

$C\times D=C\otimes D$ as k-modules and its comultiplication and counit are given by

$$\Delta(c\times d)=\sum(c_1\times c_{2(-1)}\cdot d_1)\otimes(c_{2(0)}\times d_2),\ \varepsilon(c\times d)=\varepsilon(c)\varepsilon(d),$$

for any $c\in C$, $d\in D$.

By [3], $A\#B$ is an algebra with unit $1_A\#1_B$, and $C\times D$ is a coalgebra with counit $\varepsilon\otimes\varepsilon$.

2 The New Duality Between ${}_H(A\#B)^0$ and ${}_HA^0\times{}_HB^0$

Let A be an algebra. Then, by [5],

$$A^0=\{\alpha\in A^*\mid \text{there is a cofinite ideal } I \text{ such that } \langle\alpha, I\rangle=0\}$$

is a coalgebra. If A is also a left H-module, then, by [4],

$$_HA^0=\{\alpha\in A^* \mid \text{there is a cofinite ideal } I \text{ such that } H\cdot I\subseteq I \text{ and } <\alpha, I>=0\}$$

is a coalgebra.

In this paper, our main aim is to prove that the new dual $_H(A\#B)^0$ of the generalized smash product $A\# B$ is isomorphic to the generalized smash coproduct $_HA^0\times {_HB^0}$.

Lemma 2.1 Let B be a left H-comodule algebra with comodule structure map ρ. Then

(1) B^0 is a left H^0-module coalgebra.

(2) if B is also a left H-dimodule, then $_HB^0$ is a left H^0-module coalgebra.

Proof (1) It is straight forward.

(2) To show that $_HB^0$ is a left H^0-module coalgebra, we only have to show $_HB^0$ is a left H^0-module. As a matter of fact, for any $\alpha\in H^0$, $\beta\in {_HB^0}$, then there exist cofinite ideals I and J such that

$$<\alpha, I>=0,\ H\cdot J\subseteq J,\ <\beta, J>=0.$$

Since the comodule structure map ρ of B is an algebra map and $I\otimes B+H\otimes J$ is a cofinite ideal in $H\otimes B$, $T=\rho^{-1}(I\otimes B+H\otimes J)$ is a cofinite ideal of B.

One can check that

$$\rho(H\cdot T)\subset \sum T_{(-1)}\otimes H\cdot T_{(0)}\subset I\otimes H\cdot B+H\otimes H\cdot J\subset I\otimes B+H\otimes J,$$

so $H\cdot T\subset T$ and T is a left H-submodule ideal in $H\otimes B$.

Since

$$<\alpha\cdot\beta, T>=<\rho^*(\alpha\otimes\beta), T>=<\alpha\otimes\beta, \rho(T)>\subseteq<\alpha\otimes\beta, I\otimes B+H\otimes J>=0,$$

$_HB^0$ is a left H^0-module.

Lemma 2.2 Let A be an algebra which is a left H-module, and B an algebra, and $\alpha\in(A\otimes B)^*$.

(1) If there exist cofinite ideals I and J of A and B such that

$$H\cdot I\subset I,\ <\alpha, I\otimes B+A\otimes J>=0,$$

then $\alpha\in {_HA^0}\otimes B^0$.

(2) If also

$$H\cdot J\subset J,\ <\alpha, I\otimes B+A\otimes J>=0,\quad \text{then } \alpha\in {_HA^0}\otimes {_HB^0}.$$

Proof (1) Let $\pi_1: A\to A/I$ and $\pi_2: B\to B/J$. Then there exists a unique map

$$\alpha': A/I\otimes B/J\to k,$$

such that $\alpha=\alpha'(\pi_1\otimes\pi_2)$.

Because $\dim(A/I)<+\infty$, $\dim(B/J)<+\infty$,

$$\alpha'\in(A/I\otimes B/J)^*\cong(A/I)^*\otimes(B/J)^*.$$

Let $\alpha'=\sum\alpha'_i\otimes\beta'_i\in(A/I)^*\otimes(B/J)^*$, $\alpha_i=\alpha'_i\pi_1$, $\beta_i=\beta'_i\pi_2$. Then $<\alpha_i, I>=0$, $<\beta_i, J>=0$.

By $H\cdot I\subset I$, we know that $\alpha=\sum\alpha_i\otimes\beta_i\in {_HA^0}\otimes B^0$.

(2) In a similar way, we can show it.

Lemma 2.3 Let A be a left H-module algebra, and B a left H-comodule algebra and a left H-module. Then

$$_HA^0\otimes B^0\subset {_H(A\#B)^0},$$

and so $_HA^0\times B^0$ is a subcoalgebra of $_H(A\#B)^0$.

Proof For any $\alpha\in {_HA^0}$, $\beta\in B^0$, there exist cofinite ideals I and J such that

$$<\alpha, I>=0,\ H\cdot I\subset I, \text{ and } <\beta, J>=0.$$

It is easy to verify that $I\#B$ is a cofinite left H-submodule ideal of $A\#B$, and

$$<\alpha\otimes\beta, I\#B>=0.$$

So, $\alpha\otimes\beta\in {_H(A\# B)^0}$.

Here the left H-action of $A\#B$ is given by

$$h\cdot(a\# b)=\sum h_1\cdot a\# h_2\cdot b.$$

By Proposition 1.6 given in [2], we know that ${}_HA^0$ is a left H^0-comodule coalgebra, so we have the generalized smash coproduct ${}_HA^0\times B^0$ with comultiplication as follows

$$\Delta(\alpha\times\beta)=\sum\alpha_1\times\alpha_{2(-1)}\cdot\beta_1\otimes\alpha_{2(0)}\times\beta_2.$$

Moreover, for any $a, b\in A$, $x, y\in B$, we have

$$\begin{aligned}
<\Delta(\alpha\otimes\beta), a\# x\otimes b\# y> &= \sum<\alpha\otimes\beta, a(x_{(-1)}\cdot b)\# x_{(0)}y>\\
&=\sum<\alpha, a(x_{(-1)}\cdot b)><\beta, x_{(0)}y>\\
&=\sum<\alpha_1\otimes\alpha_2, a\otimes(x_{(-1)}\cdot b)><\beta_1\otimes\beta_2, x_{(0)}\otimes y>\\
&=\sum<\alpha_1, a><\alpha_2, x_{(-1)}\cdot b><\beta_1, x_{(0)}><\beta_2, y>\\
&=\sum<\alpha_1, a><\alpha_{2(-1)}, x_{(-1)}><\alpha_{2(0)}, b><\beta_1, x_{(0)}><\beta_2, y>\\
&=\sum<\alpha_1, a><\alpha_{2(-1)}\otimes\beta_1, x_{(-1)}\otimes x_{(0)}><\alpha_{2(0)}, b><\beta_2, y>\\
&=\sum<\alpha_1, a><\alpha_{2(-1)}\cdot\beta_1, x><\alpha_{2(0)}, b><\beta_2, y>\\
&=\sum<\alpha_1\times\alpha_{2(-1)}\cdot\beta_1\otimes\alpha_{2(0)}\times\beta_2, a\# x\otimes b\# y>\\
&=<\Delta(\alpha\times\beta), a\# x\otimes b\# y>.
\end{aligned}$$

So ${}_HA^0\times B^0$ is a subcoalgebra of ${}_H(A\# B)^0$.

Theorem 2.4 Let A be a left H-module algebra, and B a left H-dimodule which is a left H-comodule algebra with $h\cdot 1_B=\varepsilon(h)1_B$. Then

$${}_H(A\# B)^0\cong{}_HA^0\times{}_HB^0$$

as coalgebras.

Proof By Lemma 2.1, we know that ${}_HB^0$ is a left H^0-comodule coalgebra. So, by Lemma 2.3,

$${}_HA^0\times{}_HB^0\subset{}_HA^0\times B^0\subset{}_H(A\# B)^0.$$

For any $\gamma\in{}_H(A\# B)^0$, there exists a cofinite ideal T such that

$$H\cdot T\subset T,\quad<\gamma, T>=0$$

Since $A\to A\# B$, $a\mapsto a\# 1_B$ and $B\to A\# B$, $x\mapsto 1_A\# x$ are algebra maps,

$$I=\{a\in A\mid a\# 1_B\in T\},\quad J=\{b\in B\mid 1_A\# b\in T\}$$

are cofinite ideals in A and B, respectively, and

$$<\gamma, I\# B+A\# J>\subset<\gamma, T>=0$$

Since for any $a\in I$, $h\in H$,

$$h\cdot a\# 1_B=\sum h_1\cdot a\#\varepsilon(h_2)1_B=h\cdot(a\# 1_B)\in T,$$

$h\cdot a\in I$, and I is a left H-submodule ideal of A.

Similarly, J is a left H-submodule ideal of B. So, by Lemma 2.2, $\gamma\in{}_HA^0\otimes_H B^0$, and hence ${}_H(A\# B)^0={}_HA^0\otimes_H B^0$.

By Lemma 2.3, we know that

$${}_H(A\# B)^0\cong{}_HA^0\times{}_HB^0$$

as coalgebras.

• Let H be a bialgebra. A bilinear form $\sigma: H\otimes H\to k$ is called a skew pairing in [6] if the followings hold:

(L1) $\sigma(x, hg)=\sum\sigma(x_1, g)\sigma(x_2, h)$;

(L2) $\sigma(xy, h)=\sum\sigma(x, h_1)\sigma(y, h_2)$;

(L3) $\sigma(x, 1_H)=\varepsilon(x)$;

(L4) $\sigma(1_H, h)=\varepsilon(h)$.

for any $h, g, x, y\in H$.

• Let σ be a skew pairing on H. If for any $h, x \in H$, holds,

(L5) $\sum \sigma(x_1, h) x_2 = \sigma(x_2, h) x_1$, holds

then we call (H, σ) a Long skew bialgebra in [6].

Example 2.5 Let H be a Hopf algebra. Then, by [7], H is a left H-module algebra whose module action is given by $h \cdot x = \sum h_1 x S(h_2)$. So, by Remark 1.1 in [2], ${}_H H^0 = H^0$.

Let $\sigma: H \otimes H \rightarrow k$ be a skew pairing. Define a left H-action of H as follows:

$$\rightharpoonup: H \otimes H \rightarrow H, \quad h \otimes x \mapsto \sum \sigma(h, x_2) x_1.$$

Then $(H, \rightharpoonup, \Delta)$ is a left H-dimodule which is a left H-comodule algebra with $h \rightharpoonup 1_H = \varepsilon(h) 1_H$.

So, we have the generalized smash product $H \# H$ whose multiplication is given by

$$(h \# x)(g \# y) = \sum h x_1 g S(x_2) \# x_3 y.$$

By Theorem 2.4, there exists an isomorphism of coalgebras as follows:

$${}_H(H \# H)^0 \cong H^0 \times_H H^0.$$

Corollary 2.6 Let A be a left H-module algebra, and B a left H-dimodule algebra. Then

$${}_H(A \# B)^0 \cong {}_H A^0 \times_H B^0$$

as coalgebras.

Proof It is straight forward by Definition 1.2 and Theorem 2.4.

Example 2.7 Let (H, σ) be a long skew bialgebra. Define a left H-action on H^* as follows:

$$\cdot: H \otimes H^* \rightarrow H^*, \quad h \otimes f \mapsto h \cdot f,$$

where $\langle h \cdot f, x \rangle = f(xh)$.

It is easy to check that $(H^*, \cdot)$ is a left H-module algebra.

Define a left H-action on H as follows:

$$\rightharpoonup: H \otimes H \rightarrow H, \quad h \otimes x \mapsto \sum \sigma(h, x_2) x_1.$$

By [4], We know that $(H, \rightharpoonup, \Delta)$ is a left H-dimodule algebra. So, by Theorem 2.4, there exists an isomorphism of coalgebras as follows:

$${}_H(H^* \# H)^0 \cong {}_H H^{*0} \times_H H^0.$$

In Theorem 2.4, if $B = H$, then B is a left H-dimodule with the trivial action and the comodule structure map Δ. It is obvious that ${}_H B^0 = {}_H H^0 = H^0$, so we obtain

Corollary 2.8 Let A be a left H-module algebra. Then there exists an isomorphism of coalgebras as follows:

$${}_H(A \# H)^0 \cong {}_H A^0 \times H^0.$$

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