Existences of Solutions for Second-Order Impulsive Differential Equations With Boundary Value Problems

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Abstract In this paper, a class of boundary value problem of second-order impulsive nonlinear differential equation was studied. By using the Krasnoselskii fixed point theorem, we show the existence of at least one positive solution with suitable conditions impose on the nonlinear term and impulsive functions, which generalize and improve some known results.

Key words impulsive differential equations; boundary value problems; fixed point theorem

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二阶脉冲边值问题解的存在性

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[摘要] 研究了一类二阶非线性脉冲微分方程边值问题解的存在性. 利用不动点定理,通过对非线性项和脉冲函数的适当假设,证明了至少一个正解的存在性,推广和改进了一些相应文献的结果.

[关键词] 脉冲微分方程,边值问题,不动点定理

0 Introduction

In this paper, we investigate the existence of solutions of a boundary value problem for the following second-order impulsive differential equation

$$\begin{cases} y''(t)+\phi(t)f(y(t))=0, & t\in[0,1],\ t\neq t_k,\ k=1,\cdots,p; \\ \triangle y(t_k)=I_k(y(t_k)), & k=1,\cdots,p; \\ \triangle y'(t_k)=-\dfrac{1}{1-t_k}I_k(y(t_k)), & k=1,\cdots,p; \\ y(0)=y(1)=0, \end{cases} \tag{1}$$

where $f, I_k\in C(\mathbf{R},\mathbf{R})$, $\phi(t):[0,1]\to[0,\infty)$ is continuous, $\triangle y(t_k)=y(t_k^+)-y(t_k^-)$ and $y(t_k^+)=y(t_k)$, $\triangle y'(t_k)=y'(t_k^+)-y'(t_k^-)$ and $y'(t_k^-)=y'(t_k)$. Let $PC[0,1]=\{x:[0,1]\to\mathbf{R},\ x$ is continuous for any $t\neq t_k$, left continuous at $t=t_k$ and right-hand limit at $t=t_k$ exists for $k=1,\cdots,p\}$.

In recent years, the study about order impulsive differential equation has arouse the investigator's interesting. For example, by the monotone iterative technique, the existence of solution for boundary value problems were obtained[1-3], and in [4, 5], using fixed point theorem, the authors show the existence of solution to second

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order impulsive boundary value problem.

In this paper, a new existence result for (1) is obtained by using a fixed point theorem, which is due to Krasnoselskii and Zabreiko[6]. Our conditions imposed f and I_k are very easy to verify.

1 Main Results

Next, we state the following well-known fixed point theorem[6], which is crucial to our proof.

Theorem 1.1 Let X be a Banach space and $F: X \to X$ be completely continuous. If there exists a bounded and linear operator $A: X \to X$ such that 1 is not a eigenvalue of A and

$$\lim_{\|u\| \to \infty} \frac{\| F(u) - A(u) \|}{\| u \|} = 0,$$

then F has a fixed point in X.

Consider the following impulsive integral equation

$$y(t) = \int_0^1 G(t, s)\Phi(s)f(y(s))\,\mathrm{d}s + (1-t)\sum_{0<t_k<t} \frac{I_k(y(t_k))}{1-t_k}, \tag{2}$$

where

$$G(t, s) = \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 < s < t < 1. \end{cases}$$

Lemma 1.1 $y \in PC[0, 1] \cap C^2[0, 1]$ is a solution of BVP (1) if and only if $y \in PC[0, 1]$ is a solution of the integral equation (2).

Proof Suppose that $y \in PC[0, 1]$ is a solution of (2), then for $t \neq t_k$, we have

$$y'(t) = \int_0^1 (1-s)\Phi(s)f(y(s))\,\mathrm{d}s - \int_0^t s\Phi(s)f(y(s))\,\mathrm{d}s - \sum_{0<t_k<t} \frac{I_k(y(t_k))}{1-t_k},$$

$$y''(t) = -(1-t)\Phi(t)f(y(t)) - t\Phi(t)f(y(t)) = -\Phi(t)f(y(t)).$$

For $t = t_k$, we have

$$\triangle y(t_k) = y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad \triangle y'(t_k) = y'(t_k^+) - y'(t_k^-) = -\frac{I_k(y(t_k))}{1-t_k},$$

and $y(0) = y(1) = 0$. Therefore y is a solution of (1).

On the other hand, if y is a solution of (1), then

$$y'(t) = y'(0) - \int_0^t \Phi(s)f(y(s))\,\mathrm{d}s - \sum_{0<t_k<t} \frac{I_k(y(t_k))}{1-t_k},$$

$$y(t) = y'(0)t - t\int_0^t \Phi(s)f(y(s))\,\mathrm{d}s + \int_0^t s\Phi(s)f(y(s))\,\mathrm{d}s + (1-t)\frac{I_k(y(t_k))}{1-t_k}. \tag{3}$$

By using (3) and $y(1) = 0$, we have

$$y'(0) = \int_0^1 \Phi(s)f(y(s))\,\mathrm{d}s - \int_0^1 s\Phi(s)f(y(s))\,\mathrm{d}s,$$

so

$$\begin{aligned} y(t) = {} & t\int_0^1 \Phi(s)f(y(s))\,\mathrm{d}s - t\int_0^1 s\Phi(s)f(y(s))\,\mathrm{d}s - t\int_0^t \Phi(s)f(y(s))\,\mathrm{d}s + \\ & \int_0^t s\Phi(s)f(y(s))\,\mathrm{d}s + (1-t)\frac{I_k(y(t_k))}{1-t_k} = \\ & \int_0^1 G(t, s)\Phi(s)f(y(s))\,\mathrm{d}s + (1-t)\frac{I_k(y(t_k))}{1-t_k}. \end{aligned}$$

The proof is complete.

Theorem 1.2 Assume that $f, I_k: \mathbf{R} \to \mathbf{R}$ are continuous and

$$\lim_{y \to \infty} \frac{f(y)}{y} = m, \quad \lim_{y \to \infty} \frac{I_k(y)}{y} = m, \quad k = 1, \cdots, p.$$

If

$$|m| < d = \left[\sup_{t\in[0,1]}\int_0^1 G(t,s)\Phi(s)\,ds + p\right]^{-1},$$

then the boundary value problem (1) has a solution y^* and $y^* \neq 0$, when $f(0)\neq 0$.

Proof Let the Banach space $X = PC[0,1]$ be endowed with the norm

$$\|y\| = \sup_{t\in[0,1]} |y(t)|.$$

Define integral operator $F: X\to X$ by

$$(Fy)(t) = \int_0^1 G(t,s)\Phi(s)f(y(s))\,ds + (1-t)\sum_{0<t_k<t}\frac{I_k(y(t_k))}{1-t_k},$$

then it is well known that F is completely continuous[4].

In order to apply Theorem 1.1 to establish the existence result of the boundary value problem (1), we consider the following boundary value problem

$$\begin{aligned} &y''(t) + m\Phi(t)y(t) = 0,\quad \triangle y(t_k) = my(t_k),\\ &\triangle y'(t_k) = -\frac{m}{1-t_k}y(t_k),\quad y(0) = y(1) = 0. \end{aligned}\tag{4}$$

Define $A: X\to X$ by

$$(Ay)(t) = m\left[\int_0^1 G(t,s)\Phi(s)y(s)\,ds + \sum_{0<t_k<t}\frac{(1-t)y(t_k)}{1-t_k}\right],\quad t\in[0,1],$$

then it is easy to that A is a completely continuous (so bounded) linear operator and that solutions of the boundary value problem (4) are fixed points of the operator A and conversely.

First, we claim that 1 is not a eigenvalue of A. In fact, if $m = 0$, then it is obvious that the boundary value (4) has no nontrivial solution.

If $m\neq 0$ and the boundary value problem (4) has a nontrivial solution y, then $\|y\| > 0$ and

$$\begin{aligned}\|y\| = \|Ay\| &= \sup_{[0,1]}\left|m\left[\int_0^1 G(t,s)\Phi(s)y(s)\,ds + \sum_{0<t_k<t}\frac{(1-t)y(t_k)}{1-t_k}\right]\right| =\\ &|m|\sup_{\in[0,1]}\left|\int_0^1 G(t,s)\Phi(s)y(s)\,ds + \sum_{0<t_k<t}\frac{(1-t)y(t_k)}{1-t_k}\right| \leqslant\\ &|m|\sup_{\in[0,1]}\left[\int_0^1 G(t,s)\Phi(s)|y(s)|\,ds + \sum_{0<t_k<t}|y(t_k)|\right] \leqslant\\ &|m|\cdot\|y\|\sup_{\in[0,1]}\left[\int_0^1 G(t,s)\Phi(s)\,ds + p\right] < \frac{1}{d}\cdot\|y\|\cdot d = \|y\|,\end{aligned}$$

which is impossible. So 1 is not a eigenvalue of A.

Next, we will prove that

$$\lim_{\|y\|\to\infty}\frac{\|F(y) - A(y)\|}{\|y\|} = 0.$$

In fact, for any $\varepsilon > 0$, since $\lim\limits_{y\to\infty}\dfrac{f(y)}{y} = m$, $\lim\limits_{y\to\infty}\dfrac{I_k(y)}{y} = m$, then there exist a number $M_1 > 0$ such that

$$|f(y) - my| < \varepsilon|y|,\quad |I_k(y) - my| < \varepsilon|y|,\quad \text{for } |y| > M_1,\ k = 1,\cdots,p.$$

Let

$$M = \max_{|y|\leqslant M_1}\{|f(y)|, |I_1(y)|, \cdots, |I_k(y)|\},$$

and choose $L > m_1$ such that

$$\frac{M + |m|M_1}{L} < \varepsilon.$$

Then for any $y\in X$ and $\|y\| > L$, we have

(i) When $t\in[0,1]$ and $|y| < M_1$, if follows that

$$|f(y) - my| \leqslant |f(y)| + |m||y| \leqslant M + |m|M_1 < \varepsilon L < \varepsilon\|y\|,$$

$$|I_k(y(t_k)) - my(t_k)| \leqslant M + |m| M_1 < \varepsilon L < \varepsilon \|y\|. \tag{5}$$

(ii) When $t \in [0, 1]$ and $|y| > M_1$, we know that

$$|f(y) - my| < \varepsilon |y| \leqslant \varepsilon \|y\|, \quad |I_k(y(t_k)) - my(t_k)| < \varepsilon |y| \leqslant \varepsilon \|y\|. \tag{6}$$

So we can conclude from (5) and (6) that for $t \in [0, 1]$,

$$|f(y) - my| < \varepsilon |y| \leqslant \varepsilon \|y\|, \quad |I_k(y(t_k)) - my(t_k)| \leqslant \varepsilon \|y\|. \tag{7}$$

From (7), we have

$$\|F(y) - A(y)\| = \sup_{t \in [0, 1]} \left| \int_0^1 G(t, s)\phi(s)[f(y) - my]\,ds + (1-t)\sum_{0<t_k<t} \frac{I_k(y) - my}{1 - t_k} \right| \leqslant$$

$$\sup_{t \in [0, 1]} \left[\int_0^1 G(t, s)\phi(s)|f(y) - my|\,ds + \sum_{0<t_k<t} \frac{|I_k(y) - my|}{1 - t_k} \right] \leqslant$$

$$\varepsilon \|y\| \cdot \sup_{t \in [0, 1]} \left[\int_0^1 G(t, s)\phi(s)\,ds + p \right] = \frac{\varepsilon}{d} \|y\|.$$

So

$$\lim_{\|y\| \to \infty} \frac{\|F(y) - A(y)\|}{\|y\|} = 0.$$

Then, it follows from Theorem 1.1 that F has a fixed point $y^* \in X$, i.e., y^* is a solution of the boundary value problem (1). Further, we can assert that y^* is nontrivial when $f(0) \neq 0$.

2 Example

Consider the boundary value problem

$$\begin{cases} y''(t) + f(y(t)) = 0, & t \in (0, 1), \ t \neq t_1 = 1/2, \\ \triangle y(t_1) = I_1(y(t_1)), & \triangle y'(t_1) = -2I_1(y(t_1)), \\ y(0) = y(1) = 0. \end{cases} \tag{8}$$

By Theorem 1.2, it is easy to see that if f, I_k: $\mathbf{R} \to \mathbf{R}$ are continuous and

$$\left| \lim_{s \to \infty} \frac{f(s)}{s} \right| < 9/8, \quad \left| \lim_{s \to \infty} \frac{I_k(s)}{s} \right| < 9/8,$$

then the boundary value problem (8) has a solution. Since a simple calculation shows that $d = 9/8$.

Remark 2.1 The results of [4, 5] cannot appiled to the boundary value problem (8) since f, I_k may not be sublinear here.

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