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Generalized Fuzzy p – Pseudonorm and Locally Semi-Convex I – Topological Vector Spaces

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Abstract In this paper, we give a new definition of locally semi-convex I – topological vector spaces and rename locally semi-convex fuzzy topological linear spaces as locally semi-convex I – topological vector spaces of (QL) – type. The relation between these two definitions is studied. We introduce the notion of generalized fuzzy p – pseudonorm, and prove that every locally semi-convex I – vector topology can be determined by a family of generalized fuzzy p – pseudonorms.

Key words semi-convex fuzzy set, locally semi-convex I – topological vector space, generalized fuzzy p – pseudonorm
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广义模糊 p – 伪范数与局部半凸 I – 拓扑向量空间

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[摘要] 给出局部半凸 I – 拓扑向量空间的一个新定义,并重新命名“局部半凸模糊拓扑线性空间”为“(QL) – 型局部半凸 I – 拓扑向量空间”,研究这两种定义之间的关系,引入广义模糊 p – 伪范数的概念,证明每个局部半凸 I – 拓扑向量空间可通过一族广义模糊 p – 伪范数来刻画.

[关键词] 半凸模糊集, 局部半凸 I – 拓扑向量空间, 广义模糊 p – 伪范数

The concept of fuzzy topological vector space was introduced rationally by Katsaras in 1981[1]. Since then, many researches on fuzzy topological vector spaces have been carried out[2-15]. In 1992, Fang and Yan[4] introduced and studied the locally semi-convex fuzzy topological vector spaces, which are a special subclass of (QL)–type fuzzy topological vector spaces[9]. This leads naturally to the following question: Can we define the local semi-convexity to more general fuzzy topological vector spaces?

According to the standardized terminology of Höhle U and Rodabaugh S E[16], fuzzy topological vector space have been called I–topological vector space, where $I=[0,1]$. In this paper, we give a new definition of the locally semi-convex I–topological vector space, and prove that the locally semi-convex I–topological vector space defined by Fang and Yan[4] is a special case of it. We also give a characterization of the locally semi-convex I–topological vector space by means of a family of generalized fuzzy p–pseudonorms.

1 Preliminaries

Throughout this paper, let X be a vector space over the field $\mathbf{K}$ ($\mathbf{R}$ or $\mathbf{C}$), θ denote the zero element of X, $I=[0,1]$ and I^X denote a family of all fuzzy subsets of X. A fuzzy subset which take the constant value r on X

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$(0\leqslant r\leqslant 1)$ is denoted by $\underline{r}$. A fuzzy subset of X is called a fuzzy point[17], denoted by x_λ, if it takes value 0 at $y\in X\setminus\{x\}$ and its value at x is λ. The set of all fuzzy points on X is denoted by $\mathrm{Pt}(I^X)$. A fuzzy point x_λ is said to be quasi-coincident with a fuzzy subset U, denoted by $x_\lambda \tilde{\in} U$, if $U(x) > 1-\lambda$. Other notions not mentioned here, for example, I-topology, Q-neighborhood (base) of a fuzzy point x_λ, we refer to [17]. We appoint $\inf\varnothing = +\infty$.

Definition 1[1] Let $A, B\in I^X$ and $k\in\mathbf{K}$. Then $A+B$ and kA are defined by

$$(A+B)(x) = \vee\{A(s)\wedge B(t): s+t=x\},$$
$$(kA)(x) = A(x/k),\quad \text{whenever } k\neq 0,$$
$$(0A)(x) = \begin{cases} \vee_{z\in X}A(z), & \text{if } x=\theta, \\ 0, & \text{if } x\neq\theta, \end{cases}$$

respectively. In particular, for $x_\lambda, y_\mu\in \mathrm{Pt}(I^X)$, we have

$$x_\lambda + y_\mu = (x+y)_{\lambda\wedge\mu},\quad kx_\lambda = (kx)_\lambda.$$

Definition 2[1] A stratified I-topology $\mathscr{T}$ on X is said to be an I-vector topology, if the following two mappings are continuous

$$f: X\times X\to X,\quad (x, y)\to x+y,$$
$$g: \mathbf{K}\times X\to X,\quad (k, x)\to kx,$$

where $\mathbf{K}$ is equipped with the I-topology induced by the usual topology, $X\times X$ and $\mathbf{K}\times X$ are equipped with the corresponding product I-topologies.

A vector space X with an I-vector topology $\mathscr{T}$, denoted by $(X, \mathscr{T})$, is called an I-topological vector space (for short, an I-tvs).

Definition 3[9] An I-tvs $(X, \mathscr{T})$ is said to be an I-tvs of (QL)-type, if there exists a family $\mathscr{U}$ of fuzzy sets on X such that for each $\lambda\in(0, 1]$,

$$\mathscr{U}_\lambda = \left\{ U\cap \underline{r}: U\in\mathscr{U}, r\in(1-\lambda, 1]\right\}$$

is a Q-neighborhood base of θ_λ in $(X, \mathscr{T})$. The family $\mathscr{U}$ is called a Q-prebase for $\mathscr{T}$.

Lemma 1[3] Let $(X, \mathscr{T})$ be an I-tvs and $\mathscr{U}_\lambda$ a Q-neighborhood base of θ_λ in X $(\lambda\in(0, 1])$. Then it has the following properties:

(1) If $U\in\mathscr{U}_\lambda$ or $U=\underline{r}\,(r>1-\lambda)$, then there exists $\lambda_0\in(0,\lambda)$ such that for any $\mu\in[\lambda_0, 1]$ there exists $V\in\mathscr{U}_\mu$ with $V\subset U$.

(2) If $U, V\in\mathscr{U}_\lambda$, then there exists $W\in\mathscr{U}_\lambda$ such that $W\subset U\cap V$.

(3) If $U\in\mathscr{U}_\lambda$, then there exists $V\in\mathscr{U}_\lambda$ such that $V+V\subset U$.

(4) If $U\in\mathscr{U}_\lambda$, then there exists $V\in\mathscr{U}_\lambda$ such that $tV\subset U$ for all $t\in\mathbf{K}$ with $|t|\leqslant 1$.

(5) If $U\in\mathscr{U}_\lambda$ and $x\in X$, then there exists a positive number α such that $x_\lambda\tilde{\in}\alpha U$.

Conversely, let X be a vector space over $\mathbf{K}$, and $\mathscr{U}_\lambda(\lambda\in(0, 1])$ a family of fuzzy sets on X satisfies the above conditions (1) ~ (5). Then there exists a unique I-topology $\mathscr{T}$ on X such that $(X, \mathscr{T})$ is an I-tvs and $\mathscr{U}_\lambda$ is a Q-neighborhood base of θ_λ.

Definition 4[4] Let X be a vector space. A fuzzy subset U of X is called semi-convex fuzzy set iff there exists $k\geqslant 1$, such that $U+U\subset kU$, k is called semi-convex coefficient of U.

Obviously, every constant value fuzzy set $\underline{r}$ is semi-convex. Moreover, we easily prove the following lemma.

Lemma 2 If $U_i\,(i=1, \cdots, n)$ are balanced and semi-convex fuzzy sets on X, then $\bigcap_{i=1}^n U_i$ is also balanced and semi-convex.

2 The Relation Between Two Definitions of Locally Semi-Convex I-tvs

Definition 5[4] An I-tvs $(X, \mathscr{T})$ is called locally semi-convex if there exists a family $\mathscr{U}$ of balanced and semi-convex fuzzy sets on X such that for each $\lambda\in(0, 1]$,

$$\mathscr{U}_\lambda = \{U \cap \underline{r}: U \in \mathscr{U}, r \in (1-\lambda, 1]\}$$

is a Q-neighborhood base of θ_λ.

Remark 1 Obviously, an L-tvs $(X, \mathscr{T})$ is locally semi-convex in the sense of Definition 5, if and only if $(X, \mathscr{T})$ is an L-tvs of (QL)-type and has a balanced, semi-convex Q-prebase for $\mathscr{T}$.

Now, we give a new definition of locally semi-convex L-tvs, as follows.

Definition 6 An L-tvs $(X, \mathscr{T})$ is called locally semi-convex if it has a Q-neighborhood base of θ_λ consisting of balanced and semi-convex fuzzy sets for each $\lambda \in (0, 1]$.

Remark 2 For distinction, we now rename the locally semi-convex L-tvs in the sense of Definition 5 as "the locally semi-convex L-tvs of (QL)-type". By Definition 5 and 6, it is easy to see that every locally semi-convex L-tvs of (QL)-type is certainly locally semi-convex. But the following example show that the converse is false.

Example 1 Let $X = \mathbf{R}$. For each $\lambda \in (0, 1]$, we define $\mathscr{U}_\lambda$ as follows:

If $0 < \lambda \leqslant \frac{1}{2}$, then $\mathscr{U}_\lambda = \{\underline{r}: r \in (1-\lambda, 1]\}$;

If $\frac{1}{2} < \lambda \leqslant 1$, then $\mathscr{U}_\lambda = \{(-a, a) \cap \underline{t}: 1-\lambda < t \leqslant 1, a > 0\}$.

Obviously, $(-a, a)$ is balanced. From $(-a, a) + (-a, a) = 2(-a, a)$, we know that $(-a, a)$ is semi-convex. So, by Lemma 2, $(-a, a) \cap \underline{t}$ $(a > 0, 1-\lambda < t \leqslant 1)$ is balanced and semi-convex. This shows that every member in $\mathscr{U}_\lambda$ is balanced and semi-convex for all $\lambda \in (0, 1]$. Moreover, it is easy to verify that $\mathscr{U}_\lambda$ satisfies the conditions (1) - (5) in Lemma 1. Hence there exists a unique L-topology on X, denoted by $N_c(T)$, such that $(X, N_c(T))$ is a locally semi-convex L-tvs and $\mathscr{U}_\lambda$ is a Q-neighborhood base of θ_λ. But, we can prove that $(X, N_c(T))$ is not of (QL)-type, which implies $(X, N_c(T))$ is not a locally semi-convex L-tvs of (QL)-type.

In fact, assume that $(X, N_c(T))$ is of (QL)-type. Then there exists a family of fuzzy sets $\mathscr{B}$ such that for each $\lambda \in (0, 1]$, $\mathscr{B}_\lambda = \{B \cap \underline{r}: B \in \mathscr{B}, r \in (1-\lambda, 1]\}$ is a Q-neighborhood base of θ_λ. Let $B \in \mathscr{B}$. Since for each $\lambda \in (0, \frac{1}{2}]$, B is a Q-neighborhood of θ_λ and $\mathscr{U}_\lambda = \{\underline{r}: r \in (1-\lambda, 1]\}$ is a Q-neighborhood base of θ_λ, there exists $r_0 \in (1-\lambda, 1]$ such that $\underline{r_0} \subset B$, which implies that $B = X$. This shows that for each $\lambda \in (0, 1]$, $\{\underline{r}: r \in (1-\lambda, 1]\}$ is a Q-neighborhood base of θ_λ. On the other hand, since for each $\lambda \in (\frac{1}{2}, 1]$, $U = (-1, 1)$ is a Q-neighborhood of θ_λ, there exists $r \in (1-\lambda, 1]$ such that $\underline{r} \subset U$, which is a contradiction. Therefore $(X, N_c(T))$ is not of (QL)-type.

3 Generalized Fuzzy p-Pseudonorm and Locally Semi-Convex L-tvs

Definition 7[18] A nonempty set D is said to have the stratified structure $\{D_\lambda: \lambda \in (0, 1]\}$ if $D = \bigcup_{\lambda \in (0,1]} D_\lambda$, and satisfies the following conditions:

(D-1) $0 < \lambda < \mu \leqslant 1$ implies that $D_\lambda \subset D_\mu$;

(D-2) For each $d \in D_\lambda$, there exists $\lambda_0 \in (0, \lambda)$ such that $d \in D_{\lambda_0}$.

By Definition 7, it is easy to obtain the following lemma.

Lemma 3 If $D = \bigcup_{\lambda \in (0,1]} D_\lambda$ has the stratified structure, then for each $d \in D_\lambda$, there exists some $\lambda_0 \in (0, \lambda)$ such that $d \in D_\mu$ for all $\mu \in [\lambda_0, 1]$.

Definition 8 A mapping $f: \mathrm{Pt}(I^X) \to [0, +\infty)$ is called a generalized fuzzy p-pseudonorm on X, if it satisfies the following conditions:

(GP-1) There exists $\lambda \in (0, 1]$, such that $f(\theta_\lambda) = 0$ and $f(x_\lambda) < +\infty$, $\forall x \in X$;

(GP-2) There exists $p \in (0, 1]$ such that $f(\alpha x_\lambda) = |\alpha|^p f(x_\lambda)$, $\forall x_\lambda \in \mathrm{Pt}(I^X)$, $\alpha \in \mathbf{K}$, $\alpha \neq 0$;

(GP-3) For each $\lambda \in (0, 1]$, $f(x_\lambda + y_\lambda) \leqslant f(x_\lambda) + f(y_\lambda)$, $x, y \in X$;

(GP-4) $f(\theta_\lambda) \leqslant f(x_\lambda)$, $f(x_\lambda) = \inf_{0<\mu<\lambda} f(x_\mu)$, $\forall x \in X$.

Lemma 4 Let f be a generalized fuzzy p-pseudonorm on X and $t > 0$. Define a fuzzy set $U_{f,t}$ on X by

$$U_{f,t}(x) = \sup\{1-\lambda: f(x_\lambda) < t\}. \tag{1}$$

Then $U_{f,t}$ has the following properties:

(i) $x_\lambda \in U_{f,t} \Leftrightarrow f(x_\lambda) < t$;

(ii) $U_{f,t} = \bigcup_{0<s<t} U_{f,s}$;

(iii) $U_{f,t} + U_{f,t} \subset 2^{\frac{1}{p}} U_{f,t}$;

(iv) $U_{f,\frac{t}{2}} + U_{f,\frac{t}{2}} \subset U_{f,t}$.

Proof (1) Let $x_\lambda \tilde{\in} U_{f,t}$, i.e. $U_{f,t}(x) > 1-\lambda$. By (1), there exists $\lambda_0 \in (0, \lambda)$, such that $U_{f,t}(x) > 1-\lambda_0$ and $f(x_{\lambda_0}) < t$. By (GP-4), we have $f(x_\lambda) \leqslant f(x_{\lambda_0}) < t$. Conversely, let $f(x_\lambda) < t$, then there exists $0 < \mu < \lambda$, such that $f(x_\mu) < t$ by (GP-4). It follows from (1) that $U_{f,t}(x) \geqslant 1-\mu > 1-\lambda$, i.e., $x_\lambda \in U_{f,t}$.

By using (i), it is easy to verify (ii) ~ (iv). The proof is omitted.

Theorem 1 Let X be a vector space over $\mathbf{K}$, and let $\{\|\cdot\|_d: d\in D\}$ be a family of generalized fuzzy p_d-pseudonorms $(0 < p_d \leqslant 1, d \in D)$, where D has the stratified structure $\{D_\lambda: \lambda \in (0,1]\}$ and satisfies the condition:

(GP-1)′ When $d \in D_\lambda$, $\|\theta_\lambda\|_d = 0$ and $\|x_\lambda\|_d < +\infty$ for all $x \in X$.

Then there exists a unique I-topology $\mathscr{T}$ on X such that $(X, \mathscr{T})$ is a locally semi-convex I-tvs, and

$$\mathscr{U}_\lambda = \left\{ \left(\bigcap_{i=1}^{n} U_{d_i,t} \right) \cap \underline{r}: t > 0, r \in (1-\lambda, 1], d_i \in D_\lambda, i = 1, \cdots, n; n \in \mathbf{N} \right\} \tag{2}$$

is a Q-neighborhood base of θ_λ, where $U_{d_i,t} = U_{\|\cdot\|_{d_i},t}$ is defined by (1). The I-topology $\mathscr{T}$ is called the I-topology determined by the family of generalized fuzzy p_d-pseudonorms $\{\|\cdot\|_d: d \in D\}$.

Proof We first prove that $\{\mathscr{U}_\lambda\}_{\lambda\in(0,1]}$ satisfies the conditions (1) ~ (5) in Lemma 1.

(1) Let $U = (\bigcap_{i=1}^{n} U_{d_i,t}) \cap \underline{r} \in \mathscr{U}_\lambda$, then $d_i \in D_\lambda$ $(i = 1, \cdots, n)$, $r > 1-\lambda$. From Lemma 3, there exists $\lambda_* \in (0, \lambda)$, such that $d_i \in D_\mu$ for all $\mu \in [\lambda_*, 1]$, $i = 1, \cdots, n$. We put $V = (\bigcap_{i=1}^{n} U_{d_i,\frac{t}{2}}) \cap \underline{r}$ and take $\lambda_0 \in (\max\{\lambda_*, 1-r\}, \lambda)$, then $V \in \mathscr{U}_\mu$ and $V \subset U$ for all $\mu \in [\lambda_0, 1]$.

Let $U = \underline{r}$ with $r \in (1-\lambda, 1]$. Arbitrarily taking a $\lambda_0 \in (1-r, \lambda)$, then for any $\mu \in [\lambda_0, 1]$, $t > 0$ and $d_i \in D_\mu$ $(i = 1, \cdots, n)$, $V = (\bigcap_{i=1}^{n} U_{d_i,t}) \cap \underline{r} \in \mathscr{U}_\mu$ and $V \subset \underline{r}$.

(2) Let $U = (\bigcap_{i=1}^{n} U_{d_i,t}) \cap \underline{r_1}$, $V = (\bigcap_{j=1}^{m} U_{e_j,s}) \cap \underline{r_2} \in \mathscr{U}_\lambda$, where $d_i, e_j \in D_\lambda$, $t, s > 0$ and $r_1, r_2 \in (1-\lambda, 1]$. Put $r = \min\{r_1, r_2\}$, $\sigma = \min\{s, t\}$ and

$$U_{c_k,\sigma} = \begin{cases} U_{d_k,\sigma}, & k = 1, \cdots, n, \\ U_{e_{k-n},\sigma}, & k = n+1, \cdots, n+m. \end{cases}$$

Then we have $W = (\bigcap_{k=1}^{m+n} U_{c_k,\sigma}) \cap \underline{r} \in \mathscr{U}_\lambda$ and

$$W = (\bigcap_{k=1}^{m+n} U_{c_k,\sigma}) \cap \underline{r} \subset [(\bigcap_{i=1}^{n} U_{d_i,t}) \cap \underline{r_1}] \cap [(\bigcap_{j=1}^{m} U_{e_j,s}) \cap \underline{r_2}] = U \cap V.$$

(3) Let $U = (\bigcap_{i=1}^{n} U_{d_i,t}) \cap \underline{r} \in \mathscr{U}_\lambda$, then $d_i \in D_\lambda$ $(i = 1, \cdots, n)$, $r > 1-\lambda$. By Lemma 4(iv), $U_{d_i,\frac{t}{2}} + U_{d_i,\frac{t}{2}} \subset U_{d_i,t}$ $(i = 1, \cdots, n)$. Put $V = (\bigcap_{i=1}^{n} U_{d_i,\frac{t}{2}}) \cap \underline{r}$, then $V \in \mathscr{U}_\lambda$ and $V + V \subset U$.

(4) Let $U = (\bigcap_{i=1}^{n} U_{d_i,t}) \cap \underline{r} \in \mathscr{U}_\lambda$, then $d_i \in D_\lambda$ $(i = 1, \cdots, n)$, $r \in (1-\lambda, 1]$. We will prove that U is balanced. To this end, we need only to show that for each $d \in D_\lambda$ and $t > 0$, $U_{d,t}$ is balanced.

In fact, if $0 < |\alpha| \leqslant 1$, then we have the following implications:

$$x_\lambda \tilde{\in} \alpha U_{d,t} \Rightarrow \|\frac{1}{\alpha} x_\lambda\|_d < t \Rightarrow \frac{1}{|\alpha|^{p_d}} \|x_\lambda\|_d < t \Rightarrow \|x_\lambda\|_d < |\alpha|^{p_d} t \leqslant t \Rightarrow x_\lambda \tilde{\in} U_{d,t},$$

so, $\alpha U_{d,t} \subset U_{d,t}$; If $\alpha = 0$, then $0 \cdot U_{d,t} = \theta_{\sup_{x\in X} U_{d,t}(x)}$. For $x \in X$, if $U_{d,t}(x) = \mu > 0$, then $\forall\, 0 < \varepsilon < \mu$, we have $x_{1-\mu+\varepsilon} \in U_{d,t}$. Therefore, $\|x_{1-\mu+\varepsilon}\|_d < t$. By (GP-4), $\|\theta_{1-\mu+\varepsilon}\|_d \leqslant \|x_{1-\mu+\varepsilon}\|_d < t$. By Lemma 4, we have $U_{d,t}(\theta) > \mu - \varepsilon$, which shows that $U_{d,t}(\theta) \geqslant \mu = U_{d,t}(x)$ $(\forall x \in X)$, and so $0 \cdot U_{d,t} \subset U_{d,t}$. Hence $U_{d,t}$

is balanced.

(5) Let $U=(\bigcap_{i=1}^{n}U_{d_i,t})\cap \underline{r}\in\mathscr{U}_\lambda$, where $d_i\in D_\lambda$, $r\in(1-\lambda,1]$ and $t>0$. By (GP-1)$'$, $\|x_\lambda\|_{d_i}<+\infty$, taking $\alpha=\max\{[(\|x_\lambda\|_{d_i}+1)/t]^{\frac{1}{p_{d_i}}}: i=1,\cdots,n\}$, it follows from (GP-2) that

$$\|\frac{1}{\alpha}x_\lambda\|_{d_i}=\frac{1}{\alpha^{p_{d_i}}}\|x_\lambda\|_{d_i}\leqslant\frac{t}{\|x_\lambda\|_{d_i}+1}\|x_\lambda\|_{d_i}<t,\quad i=1,\cdots,n.$$

So we have $\frac{1}{\alpha}x_\lambda\in(\bigcap_{i=1}^{n}U_{d_i,t})\cap\underline{r}$, i.e., $x_\lambda\in\alpha[(\bigcap_{i=1}^{n}U_{d_i,t})\cap\underline{r}]=\alpha U$.

Thus, from Lemma 1 we know that there exists a unique L-topology $\mathscr{T}$ on X such that $(X,\mathscr{T})$ is an L-tvs and $\mathscr{U}_\lambda$ is a Q-neighborhood base of θ_λ.

By the proof of (4), we have known that every member in $\mathscr{U}_\lambda$ is balanced. In the following we will prove that every member in $\mathscr{U}_\lambda$ is semi-convex. From Lemma 4(ii), we have $U_{d,t}+U_{d,t}\subset kU_{d,t}$, where $k=2^{\frac{1}{p_d}}$, hence $U_{d,t}$ is semi-convex. So, by Lemma 2, we conclude that every member in $\mathscr{U}_\lambda$ is semi-convex. Therefore $(X,\mathscr{T})$ is a locally semi-convex L-tvs.

Let U be a balanced and semi-convex fuzzy subset on X. By the balance and semi-convexity of U, there exists $K=K_U>3$, such that $U+U+U\subset KU$. We put $p_U=\log_K 2$ and define Φ_U: $\mathrm{Pt}(L^X)\to[0,\infty)$ by

$$\Phi_U(x_\lambda)=\inf\{t^{p_U}: x_\lambda\in tU,\quad t>0\}. \tag{3}$$

Lemma 5 Let U be a balanced and semi-convex fuzzy subset on X. Define the mapping $\|x_\lambda\|_U$: $\mathrm{Pt}(L^X)\to[0,\infty)$ as follows:

$$\|x_\lambda\|_U=\inf\left\{\sum_{i=1}^{n}\Phi_U(x^{(i)}_{\lambda_i}):\ \sum_{i=1}^{n}x^{(i)}=x,\ \lambda=\wedge_{i=1}^{n}\lambda_i,\ n\in\mathbf{N}\right\}, \tag{4}$$

where Φ_U is defined by (3). Then

$$\frac{1}{2}\Phi_U(x_\lambda)\leqslant\|x_\lambda\|_U\leqslant\Phi_U(x_\lambda),\ \forall x_\lambda\in\mathrm{Pt}(L^X).$$

Proof It is obvious that $\|x_\lambda\|_U\leqslant\Phi_U(x_\lambda)$. If $\|x_\lambda\|_U=+\infty$, then $\Phi_U(x_\lambda)=+\infty$, and so $\frac{1}{2}\cdot\Phi_U(x_\lambda)=+\infty=\|x_\lambda\|_U$. If $\|x_\lambda\|_U<+\infty$, then as the proof of Proposition 1.2 in [2], we can prove that $\frac{1}{2}\cdot\Phi_U(x_\lambda)\leqslant\|x_\lambda\|_U$.

From Theorem 3.1 and Example 4.2 in [18], we have

Lemma 6 Let $(X,\mathscr{T})$ be a locally semi-convex L-tvs and $\mathscr{U}_\lambda$ be a balanced and semi-convex Q-neighborhood base of $\theta_\lambda(\lambda\in(0,1])$, and let $D_\lambda=\bigcup_{0<\alpha<\lambda}\mathscr{U}_\alpha$. Then D_λ is still a balanced and semi-convex Q-neighborhood base of θ_λ, and D has the stratified structure $\{D_\lambda: \lambda\in(0,1]\}$.

Theorem 2 Let $(X,\mathscr{T})$ be a locally semi-convex L-tvs. Then $\mathscr{T}$ can be determined by a family of generalized fuzzy p_d-pseudonorms $\{\|\cdot\|_d: d\in D\}$ satisfying the condition (GP-1)$'$, where the set D has the stratified structure $\{D_\lambda: \lambda\in(0,1]\}$.

Proof Since $(X,\mathscr{T})$ is a locally semi-convex L-tvs, it has a Q-neighborhood base $\mathscr{U}_\lambda$ of θ_λ consisting of balanced and semi-convex fuzzy sets for each $\lambda\in(0,1]$. Without loss of generality, by Lemma 6, we can suppose that $\mathscr{U}_\lambda\subset\mathscr{U}_\mu$ when $0<\lambda<\mu\leqslant1$, and $\mathscr{U}=\bigcup_{\lambda\in(0,1]}\mathscr{U}_\lambda$ has the stratified structure $\{\mathscr{U}_\lambda: \lambda\in(0,1]\}$. By Lemma 5, for each $U\in\mathscr{U}$, we define Φ_U, $\|\cdot\|_U$: $\mathrm{Pt}(L^X)\to[0,\infty)$ by (3) and (4) respectively. Now we prove that $\{\|\cdot\|_U: U\in\mathscr{U}\}$ is the family of generalized fuzzy p_d-pseudonorms on X satisfying the condition (GP-1)$'$.

(GP-1)$'$ If $U\in\mathscr{U}_\lambda$, then $\theta_\lambda\tilde{\in}tU$ for all $t>0$, so $\Phi_U(\theta_\lambda)=0$, thus $\|\theta_\lambda\|_U=0$. Moreover, by (5) of Lemma 1, for each $x\in X$, there exists $\alpha>0$ such that $x_\lambda\in\alpha U$, which implies that $\Phi_U(x_\lambda)\leqslant\alpha^{p_U}<+\infty$, by Lemma 5, we have $\|x_\lambda\|_U<+\infty$.

(GP-2) Let $U\in\mathscr{U}$ and $\alpha\neq0$. Then we can prove $\Phi_U(\alpha x_\lambda)=|\alpha|^{p_U}\Phi_U(x_\lambda)$. In fact, by the balance of

U, we have $x_\lambda \tilde{\in} (t/\alpha)U \Leftrightarrow x_\lambda \tilde{\in} (t/|\alpha|)U$, so

$$\Phi_U(\alpha x_\lambda) = \inf\{t^{p_U} \mid \alpha x_\lambda \in tU, \ t>0\} = |\alpha|^{p_U} \inf\left\{\left(\frac{t}{|\alpha|}\right)^{p_U} : x_\lambda \tilde{\in} \frac{t}{|\alpha|}U, \ t>0\right\} = |\alpha|^{p_U}\Phi_U(x_\lambda).$$

This implies that $\|\alpha x_\lambda\|_U = |\alpha|^{P_U}\|x_\lambda\|_U$.

(GP-3) Without loss of generality, suppose $\|x_\lambda\|_U$, $\|y_\lambda\|_U < +\infty$. By (4), for any $\varepsilon > 0$, there exist $\sum_{i=1}^{n} x^{(i)} = x$, $\sum_{j=1}^{m} y^{(j)} = y$, $\sum_{i=1}^{n}\lambda^{(i)} = \lambda$, $\sum_{j=1}^{m}\mu^{(j)} = \lambda$ such that $\sum_{i=1}^{n}\Phi_U(x^{(i)}_{\lambda_i}) < \|x_\lambda\|_U + \varepsilon/2$, $\sum_{j=1}^{m}\Phi_U(y^{(j)}_{\mu_j}) < \|y_\lambda\|_U + \varepsilon/2$. Note that $\sum_{i=1}^{n}x^{(i)} + \sum_{j=1}^{m}y^{(j)} = x+y$, $\sum_{i=1}^{n}\lambda^{(i)} \wedge \sum_{j=1}^{m}\mu^{(j)} = \lambda$, therefore,

$$\|x_\lambda + y_\lambda\|_U \leqslant \sum_{i=1}^{n}\Phi_U(x^{(i)}_{\lambda_i}) + \sum_{j=1}^{m}\Phi_U(y^{(j)}_{\mu_j}) < \|x_\lambda\|_U + \|y_\lambda\|_U + \varepsilon.$$

By the arbitrariness of ε, we have $\|x_\lambda + y_\lambda\|_U \leqslant \|x_\lambda\|_U + \|y_\lambda\|_U$.

(GP-4) For each $U \in \mathscr{U}$, U is normal since it is balanced, and so $x_\lambda \in tU$ implies that $\theta_\lambda \in tU$ for all $x \in X$ and $\lambda \in (0, 1]$. By (3), we have $\Phi_U(x_\lambda) \leqslant \Phi_U(\theta_\lambda)$, and then $\|x_\lambda\|_U \geqslant \|\theta_\lambda\|_U$ by (4).

Obviously, for each $\mu \in (0, \lambda)$, $x_\mu \tilde{\in} tU \Rightarrow x_\lambda \tilde{\in} tU$, and so $\Phi_U(x_\lambda) \leqslant \Phi_U(x_\mu)$. Note that (4) is equivalent to the following expression

$$\|x_\lambda\|_U = \inf\left\{\sum_{i=1}^{n}\Phi_U(x^{(i)}_{\lambda_i}) : \exists\, 1 \leqslant i_0 \leqslant n, \ \lambda_{i_0} = \lambda, \ i \leqslant i_0, \ \lambda_i = 1, \right.$$
$$\left. \sum_{i=1}^{n}x^{(i)} = x, \ n \in \mathbf{N}\right\}. \tag{5}$$

Hence $\|x_\lambda\|_U \leqslant \|x_\mu\|_U$.

If $\|x_\lambda\|_U = +\infty$, then we have $\|x_\mu\|_U = \infty$ for each $0 < \mu < \lambda$, then $\|x_\lambda\|_U = \inf_{0<\mu<\lambda}\|x_\mu\|_U$ holds. If $\|x_\lambda\|_U < +\infty$, then there exists $n_0 \in \mathbf{N}$ for all $\varepsilon > 0$, such that $\sum_{i=1}^{n_0}\Phi_U(x^{(i)}_{\lambda_i}) < \|x_\lambda\|_U + \varepsilon/2$, where $\sum_{i=1}^{n_0}x^{(i)} = x$, $\wedge_{i=1}^{n_0}\lambda^{(i)} = \lambda$. By the definition of Φ_U, there exists $t_i > 0$, such that $t_i^{P_U} < \Phi_U(x^{(i)}_{\lambda_i}) + \varepsilon/2(n_0 + 1)$ and $x^{(i)}_{\lambda_i} \in t_iU$, so $t_iU(x^{(i)}) > 1 - \lambda_i$. Then there exists $0 < \mu_i < \lambda_i$ satisfying $t_iU(x^{(i)}) > 1 - \mu_i$, $i = 1, \cdots, n$, then $x^{(i)}_{\mu_i} \tilde{\in} t_iU$, therefore $\Phi_U(x^{(i)}_{\mu_i}) \leqslant t_i^{P_U}$. Put $\mu = \wedge_{i=1}^{n_0}\mu^{(i)}$, obviously, $0 < \mu < \lambda$ and

$$\|x_\mu\|_U \leqslant \sum_{i=1}^{n_0}\Phi_U(x^{(i)}_{\mu_i}) \leqslant \sum_{i=1}^{n_0}t_i^{P_U} < \sum_{i=1}^{n_0}\Phi_U(x^{(i)}_{\lambda_i}) + n_0\varepsilon\backslash 2(n_0+1) < \|x_\lambda\|_U + \varepsilon.$$

So $\|x_\lambda\|_U = \inf_{0<\mu<\lambda}\|x_\mu\|_U$.

Thus, from Theorem 1, there exists a unique I-topology $\mathscr{T}$ on X such that $(X, \mathscr{T})$ is a locally semi-convex I-tvs and

$$\tilde{\mathscr{U}}_\lambda = \left\{\left(\bigcap_{i=1}^{n}\tilde{U}_{U_i\, t}\right) \cap \underline{r}: \ t>0, r \in (1-\lambda, 1], U_i \in \mathscr{U}_\lambda, i = 1, \cdots, n; \ n \in \mathbf{N}\right\} \tag{6}$$

is a balanced and semi-convex Q-neighborhood base of θ_λ, where $\tilde{U}_{U_i, t}$ is defined by (1), i.e., $\tilde{U}_{U_i\, t}(x) = \sup\{1-\lambda: \|x_\lambda\|_{U_i} < t\}$. Now we prove that $\tilde{\mathscr{T}} = \mathscr{T}$.

For each $U \in \mathscr{U}_\lambda$ and $t > 0$ and $0 < p_U \leqslant 1$, if $x_\lambda \tilde{\in} \tilde{U}_{U, \frac{t}{2}}$, then $\|x_\lambda\|_U < \frac{t}{2}$. By Lemma 5, $\frac{1}{2}\Phi_U(x_\lambda) \leqslant \|x_\lambda\|_U < \frac{t}{2}$, and so $\Phi_U(x_\lambda) < t$. From the definition of Φ_U, there exists $a^{p_U} < t$ such that $x_\lambda \tilde{\in} aU$, which implies that $x_\lambda \in t^{\frac{1}{p_U}}U$. This shows that $\tilde{U}_{U, t/2} \subset t^{\frac{1}{p_U}}U$. In particular, we have $\tilde{U}_{U, \frac{1}{2}} \cap \underline{r} \subset \tilde{U}_{U, \frac{1}{2}} \subset U$ for each $r \in (1-\lambda, 1]$. Notice that $\tilde{U}_{U, \frac{1}{2}} \cap \underline{r} \in \tilde{\mathscr{U}}_\lambda$, hence $\mathscr{T} \subset \tilde{\mathscr{T}}$.

On the other hand, for each $\left(\bigcap_{i=1}^{n}\tilde{U}_{U_i\, t}\right) \cap \underline{r} \in \tilde{\mathscr{U}}_\lambda$, then $U_i \in \mathscr{U}_\lambda$, $t > 0$ and $r \in (1-\lambda, 1]$. Taking natural number $k > t + 1$ and $p = \wedge_{i=1}^{n}p_{U_i}$, put $\delta = \left(\frac{t}{k}\right)^{\frac{1}{p}}$, we can prove

$$\mathscr{U}_i \subset \tilde{U}_{U_i, t}, \quad i = 1, \cdots, n. \tag{7}$$

In fact, if $x_\lambda \in \mathscr{U}_i$, then we have $\Phi_{U_i}(x_\lambda) \leqslant \delta^{U_i} = \left(\frac{t}{k}\right)^{\frac{p_{U_i}}{p}} \leqslant \frac{t}{k} < t$. By Lemma 5, we have $\| x_\lambda \|_{U_i} < t$. By Lemma 4, we infer $x_\lambda \in \tilde{U}_{U_i, t}$, (7) is proved.

So, we have $(\bigcap_{i=1}^{n} \mathscr{U}_i) \cap \underline{r} \subset (\bigcap_{i=1}^{n} \tilde{U}_{U_i, t}) \cap \underline{r}$. Notice that $(\bigcap_{i=1}^{n} \mathscr{U}_i) \cap \underline{r}$ is a Q-neighborhood of θ_λ in $(X, \mathscr{T})$, so $\mathscr{T} \subset \mathscr{T}$. Therefore $\mathscr{T} = \mathscr{T}$.

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