Some Remarks on Star-Lindelöf Spaces

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Abstract A space X is star-Lindelöf if for every open cover $\mathscr{U}$ of X, there exists a countable subset F of X such that $St(F, \mathscr{U}) = X$. In this note, we discuss the relationship between star-Lindelöf spaces and related spaces, and give an example showing the product of two countable compact spaces is not star-Lindelöf.

Key words countably compact, Lindelöf, star-Lindelöf

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关于 Star-Lindelöf空间的注释

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[摘要] 一个空间称为 star-Lindelöf如果对于 X 的任意开覆盖 $\mathscr{U}$, 在 X 中存在一个可数子集 F 使得 $St(F, \mathscr{U}) = X$. 在这个注释中, 我们讨论 star-Lindelöf空间与相关拓扑空间关系, 并且给出两个可数紧空间的积不是 star-Lindelöf空间的例子.

[关键词] 可数紧, Lindelöf, star-Lindelöf

By a space, we mean a topological space. Let us recall that a space X is countably compact if every countable open cover of X has a finite subcover. Fleischman[1] defined a space X to be starcompact if for every open cover $\mathscr{U}$ of X, there exists a finite subset F of X such that $St(F, \mathscr{U}) = X$, where $St(F, \mathscr{U}) = \cup\{U \in \mathscr{U}: U \cap F \neq \emptyset\}$. He proved that every countably compact space is starcompact. As generation of starcompactness, the following classes of spaces were given.

Definition 1[2] A space X is star-Lindelöf if for every open cover $\mathscr{U}$ of X, there exists a countable subset F of X such that $St(F, \mathscr{U}) = X$.

Definition 2[3] A space X is $*\ \sigma$-compact if X has a σ-compact dense subset.

In [4], a star-Lindelöf space is called strongly star-Lindelöf, in [5], a star-Lindelöf space is called $*$ Lindelöf.

From the above definition, it is not difficult to see that every countably compact space is star-Lindelöf, every Lindelöf space is star-Lindelöf. A space satisfies the countable chain condition (CCC) provided it does not contain an uncountable collection of pairwise disjoint open sets.

In [5], Dai Mumin asked the following questions on star-Lindelöf spaces.

Question 1 Is CCC space star-Lindelöf?

Question 2 Is the product of two star-Lindelöf spaces star-Lindelöf?

In [5], Chen Haiyan, Zhang Dingwei and Wu Wei asked the following question.

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Question 3 Does there exist a* σ-compact space which is not star-Lindelöf?

The purpose of this note is to give some notes on the above questions.

Moreover, the cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal, ω_1 the first uncountable cardinal and c the cardinality of the set of all real numbers. Other terms and symbols that we do not define will be used as in [6].

1 Some Examples on Star-Lindelöf Spaces

In [4], van Douwen, Reed, Roscoe, et al constructed the following example which gives a negative answer to the first question.

Example 1[4] There exists a CCC Moore space which is not star-Lindelöf.

On the second question, there are several solutions. First, we give a new solution. For a Tychonoff space X, the symbol βX means the Cech-Stone compactification of a space X.

Example 2 There exist two countably compact spaces X and Y such that $X \times Y$ is not star-Lindelöf.

Proof Let D be a discrete space of cardinality $\mathfrak{c}$. We shall define two countably compact subspaces X and Y of βD such that $X \cup Y = \beta D$, $X \cap Y = D$ and $X \times Y$ is not star-Lindelöf.

For every $M \subseteq \beta D$, let $\mathscr{P}(M)$ denote the family of all countably infinite subsets of M and let f be a function assigning to every member A of $\mathscr{P}(\beta D)$ an accumulation point of the set A in the space βD.

Let $X_0 = D$ and

$$X_\alpha = (\bigcup_{\gamma<\alpha} X_\gamma) \cup f[\mathscr{P}(\bigcup_{\gamma<\alpha} X_\gamma))] \text{ for } 0 < \alpha < \omega_1.$$

By transfinite induction, we define a transfinite sequence $X_0, X_1, \cdots, X_\alpha, \cdots$ of subsets of βD, where $\alpha < \omega_1$. Let

$$X = \bigcup_{\alpha<\omega_1} X_\alpha,$$

then X is countably compact, since for every $A \in \mathscr{P}(X)$ is contained in some X_α and thus has an accumulation point in $X_{\alpha+1}$ and in X. By transfinite induction, it is not difficult to see that $|X_\alpha| \leqslant \mathfrak{c} \cdot \mathfrak{c} + (\mathfrak{c} \cdot \mathfrak{c})^{\aleph_0} = \mathfrak{c}$ for each $\alpha < \omega_1$, thus $|X| \leqslant \mathfrak{c}$.

Let

$$Y = D \cup (\beta D \setminus X),$$

then Y is countably compact. For every $A \in \mathscr{P}(Y)$, we have $|A| = 2^{\mathfrak{c}}$, since every infinite closed set in $\beta(D)$ has the cardinality $2^{\mathfrak{c}}$[7]. Thus every countably infinite subset of Y has an accumulation point in Y, hence Y is countably compact.

To show that $X \times Y$ is not star-Lindelöf. Since the diagonal $\{\langle d, d\rangle : d \in D\}$ is a discrete open and closed subset of $X \times Y$ with the cardinality $\mathfrak{c}$, then it is not star-Lindelöf, hence $X \times Y$ is not star-Lindelöf, star-Lindelöfness is preserved by open and closed subsets.

On the second question, in [4], van Douwen, Reed, Roscoe, et al constructed the following two examples which give negative answers to the second question.

Example 3[4] There exist a Lindelöf space X and a star-Lindelöf space Y such that $X \times Y$ is not star-Lindelöf.

Example 4[4] There exist a compact space X and a star-Lindelöf space Y such that $X \times Y$ is not star-Lindelöf.

On the third question, we give a positive answer.

Example 5 There exists a* σ-compact space which is not star-Lindelöf.

Proof Let D be a discrete space of the cardinality $\mathfrak{c}$. Define

$$X = (\beta(D) \times (\omega + 1)) \setminus ((\beta(D) \setminus D) \times \{\omega\}).$$

Then, X is* σ-compact, since $\beta(D) \times \omega$ is a σ-compact dense subset of X.

Next, we show that X is not star-Lindelöf. Since $|D| = \mathfrak{c}$, then we can enumerate D as $\{d_\alpha: \alpha < \mathfrak{c}\}$. Let $U_\alpha = \{d_\alpha\} \times [0, \omega]$ for each $\alpha < \mathfrak{c}$.

Let us consider the open cover

$$\mathscr{U} = \{U_\alpha: \alpha < \mathfrak{c}\} \cup \{\beta(D) \times \{n\}: n \in \omega\}$$

of X. Let F be a countable subset of X. Then, there exists a α_0 such that $F \cap U_\alpha = \emptyset$ for each $\alpha > \alpha_0$ by the construction of $\mathscr{U}$. If we pick $\alpha' > \alpha_0$, then $\langle \alpha', \omega \rangle \notin St(F, \mathscr{U})$, since $U_{\alpha'}$ is the only element of $\mathscr{U}$ containing $\langle \alpha', \omega \rangle$ and $U_{\alpha'} \cap F = \emptyset$. This shows that X is not star-Lindelöf.

[References]

[1] Fleischman W M. A new extension of countable compactness[J]. Fund Math, 1970, 67(1): 1-9.

[2] Matveev M V. A survery on star-covering properties[J/OL]. [1998-06-30]. http://at.yorku.ca/topology1

[3] Chen Hanyan, Zhang Dingwei, Wu Lei. A class of topological spaces which contain σ-compact spaces and separable spaces[J]. Guangxi Sciences, 2004, 11(4): 296-299.

[4] van Douwen E K, Reed G M A, Roscoe A W, et al. Star covering properties[J]. Topology Appl, 1991, 39(1): 71-103.

[5] Dai Mumin. A class of topological spaces containing Lindelöf spaces and separable spaces[J]. Chinese Annals of Mathematics, 1983, 4A(5): 571-575.

[6] Engelking R. General Topology[M]. Rev. ed. Berlin: Heldermann Verlag, 1989.

[7] Walker R C. The Stone-Čech Compactification[M]. Berlin: Springer, 1974.

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