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Some Notes on Continuous Posets

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Abstract In this paper, we investigate the order structure-continuous poset, a generalization of a continuous dcpo (i e, domain). The Cartesian product of continuous posets is studied. Some other properties of continuous posets and algebraic posets are given. Some equivalent characterization of continuous posets is given.

Key words: continuous poset, Cartesian product, algebraic poset

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关于连续偏序集的注记

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[摘要] 研究了 domain的推广 连续偏序集的 Cartesian积以及连续偏序集和代数偏序集的一些性质. 给出了连续偏序集的若干等价刻画.

[关键词] 连续偏序集, Cartesian积, 代数偏序集

The notation of continuous lattices as a model for the semantics of programming languages were introduced by Scott D in [1] in 1972. Later, a more general notation of continuous dcpos (i e, domain) was introduced and thoroughly studied in [2]. In this paper, our work is based on an even general notation of continuous posets, which was called a precontinuous poset and defined in [3]. We investigate the construction of new continuous posets from known ones by means of forming product (with pointwise order) and disjoint sums. Then we give some other properties of continuous posets and algebraic posets.

Recall some notations concerned in this paper. For unexplained notations and concepts, please refer to [2, 4].

Definition 1[2] Let P be a poset. For any two elements x and y in P, we write x y, if for any directed subset D P with D existing and y D, there exists z D such that x z. An element satisfying x x is called to be compact. The set of compact elements of P is denoted as $K(P)$.

Definition 2[2] (1) A poset P is called a continuous poset (resp. an algebraic poset) if for each a P, the set $\{x$ $P: x$ $a\}$ (resp. a $K(P)$) is a directed set and $a =$ $\{x$ $P: x$ $a\}$ (resp. $a=$ $\{$ a $K(P)\}$).

(2) A directed complete partially ordered set (in short, dcpo) which is continuous is called a domain.

The following properties are well known.

Lemma 1 Let P be a poset, x, y, z P. Then

(1) x y implies x y;

(2) x y z w implies x w;

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(3) If y x and z x, then y z x whenever y z exists in P;

(4) The way-below relation on a continuous poset has strongly interpolation property, i. e., if x y with x y, then there is z with x z such that x z y (see [2] for the proof of the strongly interpolating property of a domain).

There are many interesting continuous posets which are not domains. Some examples are here.

Example 1 (1) Every chain C is a continuous poset. x, y C, if x y, then $x = 0$ or $x < y$ or $x = y$. Then for each element x in C, $\{y: y \quad x\} = x$. Hence, the real number set R, the rational number set Q and the national number set N are continuous posets but not domains.

(2) For any set X, let $P_0(X) = \{A \quad X: A$ is finite$\}$. Then $(P_0(X), \quad)$ is a continuous poset but not a domain in general. This follows that for each A $P_0(X)$, A A. However $P_0(X)$ is not directed complete unless X is a finite set. In general, if m is a cardinal and $m < |X|$, then $P_m(X) = \{A \quad X: \ |A| \quad m\}$ is a continuous poset with respect to but not a domain.

(3) Let P be a continuous poset, and let A be a lower set in P (i. e. $A = \quad A$). Assume that for every directed subset D A for which the least upper bound of D exists in A, and it is also the least upper bound of D in P (i. e. ${}_A D = {}_P D$ whenever ${}_A D$ exists). Then A is a continuous poset. For a A, x, y A satisfying x a and y a, then there exists z P such that x, y z a. Since A is a lower set, z A. Then a is directed for each a A. In fact, x ${}_P a$ implies x ${}_A a$ for x, a A. For each directed D A for which ${}_A D$ exists, a ${}_A D = {}_P D$, by x ${}_P a$ there exists d D such that x d, thus x ${}_A a$. Thus for each a A, $\{x: x \quad {}_P a\}$ $\{x: x \quad {}_A a\}$ a, hence $a = {}_A\{x: x \quad {}_A a\}$.

Let P be a poset. We say the distributive law holds for all directed subsets which suprema exist in P if for any nonempty family of elements in P $\{x_{jk}: j \quad J, \ k \quad K(j)\}$ for which $\{x_{jk}: k \quad K(j)\}$ is directed and has a supremum for all j J, the following identity holds:

$$_{j \quad J k \quad K(j)} x_{jk} = {}_{f \quad {}_{j \quad J} K(j) j \quad J}\, x_{j\, f(j)}\,.$$

Proposition 1 Suppose a poset P satisfy the following three conditions:

(1) Every nonempty subset of P has an infimum;

(2) The distributive law holds for all directed subsets which suprema exist;

(3) For each x P, x is directed.

Then P is a continuous poset.

Proof For each a P, let J be the set of all directed subset j of P with sup j a, $\{a\}$ J, thus J . For each j J, let $K(j) = j$. In other words, j is indexing itself. Further, consider the family of element $x_{jk} = k$ for j J and k $K(j)$.

Suppose f ${}_{j \quad J} K(j)$, and let $t = {}_{j \quad J} x_{j, f(j)} = {}_{j \quad J} f(j)$. We claim t a. In fact, if D is a directed subset of P with a $\sup D$, then D J. $t = {}_{j \quad J} f(j)$ $f(D)$ $K(D) = D$. Since $\{a\}$ J, ${}_{j \quad J \quad k \quad K(j)} x_{jk} = a$. Also, ${}_{f \quad {}_j \, {}_J K(j) \quad j \quad J} x_{j, f(j)} = {}_{j \quad J \quad k \quad K(j)} x_{jk} = a$. Thus $\{x \quad P: x \quad a\}$ a, hence $a = \{x \quad P: x \quad a\}$.

1 The Cartesian Product of Continuous Posets

In this section we investigate the construction of new continuous posets from known ones by means of forming products (with pointwise order).

Proposition 2 If $\{P_i: i \quad I\}$ is a family of continuous posets with least element 0, then the Cartesian product ${}_{i \quad I} P_i$ is also a continuous poset. For element $x = (x_i)_{i \quad I}$ and $y = (y_i)_{i \quad I}$ in ${}_{i \quad I} P_i$ the way-below relation is given by

x y iff x_i y_i for all i I and $x_i = 0$ for all but finitely many i I.

Proof Let us first show that the characterization of the way-below relation holds in any product of continuous posets P_i with 0.

Suppose first that $x \ll y$. For every finite set $F \subseteq I$ define y^F to be the element of $\prod_{i \in I} P_i$ with $y_i^F = y_i$ for all $i \in F$ and $y_i^F = 0$ for $i \notin F$. The family of the y^F is directed and its supremum is y. As $x \ll y$, there is some finite subset $F \subseteq I$ such that $x \leqslant y^F$, whence $x_i = 0$ for all $i \notin F$. In order to show that $x_i \ll y_i$ for all $i \in I$, fix i and consider any directed set D for which $\sup D$ exists in P_i satisfying $y_i \leqslant \sup D$. To every $d \in D$ we associate the element $\bar{d} \in \prod_{i \in I} P_i$ defined by $\bar{d}_i = d$ and $\bar{d}_j = y_j$ for all $j \neq i$. The family $\overline{(d)}_{d \in D}$ is directed and $y \leqslant \sup_{d \in D} \bar{d}$. As $x \ll y$, there is some $d \in D$ such that $x \leqslant \bar{d}$, whence $x_i \leqslant d$.

For the converse, suppose that $x_i \ll y_i$ for all $i \in I$ and there is a finite set $F \subseteq I$ such that $x_i = 0$ for $i \notin F$. Let D be any directed set in $\prod_{i \in I} P_i$ for which $\sup D$ exists such that $y \leqslant \sup D$. Then $y_i \leqslant \sup_{d \in D} d_i$ for every $i \in I$. As $x_i \ll y_i$ for all $i \in I$, there is a $d^i \in D$ such that $x_i \leqslant d_i^i$ ($=i$th component of d^i). As D is directed, there is a $d \in D$ such that $d^i \leqslant d$ for all $i \in F$. Thus $x_i \leqslant d_i$ for $i \in F$. As $x_i = 0$ for all $i \notin F$, we conclude that $x \leqslant d$. This prove that $x \ll y$.

If P_i are continuous for all $i \in I$, the set of all $x \ll y$ is easily seen to be directed and to have y as its supremum by the above characterization of the way-below relation. Hence, $\prod_{i \in I} P_i$ is continuous.

Remark 1 From the above proposition, it is obvious that the Cartesian product $P_1 \times P_2 \times \cdots \times P_n$ of finite many continuous posets $P_1, P_2, \cdots, P_n$ is a continuous poset. Furthermore, for $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in $P_1 \times P_2 \times \cdots \times P_n$ the way-below relation is given by

$$x \ll y \text{ iff } x_i \ll y_i \text{ for all } i = 1, 2, \cdots, n.$$

Corollary 1 The Cartesian product $\prod_{i \in I} P_i$ of a family of chains P_i with least element 0 is continuous.

Proposition 3 Every antichain is a continuous poset without least element, and any product of antichains is an antichain, hence a continuous poset.

Proof For each $a \in P$ which is an antichain, $\{x \in P: x \ll a\} = \{a\}$, hence every antichain is a continuous poset.

Remark 2 This shows that $\prod_{i \in I} P_i$ is a continuous poset does not imply that each of P_i has least element.

By a discrete union of posets P_j, $j \in J$, we mean a disjoint union of P_j such that elements in different components P_j are incomparable.

Proposition 4 The Cartesian product $\prod_{i \in I} P_i$ of a family of continuous posets $\{P_i : i \in I\}$ is again continuous if all the continuous posets P_i are discrete unions of continuous posets with least element 0.

Proof Suppose for any $i \in I$, $P_i = \sqcup_{j \in j(i)} P_{ij}$, the discrete union of continuous posets $\{P_{ij} : j \in j(i)\}$ with least element 0. Let $x \in \prod_{i \in I} P_i$ and $i \in I$. Then there exists $j(x, i) \in j(i)$ such that $x_i \in P_{i, j(x,i)}$.

With similar method to Proposition 2, one can prove that for element $x = (x_i)$ and $y = (y_i)$ in $\prod_{i \in I} P_i$, $x \ll y$ iff for all $i \in I$, $j(x, i) = j(y, i)$, $x_i \ll y_i$ and $x_i = 0_{i, j(x,i)}$ for all but finitely many $i \in I$.

From the characterization of way-below relation in $\prod_{i \in I} P_i$, we have $\bigwedge^{\uparrow} \{x \in \prod_{i \in I} P_i : x \ll y\} = y$ for all $y \in \prod_{i \in I} P_i$. Hence $\prod_{i \in I} P_i$ is a continuous poset.

2 Some Properties of Continuous Posets

Let S be a poset. Adjoin an identity by forming $S^1 = S \cup 1$ with an element $1 \notin S$ and $x \leqslant 1$ for all $x \in S$.

Let P and Q be posets. Define the following five kinds of "disjoint" sums:

(1) (Disjoint sum) $P \sqcup Q$, the disjoint union of P and Q (with the obvious partial ordering: the elements $x \in P$ and $y \in Q$ are incomparable);

(2) (Coalesced sum) $P \oplus Q$, the disjoint sum $P \sqcup Q$ with the bottom elements identified, if they have

them;

(3) (Separated sum) $P+Q=(P\oplus Q)_\perp$, that is, the disjoint sum with a new bottom element adjoined;

(4) $P+_1Q=(P\oplus Q)^1$;

(5) $P+_2Q=P\oplus Q$ with the 1 elements identified, if they have them.

Proposition 5 Let P and Q be continuous posets, then $P\sqcup Q$, $P\oplus Q$ and $P+Q$ are all continuous posets.

Proof (1) Let $a\in P\sqcup Q$, $a\in P$ or $a\in Q$. If $a\in P$, then $\{x\in P\sqcup Q: x\ll_{P\sqcup Q}a\}=\{x\in P: x\ll_P a\}$. So $\bigvee_{P\sqcup Q}\{x\in P\sqcup Q: x\ll_{P\sqcup Q}a\}=\bigvee_{P\sqcup Q}\{x\in P: x\ll_P a\}=\bigvee_P\{x\in P: x\ll_P a\}=a$.

If $a\in Q$, we also have $\bigvee_{P\sqcup Q}\{x\in P\sqcup Q: x\ll_{P\sqcup Q}a\}=a$.

If $a\in P\sqcup Q$, then $a\in P$ or $a\in Q$. Without loss of generality, suppose $a\in P$, then $\Downarrow_{P\sqcup Q}a=\Downarrow_P a$ is directed. Hence $P\sqcup Q$ is continuous poset.

(2) Suppose both P and Q have bottom elements. If $a=0_{P\oplus Q}$, then $\{x\in P\oplus Q: x\ll_{P\oplus Q}a\}=0_{P\oplus Q}$. Hence, $\bigvee_{P\oplus Q}\{x\in P\oplus Q: x\ll_{P\oplus Q}a\}=0_{P\oplus Q}$. Otherwise, without loss of generality, let $a\in P-\{0_P\}$, then $\{x\in P\oplus Q: x\ll_{P\oplus Q}a\}=\{x\in P\oplus Q: x\ll_P a\}\cup\{0_{P\oplus Q}\}$. So

$\bigvee_{P\oplus Q}\{x\in P\oplus Q: x\ll_{P\oplus Q}a\}=\bigvee_{P\oplus Q}\{\{x\in P\oplus Q: x\ll_P a\}\cup\{0_{P\oplus Q}\}\}=\bigvee_P\{x\in P: x\ll_P a\}=a$.

If $a\in P\oplus Q-\{0_{P\oplus Q}\}=P-\{0_P\}$, $\Downarrow_{P\oplus Q}a=\Downarrow_P a\cup\{0_{P\oplus Q}\}$ is directed. Hence $P\oplus Q$ is continuous poset.

(3) Let $a\in\{P+Q-\{0_{P+Q}\}\}$. Then $a\in P$ or $a\in Q$. If $a\in P$, then $\{x\in\{P+Q: x\ll_{P+Q}a\}=\{x\in P: x\ll_P a\}\cup\{0_{P+Q}\}$ is directed and have a as its supremum.

Similarly, if $a\in Q$, then $\{x\in\{P+Q\}: x\ll_{P+Q}a\}=\{x\in Q: x\ll_Q a\}\cup\{0_{P+Q}\}$ is directed and have a as its supremum. Hence $P+Q$ is continuous poset.

Example 2 Let P and Q be continuous posets, then $P+_1Q$ and $P+_2Q$ need not be continuous.

(1) Let $P=Q=N$. Then both P and Q are continuous. Let $a\in P-\{0_P\}$, put $C=Q-\{0_Q\}$ satisfying that C is directed and $\bigvee_{P+_1Q}C=1_{P+_1Q}$. But for any $c\in C$, $x\not\leq c$ for each $x\in P-\{0_P\}$; the case is similar for $x\in Q-\{0_Q\}$. Thus $a=\{0_{P+_1Q}\}$, whence $P+_1Q$ is not a continuous poset.

(2) Let $P=Q=[0,1]$. Then both P and Q are continuous. Let $a\in P-\{0_P,1_P\}\subseteq P+_2Q-\{0_{P+_2Q},1_{P+_2Q}\}$, put $B=Q-\{0_Q,1_Q\}$ satisfying that B is directed and $\bigvee_{P+_2Q}B=1_{P+_2Q}$. But for any $b\in B$, $a\not\leq b$. Thus, $a\ll 1_{P+_2Q}$ is not right. Similarly, we have for all $b\in Q-\{0_Q,1_Q\}$, $b\ll_{P+_2Q}1_{P+_2Q}$ is not right. So $\{x\in_{P+_2Q}: x\ll_{P+_2Q}1_{P+_2Q}\}=\{0_{P+_2Q}\}$, whence $P+_2Q$ is not continuous.

Remark 3[2] If P and Q are domains, then $P\sqcup Q$, $P\oplus Q$, $P+Q$ and $P+_1Q$ are domains, $P+_2Q$ need not be a domain.

Put $Id\ P=\{I\subseteq P: I$ is an ideal such that $\sup I$ exists$\}$.

We say that a function $g: S\to T$ into a poset is cofinal if for all $t\in T$ there is an $s\in S$ such that $t\leqslant g(s)$, i.e., if $g^{-1}(\uparrow t)\neq\emptyset$ for all $t\in T$.

Lemma 2[2] Let $g: S\to T$ be a function between posets. Assume that the following conditions are satisfied:

(1) S is a complete lattice, or S is a complete semilattice and g is cofinal;

(2) g preserves all existing infs.

Then g has a lower adjoint $d: T\to S$ given by the formula: $d(t)=\inf g^{-1}(\uparrow t)$ or $d(t)=\min g^{-1}(\uparrow t)$.

Proposition 6 For a poset P, the following conditions are equivalent:

(1) P is continuous;

(2) for each $x\in P$, the set $\Downarrow x$ is the smallest ideal I with $x\leqslant\sup I$;

(3) for each $x\in P$, there is a smallest ideal I with $x\leqslant\sup I$;

(4) the sup map $r=(I\to\sup I): IdP\to P$ has a lower adjoint;

These conditions imply

(5) the sup map $r: IdP\to P$ preserves all existing infs.

and if P is a complete semilattice or complete lattice, then all five conditions are equivalent.

Proof (1)$\Rightarrow$(2): If P is continuous, then for each $x \in P$, $\Downarrow x \in IdP$ and $x = \sup \Downarrow x$. For each ideal I with $x \leqslant \sup I$, $\forall y \ll x$, $\exists i \in I$ such that $y \leqslant i$, then $y \in I$, thus $\Downarrow x \subseteq I$.

(2)$\Rightarrow$(3): Trivially.

(3)$\Rightarrow$(1): If $J(x) = \{I \in Id\ P : x \leqslant \sup I\}$ has a smallest element M, then $M \subseteq I$ for all $I \in J(x)$ and thus $M \subseteq \cap J(x) \subseteq M$. We can show that $\cap J(x) = \Downarrow x$. On one hand, $\forall y \ll x$ with $x \leqslant \sup I$, then $\exists i \in I$ such that $y \leqslant i$, then $y \in I$, thus $\Downarrow x \subseteq \cap J(x)$. On other hand, $\forall y \in \cap J(x)$, then $y \in I$ for each I with $x \leqslant \sup I$. For each directed set D with $\vee D$ existing, we take $I = \downarrow D$, then $x \leqslant \vee D = \vee I$, by $y \in I$, then there exists $d \in D$ such that $y \leqslant d$. Thus $y \ll x$. Thus $\Downarrow x$ is the smallest ideal, and $x = \sup \Downarrow x$, hence P is continuous.

(3) iff (4): The map r has a lower adjoint iff $\min r^{-1}(\uparrow x)$ exists for all x. But $\min r^{-1}(\uparrow x)$ is precisely the smallest element of $J(x)$.

(4)$\Rightarrow$(5): The sup map preserves infs.

(5)$\Rightarrow$(4): This follows from Lemma 2, as the sup map $r: IdP \to P$ clearly is cofinal.

Let P be a poset and $S \subseteq P$, $Id\ S = \{I \subseteq S : I$ is an ideal such that $\sup I$ exists in $P\}$.

Proposition 7 Let P be an algebraic poset, $S = K(P)$, then the map $\phi: x \mapsto \downarrow x \cap S, P \to Id\ S$ is an isomorphism.

Proof To prove that $\phi: x \mapsto \downarrow x \cap S, P \to Id\ S$ is a bijective, we claim that $\sup: I \mapsto \sup I$, $Id\ S \to P$ is the inverse of this map.

Since $\sup(\downarrow x \cap S) = x$ by the definition of algebraic poset, it suffices to show that $\downarrow(\sup I) \cap S = I$ for each $I \in Id\ S$. $\supseteq$ is clear so we must show $\subseteq$. Let $k \in \downarrow(\sup I) \cap S$, that means $k \ll k \leqslant \sup I$. Thus there exists an $a \in I$ such that $k \leqslant a$. Since I is a lower set in S, we have $k \in I$. The two maps are clearly order-preserving, whence ϕ is an isomorphism.

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