A Simple Alternating Direction Method for Linear Variational Inequality Problems

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Abstract The alternating direction methods for solving variational inequality problems needs to solve several subproblems, which are also variational inequalities. Thus, the efficiency of this type of methods is influenced by the methods for solving the subproblems. In this paper, we propose a simple alternating direction method. It needs only to perform some matrix-vector productions and projection onto a simple set. Under mild assumption, we show the global convergence of the method. Some preliminary computational results are reported, showing the efficiency of the proposed method.

Key words alternating direction method, linear variational inequality problem, global convergence

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解线性变分不等式问题的一个简单交替方向法

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[摘要] 解变分不等式的交替方向法每一步需要解一个(几个)变分不等式子问题, 算法的有效性受这些子问题的影响很大. 本文提出了一个解线性变分不等式的简单的交替方向法. 在每一步迭代中, 只需要做矩阵-向量乘法和到简单集合的投影, 使得算法的效率得到保证. 在适当的条件下证明了算法的全局收敛性. 初步的数值结果表明, 我们的新算法较原有同类算法有所改进.

[关键词] 交替方向法, 线性变分不等式, 全局收敛

0 Introduction

We consider the linear variational inequality problem: Find $x^* \in S$, such that

$$(x - x^*)^T (Hx^* + c) \geqslant 0, \quad \forall x \in S, \tag{1}$$

where $S \subset \mathbf{R}^n$ is a nonempty closed convex subset of $\mathbf{R}^n$ and $H \in \mathbf{R}^{n\times n}$ is a matrix and $c \in \mathbf{R}^n$. In many practical applications, S has the following structure:

$$S = \{x \in \mathbf{R}^n \mid Ax = b, \ x \in K\},$$

where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, and K is a simple nonempty closed convex subset of $\mathbf{R}^n$.

By attaching a Lagrange multiplier vector $y \in \mathbf{R}^m$ to the linear constraint $Ax = b$, we get an equivalent form of the variational inequality problem (1): Find $u^* \in \Omega$, such that

$$(u - u^*)^T F(u^*) \geqslant 0, \quad \forall u \in \Omega \tag{2}$$

where

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \ F(u) = \begin{pmatrix} (Hx + c) - A^T y \\ Ax - b \end{pmatrix}, \ \Omega = K \times \mathbf{R}^m. \tag{3}$$

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In the following, we will always assume that the solution set of problem (2)-(3) is nonempty. For solving this problem with H being symmetric, He and Zhou[1] proposed the following alternating direction method: Given $(x^k, y^k)\in\Omega$, compute the temporal point $\tilde{u}^k=(\tilde{x}^k,\tilde{y}^k)$ via

$$\tilde{x}^k = P_K[x^k-(Hx^k+A^T(Ax^k-b)-A^Ty^k+c)],$$
$$\tilde{y}^k = y^k-(A\tilde{x}^k-b).$$

Then, set

$$r(u^k)=\begin{pmatrix} x^k-\tilde{x}^k \\ Ax^k-b\end{pmatrix},\quad d(u^k)=\begin{pmatrix} x^k-\tilde{x}^k \\ y^k-\tilde{y}^k\end{pmatrix}.$$

Finally, compute the stepsize $\rho(u^k)$

$$\rho(u^k)=\frac{\|x^k-\tilde{x}^k\|^2+\|Ax^k-b\|^2}{\|x^k-\tilde{x}^k\|^2_{I+H+A^TA}+\|y^k-\tilde{y}^k\|^2},$$

and get the next iterate

$$u^{k+1}=u^k-\gamma\rho(u^k)(u^k-\tilde{u}^k).$$

Their method is more attractive than the classical alternating direction methods[2-7], instead of solving the structurally difficult variational inequality problem in the classical methods, they only make a projection to the simple set K and calculate some matrix-vector products to get the next iterate u^{k+1}. This is advantageous especially for large scale problems. Their method was then extended to solving linear variational inequality problems by adopting an alternative stepsize rule[8].

In this paper, we propose a new alternating direction method, which is as simple as [1] and [8]. Under similar conditions as those in [1, 8], we prove the global convergence of the method.

1 The New Algorithm

We now present our method formally.

Algorithm. Given $0<\tau<2$ and $(x^k, y^k)\in\Omega$, compute the temporal point $\tilde{x}^k\in K$ via

$$\tilde{x}^k = P_K\left[\frac{1}{\mu}(Bx^k-c+A^Ty^k+A^Tb)\right],\tag{4}$$

where $B=\mu I-(H+A^TA)$, $\mu>\|H+A^TA\|$ is a constant. Then, compute the stepsize by

$$\rho_k=\frac{(x^k-\tilde{x}^k)^TB(x^k-\tilde{x}^k)+\|A\tilde{x}^k-b\|^2}{\|B(x^k-\tilde{x}^k)\|^2+\|A\tilde{x}^k-b\|^2}.\tag{5}$$

Finally, find the next iterative point by

$$x^{k+1}=P_K[x^k-\tau\rho_kB(x^k-\tilde{x}^k)],\tag{6}$$
$$y^{k+1}=y^k-\tau\rho_k(A\tilde{x}^k-b).\tag{7}$$

Notice that (4) is equivalent to finding $\tilde{x}^k$, such that

$$(x'-\tilde{x}^k)^T\{B(\tilde{x}^k-x^k)+(H\tilde{x}^k+c)-A^T[y^k-(A\tilde{x}^k-b)]\}\geqslant 0,\ \forall x'\in K.\tag{8}$$

Let x^* and y^* be an arbitrary solution of (2). Then, setting $x'=x^*$ in (8), we have

$$(x^*-\tilde{x}^k)^T\{B(\tilde{x}^k-x^k)+(H\tilde{x}^k+c)-A^T[y^k-(A\tilde{x}^k-b)]\}\geqslant 0\tag{9}$$

Since $\tilde{x}^k\in K$,

$$(\tilde{x}^k-x^*)^T\{(Hx^*+c)-A^Ty^*\}\geqslant 0\tag{10}$$

Adding (9) and (10), we have

$$(\tilde{x}^k-x^*)^T\{-B(\tilde{x}^k-x^k)+H(x^*-\tilde{x}^k)-A^T(y^*-y^k)-A^T(A\tilde{x}^k-b)\}\geqslant 0$$

Since H is positive semidefinite,

$$(\tilde{x}^k-x^*)^TH(x^*-\tilde{x}^k)\leqslant 0$$

Thus,

$$(\tilde{x}^k-x^*)^T\{-B(\tilde{x}^k-x^k)-A^T(y^*-y^k)-A^T(A\tilde{x}^k-b)\}\geqslant 0.$$

Using the fact that $Ax^*=b$ and by rearranging terms, we get,

$$(x^k - x^*)^{\mathrm T} B(x^k - \bar{x}^k) + (y^k - y^*)(A\bar{x}^k - b) \geqslant \| A\bar{x}^k - b \|^2 + (x^k - \bar{x}^k)^{\mathrm T} B(x^k - \bar{x}^k), \quad (11)$$

which means that $[B(x^k - \bar{x}^k);\ A\bar{x}^k - b]$ is an ascent direction of the unknown function $\frac{1}{2}\| [x - x^*;\ y - y^*] \|^2$. Here we have used the MATLAB convention that for any two column vectors $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$, $[x; y] := (x^{\mathrm T}, y^{\mathrm T})^{\mathrm T}$.

2 Global Convergence

In this section, we analyze the global convergence of the proposed algorithm. Based on (11), we have the following theorem.

Theorem 1 For any solution point u^* of (2), the sequence $\{u^k\}$ generated by the algorithm satisfies

$$\| u^{k+1} - u^* \|^2 \leqslant \| u^k - u^* \|^2 - \tau(2-\tau)\rho_k(\| A\bar{x}^k - b \|^2 + (x^k - \bar{x}^k)^{\mathrm T} B(x^k - \bar{x}^k)). \quad (12)$$

Proof For any two vectors s, $t \in \mathbf{R}^n$, we have

$$\| P_K[s] - P_K[t] \| \leqslant \| s - t \|,$$

i.e., the project operator P_K is nonexpansive. From this property, we have

$$\begin{aligned}
&\| u^{k+1} - u^* \|^2 \\
&\leqslant \| x^k - \tau\rho_k B(x^k - \bar{x}^k) - x^* \|^2 + \| y^k - \tau\rho_k(A\bar{x}^k - b) - y^* \|^2 \\
&= \| x^k - x^* \|^2 + \| y^k - y^* \|^2 - 2\tau\rho_k(x^k - x^*)^{\mathrm T} B(x^k - \bar{x}^k) - 2\tau\rho_k(y^k - y^*)^{\mathrm T}(A\bar{x}^k - b) + \\
&\quad \tau^2\rho_k^2(\| B(x^k - \bar{x}^k) \|^2 + \| A\bar{x}^k - b \|^2) \\
&\leqslant \| u^k - u^* \|^2 - 2\tau\rho_k((x^k - \bar{x}^k)^{\mathrm T} B(x^k - \bar{x}^k) + \| A\bar{x}^k - b \|^2) + \tau^2\rho_k^2(\| B(x^k - \bar{x}^k) \|^2 + \| A\bar{x}^k - \\
&\quad b \|^2) \\
&= \| u^k - u^* \|^2 - \tau(2-\tau)\rho_k[(\| A\bar{x}^k - b \|^2 + (x^k - \bar{x}^k)^{\mathrm T} B(x^k - \bar{x}^k)],
\end{aligned}$$

where the second inequality follows from (11) and the last equality follows from the definition of ρ_k.

Since B is positive definite,

$$(x^k - \bar{x}^k)^{\mathrm T} B(x^k - \bar{x}^k) \geqslant \lambda_{\min}(B) \| x^k - \bar{x}^k \|^2,$$

where $\lambda_{\min}(B)$ is the minimum eigenvalue of B. On the other hand,

$$\| B(x^k - \bar{x}^k) \|^2 \leqslant \| B \|^2 \| x^k - \bar{x}^k \|^2.$$

From the definition of ρ_k, we have

$$\rho_k \geqslant \frac{\lambda_{\min}(B) \| x^k - \bar{x}^k \|^2 + \| A\bar{x}^k - b \|^2}{\| B \|^2 \| x^k - \bar{x}^k \|^2 + \| A\bar{x}^k - b \|^2} \geqslant \frac{\min(\lambda_{\min}(B), 1)}{\max(\| B \|^2, 1)} := c, \quad (13)$$

for all $k > 0$.

We have the following main result.

Theorem 2 The sequence $\{u^k\}$ generated by the algorithm converges to a solution of the variational inequality problem (2).

Proof It follows from Theorem 1 and (13) that

$$\| u^{k+1} - u^* \|^2 \leqslant \| u^k - u^* \|^2 - \tau(2-\tau)c(\| A\bar{x}^k - b \|^2 + (x^k - \bar{x}^k)^{\mathrm T} B(x^k - \bar{x}^k)).$$

Since $c > 0$, $0 < \tau < 2$ and the fact that B is positive definite that

$$\| u^{k+1} - u^* \|^2 \leqslant \| u^k - u^* \|^2 \leqslant \cdots \leqslant \| u^0 - u^* \|^2, \quad (14)$$

which means that the generated sequence $\{u^k\}$ is bounded. Furthermore,

$$\lim_{k\to\infty} \| A\bar{x}^k - b \|^2 + (x^k - \bar{x}^k)^{\mathrm T} B(x^k - \bar{x}^k) = 0,$$

or, equivalently,

$$\lim_{k\to\infty} \| A\bar{x}^k - b \| = \lim_{k\to\infty} (x^k - \bar{x}^k)^{\mathrm T} B(x^k - \bar{x}^k) = 0.$$

Since $\{u^k\}$ is bounded, it has at least one cluster point. Let $\tilde{u}$ be a cluster point of $\{u^k\}$ and $\{u^{k_j}\}$ be the subsequence converging to $\tilde{u}$. Then,

$$\tilde{x} = \lim_{k\to\infty} x^{k_j} = \lim_{k\to\infty} \bar{x}^{k_j}$$

and

$$\|A\tilde{x}-b\| = \lim_{k\to\infty}\|A\tilde{x}^{k_j}-b\| = 0 \tag{15}$$

Since the projection operator is continuous, taking limit along the subsequence in (4) and using (15), we have

$$\tilde{x} = P_K[\tilde{x}-\frac{1}{\mu}(H\tilde{x}+c-A^T\tilde{y})],$$

which, together with $A\tilde{x}=b$ implies that $\tilde{u}$ is a solution of (2). We can take $u^* = \tilde{u}$ in (14) and

$$\|u^{k+1}-\tilde{u}\| \leqslant \|u^k-\tilde{u}\|.$$

the whole sequence $\{u^k\}$ thus converges to $\tilde{u}$.

3 Numerical Results

We implement the proposed algorithm in Matlab to solve a linear variational inequality problem and report the results.

The problem under consideration is the linear variational inequality problem with

$$F(x) = Hx + c,$$

where

$$H = \begin{bmatrix} 1 & 2 & \cdots & \cdots & 2 \\ 0 & 1 & 2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 2 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}, \quad q = (-1, -1, \cdots, -1)^T.$$

and

$$S = \{x\in \mathbf{R}^n \mid \sum_{i=1}^n x_i = 1, x_i \geqslant 0, i = 1, 2, \cdots, n.\}$$

The constraint set K is the nonnegative orthant $\mathbf{R}^n_+$ and

$$A = (1, 1, \cdots, 1), \quad b = 1.$$

This problem is a modification of the standard test problem, the linear complementarity problem LCP(H, c), i.e., LVI(H, c, S) with $S=\mathbf{R}^n_+$. The LCP(H, c) was used in many papers [10]–[12], for which Lemke's method is known to run in exponential time. The unique solution is $(0, \cdots, 0, 1)^T$. We report the computational results with the dimension varying from 8 to 2 000 and with the initial point $u^0 = (1, \cdots, 1)$. We set the parameter μ to be $30\|H+A^TA\|$. The computational results are reported in Table 1. The column 'N' denotes the dimension of the problem and the stopping criterion is

$$\|e(u^k)\| \leqslant \varepsilon,$$

where ε is set to 10^{-6}. 'IN' denotes Number of Iteration and 'CPU' denotes the CPU time in seconds. For the purpose of comparison, we also code the algorithm of Han and Lo[8], denoted as Han & Lo's method. Note also that since $K=\mathbf{R}^n_+$, the projection in the sense of the Euclidean norm is very easy to carry out. For any $z\in\mathbf{R}^n$, $P_K[z]$ is defined as componentwise

$$(P_K[z])_j = \begin{cases} z_j, & \text{if } z_j \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

Table 1 Numerical results with $u^0 = (0, \cdots, 0)^T$

		N 8	16	32	64	128	256	512	1000	2000
Han & Lo's method	IN	22	48	82	303	1 969	2 513	3 009	3 804	5 211
	CPU	0.05	0.15	0.21	0.79	6.77	12.90	31.60	84.45	193.66
Proposed Method	IN	14	16	67	136	377	746	805	854	963
	CPU	0.01	0.04	0.15	0.28	0.94	6.65	8.33	17.35	29.76

From the above table we can see that the proposed method is simple and efficient.

4 Conclusion

In this paper, we presented a simple alternating direction method for solving linear variational inequality problems. At each iteration, the method needs only a projection onto a simple set and some matrix- vector production. It is thus suitable to solve large-scale problems. We proved the global convergence of the method under the mild condition that the underlying matrix in the variational inequality problem is positive definite and the solution set is nonempty. An important future research topic is to consider if this simple method can be extended to nonlinear variational inequality problems.

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