Asymptotic Properties of Parametric Bayesian Estimation in ARFIMA Models

Hong Zhaoping Du Xiuli

(School of Mathematics and Computer Science, Nanjing Normal University, Nanjing 210046, China)

Abstract The marginal posterior distribution of the parameter in the ARFIMA models is presented by Bayes theorem and the mode of the marginal posterior distribution is choosed as the estimator. Then, followed the analysis of the asympototic properties of maximum likelihood estimation for the seasonal ARFIMA models, the consistency, efficiency and asymptotic normality of the Bayesian estimator are proved. Finally, large sample performance of the Bayesian estimates is examined by simulations. It is shown that the estimates behave well when the sample size is large enough.

Key words: Bayesian methods, ARFIMA models, posterior distribution, asymptotic properties

CLC number: O212.8 **Document code**: A **Article ID**: 1001-4616(2008)02-0016-07

ARFIMA模型参数贝叶斯估计的渐近性质

洪兆萍, 杜秀丽

(南京师范大学数学与计算机科学学院, 江苏 南京 210046)

[摘要] 首先根据贝叶斯定理得到 ARFIMA模型参数的后验边缘分布, 并选择后验边缘分布的众数作为参数的估计值. 参照季节性 ARFIMA 模型的极大似然估计的渐近性质的证明思路, 证明了模型参数的贝叶斯估计具有相合性、有效性和渐近正态性. 最后, 对参数的贝叶斯估计方法的大样本性质进行仿真模拟, 结果表明当时间序列样本足够大时, 参数的估计值越来越接近于真实值.

[关键词] 贝叶斯方法, ARFIMA模型, 后验分布, 渐近性质

The autoregressive fractionally integrated moving average (ARFIMA) (p, d, q) process $\{x_t\}$ is defined by

$$\Phi(B)(1-B)^d x_t = \theta(B)\epsilon_t, \tag{1}$$

where both p and q are integers, d is real, $\Phi(B) = 1-\Phi_1 B - \cdots - \Phi_p B^p$ and $\theta(B) = 1-\theta_1 B - \cdots - \theta_q B^q$ are the autoregressive and moving average operators, respectively, $Bx_t = x_{t-1}$, the $\{\epsilon_t\}$ are independent and identically distributed as normal random variables with mean zero and variance σ^2 and the fractional difference operator is defined by a binomial series $(1-B)^d = \sum_{j=0}^{\infty}\binom{d}{j}B^j = 1-Bd+\frac{d(d-1)}{2!}B^2-\cdots$. Under the assumption that the roots of polynomials $\Phi(B)$ and $\theta(B)$ are outside the unit circle and $|d|<\frac{1}{2}$, the ARFIMA (p, d, q) process is second order stationary and invertible. The spectral density of this process is

$$f_\Phi(\omega) = \frac{\sigma^2}{2\pi}|1-e^{-i\omega}|^{-2d}|\Phi(e^{-i\omega})|^{-2}|\theta(e^{-i\omega})|^2, \tag{2}$$

and its ACF may be written as

$$\gamma(k) = \int_{-\pi}^{\pi} f(\omega)e^{i\omega k}d\omega. \tag{3}$$

Received date: 2007-11-28.

Corresponding author: Hong Zhaoping, teaching assistant, majored in financial statistics. E-mail: zphong2002@yahoo.com.cn

Since the seminal work by Granger and Joyeux[1], estimation of long-memory time series have been received considerable attention and a number of parameter estimation procedures[2] have been proposed. Recently, the statistical analysis of ARFIMA models in the Bayesian framework has been developed[3], which takes the prior information of the parameters into consideration and is more efficient than other estimation techniques.

Until now there is no any research about the asymptotic properties of the Bayesian estimation for the ARFIMA models because of the complexity of the parameter's estimation. In this paper, we present a fully Bayesian analysis of parameter estimation. Based on MLE asymptotics we prove the consistency, efficiency and asymptotic normality of the Bayesian estimators. The proof technique is based on an approximation of the spectral density proposed by Ref [4].

1 Bayesian Estimation Based on Posterior Modes

The exact likelihood function based on n observations $X_n = (x_1, \cdots, x_n)'$ from a Gaussian ARFIMA process X_t is

$$L(X_n \mid \psi_1) = (2\pi\sigma^2)^{-\frac{n}{2}} \mid \Omega_n \mid^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2} X_n' \Omega_n^{-1} X_n\right\}, \qquad (4)$$

where $\psi_1 = (\phi, \theta, d, \sigma)'$ is an $p+q+2$ vector, $\psi = (\phi, \theta, d)$ is an $p+q+1$ vector, $\phi = (\phi_1, \cdots, \phi_p)$, $\theta = (\theta_1, \cdots, \theta_q)$, $\sigma^2 \Omega_n$ is the covariance matrix of X_n with elements γ_k. We write

$$[\Omega(\psi)]_{r,s=1,\cdots,n} = [\Omega(f_\psi)]_{r,s=1,\cdots,n} = \int_{-\pi}^{\pi} f_\psi(\omega) e^{i\omega(r-s)} d\omega.$$

Assume that σ, d and (ϕ, θ) are mutually dependent, then $\pi(\psi_1) = \pi(\sigma)\pi(\phi, \theta)\pi(d)$ and

1. a uniform prior for ϕ and θ over C_p, C_q respectively, where C denotes the set of complex numbers, so that $\pi(\phi, \theta) \propto 1$, $(\phi, \theta) \in C_p \times C_q$;

2. a uniform prior for d over $\left(-\frac{1}{2}, \frac{1}{2}\right)$, so that $\pi(d) \propto 1$, $d \in \left(-\frac{1}{2}, \frac{1}{2}\right)$;

3. an noninformative prior for σ on $\mathbf{R}^+$, so that $\pi(\sigma) \propto \frac{1}{\sigma}$, $\sigma > 0$.

Correspondingly, $\pi(\psi_1) \propto \frac{1}{\sigma}$.

By Bayes theorem,

$$\pi(\psi_1 \mid X_n) \propto L(X_n \mid \psi_1)\pi(\psi_1) \propto (\sigma^2)^{-\frac{n+1}{2}} \mid \Omega_n \mid^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2} X_n' \Omega_n^{-1} X_n\right\}.$$

Thus, the marginal posterior distribution of the parameter ψ is

$$\pi(\psi \mid X_n) \propto \int_{\sigma>0} (\sigma^2)^{-\frac{n+1}{2}} \mid \Omega_n \mid^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2} X_n' \Omega_n^{-1} X_n\right\} d\sigma \propto 2^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right) \mid \Omega_n \mid^{-\frac{1}{2}} (X_n' \Omega_n^{-1} X_n)^{-\frac{n}{2}}.$$

The log function of $\pi(\psi \mid X_n)$ is

$$\ln[\pi(\psi \mid X_n)] = \left(\frac{n}{2} - 1\right)\ln 2 + \ln\Gamma\left(\frac{n}{2}\right) - \frac{1}{2}\ln \mid \Omega_n \mid - \frac{n}{2}\ln(X_n' \Omega_n^{-1} X_n),$$

where Ω_n^{-1} is positive definite.

Then,

$$-\frac{1}{n}\ln[\pi(\psi \mid X_n)] = -\frac{1}{n}\left(\frac{n}{2} - 1\right)\ln 2 - \frac{1}{n}\ln\Gamma\left(\frac{n}{2}\right) + \frac{1}{2n}\ln \mid \Omega_n \mid + \frac{1}{2}\ln(X_n' \Omega_n^{-1} X_n).$$

Let $\ln\pi(\psi)$ be $-\frac{1}{n}\ln[\pi(\psi \mid X_n)]$, unless otherwise stated, we write $\pi(\psi \mid X_n)$ as $\pi(\psi)$, Ω_n as Ω and X_n as X herein.

we will choose the mode as the estimation of parameter ψ. The approximate Bayesian estimator ψ_n of ψ maximizes $\pi(\psi \mid X_n)$ and the log function is monotonic, so we can estimate ψ by solving the equation:

$$\nabla \ln\pi(\phi) = 0,$$

where "∇" denotes the differencing. Obviously ϕ_n has the interpretation of being the "most likely" value of ϕ, given the prior and the sample X_n. So the Bayesian estimator given above is similar with the maximum likelihood estimator and we will illustrate the asymptotic properties of the Bayesian estimator following the research on that of the maximum likelihood estimator.

2 Asymptotic Properties of Bayesian Estimation

We assume that $\phi_0 = (\Phi_0, \theta_0, d_0)'$ is the true value of ϕ. Following the conclusion in Ref [5]:

$$\sigma^2 = \frac{X'\Omega^{-1}X}{n},$$

we have $E\sigma^2 < \infty$, so that $X'\Omega^{-1}X$ is $O(n)$ as $n\to\infty$.

Theorem 1 (Consistency) $\phi_n \xrightarrow{p} \phi_0$ as $n\to\infty$.

Proof Following Ref [6], it suffices to prove that

(1) $\nabla \ln\pi(\phi_0) \xrightarrow{p} 0$, as $n\to\infty$;

(2) there exists a positive definite matrix $M(\phi_0)$ such that for all $\epsilon > 0$,

$$P(\nabla^2 \ln\pi(\phi_0) > M(\phi_0)) > 1-\epsilon;$$

(3) there exists a constant $0<M<\infty$ such that

$$E\,|\nabla^3 \ln\pi(\phi)| < M \text{ for all } \phi\in\Theta.$$

We prove that condition (1)–(3) hold in this case.

(1)
$$\nabla \ln\pi(\phi_0) = \frac{1}{2n}\mathrm{tr}\{\Omega(\phi_0)^{-1}\Omega(\nabla\phi_0)\} - \frac{1}{2}\frac{1}{X'\Omega^{-1}(\phi_0)X}\mathrm{tr}\{XX'\Omega(\phi_0)^{-1}\Omega(\nabla\phi_0)\Omega(\phi_0)^{-1}\} \to$$
$$\frac{1}{2n}\mathrm{tr}\{\Omega(\phi_0)^{-1}\Omega(\nabla\phi_0)\} - \frac{1}{2}\frac{1}{n}\mathrm{tr}\{XX'\Omega(\phi_0)^{-1}\Omega(\nabla\phi_0)\Omega(\phi_0)^{-1}\}.$$

Hence,

$$E[\nabla \ln\pi(\phi_0)] \to \frac{1}{2n}\mathrm{tr}\{\Omega(\phi_0)^{-1}\Omega(\nabla\phi_0)\} - \frac{1}{2n}\mathrm{tr}\{\Omega(\nabla\phi_0)\Omega(\phi_0)^{-1}\} \to 0$$
$$\text{as, } n\to\infty.$$

On the other hand,

$$\mathrm{var}[\nabla \ln\pi(\phi_0)] = E[\nabla \ln\pi(\phi_0)]^2 - \{E[\nabla \ln\pi(\phi_0)]\}^2 \to$$
$$\frac{1}{2n^2}\mathrm{tr}\{\Omega(\phi_0)^{-1}\Omega(\nabla\phi_0)\Omega(\phi_0)^{-1}\Omega(\nabla\phi_0)\} =$$
$$\frac{1}{2n^2}\mathrm{tr}\{\Omega(f_{\phi_0})^{-1}\Omega(\nabla f_{\phi_0})\Omega(f_{\phi_0})^{-1}\Omega(\nabla f_{\phi_0})\}.$$

By Lemma 5[6] we have

$$\frac{1}{n}\mathrm{tr}\{\Omega(f_{\phi_0})^{-1}\Omega(\nabla f_{\phi_0})\Omega(f_{\phi_0})^{-1}\Omega(\nabla f_{\phi_0})\} \to \frac{1}{2\pi}\int_{-\pi}^{\pi}[\nabla \ln f_{\phi_0}(\omega)][\nabla \ln f_{\phi_0}(\omega)]'\mathrm{d}\omega,$$
$$\text{as } n\to\infty.$$

Therefore, $\mathrm{var}[\nabla \ln\pi(\phi_0)] \to 0$ and the result holds by virtue of the Chebyshev's inequality.

(2) Observe that

$$E[\nabla^2 \ln\pi(\phi_0)] \to \frac{1}{2n}\mathrm{tr}\{\Omega(\phi_0)^{-1}\Omega(\nabla\phi_0)\Omega(\phi_0)^{-1}\Omega(\nabla\phi_0)\} =$$
$$\frac{1}{2n}\mathrm{tr}\{\Omega(f_{\phi_0})^{-1}\Omega(\nabla f_{\phi_0})\Omega(f_{\phi_0})^{-1}\Omega(\nabla f_{\phi_0})\}.$$

Hence, by Lemma 5[6] we have

$$E[\nabla^2 \ln\pi(\phi_0)] \to \frac{1}{4\pi}\int_{-\pi}^{\pi}[\nabla \ln f_{\phi_0}(\omega)][\nabla \ln f_{\phi_0}(\omega)]'\mathrm{d}\omega \equiv T(\phi_0).$$

On the other hand,

$$\operatorname{var}[\nabla^2 \ln\pi(\phi_0)] \to \frac{1}{2n^2}\operatorname{tr}\{[\Omega(f_{\phi_0})(2A^{(1)}_{\phi_0} - A^{(2)}_{\phi_0})]^2\},$$

where the matrices $A^{(1)}_{\phi_0}$ and $A^{(2)}_{\phi_0}$ are defined as in Lemma 6[6]. Then, an application of Lemma 5[6] establishes that $\operatorname{var}[\nabla^2 \ln\pi(\phi_0)] \to 0$. Thus, $\nabla^2 \ln\pi(\phi_0) \xrightarrow{p} T(\phi_0)$. Besides, since $T(\phi_0)$ is positive-definite, we can choose the positive matrix $M(\phi_0) \equiv T(\phi_0) - \kappa I$ with $0 < \kappa < \lambda_{\min}(T(\phi_0))$ and the result holds, where $\lambda_{\min}(T(\phi_0))$ denotes the minimum eigenvalue of $T(\phi_0)$.

(3) Note that

$$\begin{aligned}\nabla^3 \ln\pi(\phi) \to{} & \frac{1}{n}\operatorname{tr}\{\Omega(f_\phi)^{-1}\Omega(\nabla f_\phi)\Omega(f_\phi)^{-1}\Omega(\nabla f_\phi)\Omega(f_\phi)^{-1}\Omega(\nabla f_\phi)\} + \\ & \frac{1}{2n}\operatorname{tr}\{\Omega(f_\phi)^{-1}\Omega(\nabla^3 f_\phi)\} - \\ & \frac{3}{2n}\operatorname{tr}\{\Omega(f_\phi)^{-1}\Omega(\nabla f_\phi)\Omega(f_\phi)^{-1}\Omega(\nabla^2 f_\phi)\} + \\ & \frac{1}{2n}X'[3\Omega(f_\phi)^{-1}\Omega(\nabla^2 f_\phi)\Omega(f_\phi)^{-1}\Omega(\nabla f_\phi)\Omega(f_\phi)^{-1} + \\ & 3\Omega(f_\phi)^{-1}\Omega(\nabla f_\phi)\Omega(f_\phi)^{-1}\Omega(\nabla^2 f_\phi)\Omega(f_\phi)^{-1} - \\ & 6\Omega(f_\phi)^{-1}\Omega(\nabla f_\phi)\Omega(f_\phi)^{-1}\Omega(\nabla f_\phi)\Omega(f_\phi)^{-1}\Omega(\nabla f_\phi)\Omega(f_\phi)^{-1} - \\ & \Omega(f_\phi)^{-1}\Omega(\nabla^3 f_\phi)\Omega(f_\phi)^{-1}]X.\end{aligned}$$

But,

$$|X'\Omega(\nabla f_\phi)X| = \left|\int_{-\pi}^{\pi}\nabla f_\phi \left|\sum_{j=1}^{n} e^{i\omega j}X_j\right|^2 d\omega\right| \leqslant \int_{-\pi}^{\pi}|\nabla f_\phi| \left|\sum_{j=1}^{n} e^{i\omega j}X_j\right|^2 d\omega = X'\Omega(|\nabla f_\phi|)X.$$

Analogously,

$$|X'\Omega(\nabla^2 f_\phi)X| \leqslant X'\Omega(|\nabla^2 f_\phi|)X$$

and

$$|X'\Omega(\nabla^3 f_\phi)X| \leqslant X'\Omega(|\nabla^3 f_\phi|)X.$$

Therefore,

$$\begin{aligned}|\nabla^3 \ln\pi(\phi)| \leqslant{} & \frac{1}{n}\operatorname{tr}\{\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\} + \\ & \frac{1}{2n}\operatorname{tr}\{\Omega(f_\phi)^{-1}\Omega(|\nabla^3 f_\phi|)\} + \\ & \frac{3}{2n}\operatorname{tr}\{\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\Omega(f_\phi)^{-1}\Omega(|\nabla^2 f_\phi|)\} + \\ & \frac{1}{2n}X'[3\Omega(f_\phi)^{-1}\Omega(|\nabla^2 f_\phi|)\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\Omega(f_\phi)^{-1} + \\ & 3\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\Omega(f_\phi)^{-1}\Omega(|\nabla^2 f_\phi|)\Omega(f_\phi)^{-1} + \\ & 6\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\Omega(f_\phi)^{-1} + \\ & \Omega(f_\phi)^{-1}\Omega(|\nabla^3 f_\phi|)\Omega(f_\phi)^{-1}]X,\end{aligned}$$

and

$$\begin{aligned}E|\nabla^3 \ln\pi(\phi)| \leqslant{} & \frac{4}{n}\operatorname{tr}\{\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\} + \\ & \frac{1}{n}\operatorname{tr}\{\Omega(f_\phi)^{-1}\Omega(|\nabla^3 f_\phi|)\} + \\ & \frac{9}{2n}\operatorname{tr}\{\Omega(f_\phi)^{-1}\Omega(|\nabla f_\phi|)\Omega(f_\phi)^{-1}\Omega(|\nabla^2 f_\phi|)\}.\end{aligned}$$

Now, by an argument similar to Ref[7], we conclude that the first summand is bounded by

$$\frac{1}{n}\mathrm{tr}\{[\Omega(K|\omega|^{-\alpha+\epsilon})^{-1}\times\Omega(K|\omega|^{-\alpha-\epsilon})]^{3}\},$$

where $\epsilon>0$ can be chosen arbitrarily small. By Lemma 5[6], this term convergers to

$$K\int_{-\pi}^{\pi}|\omega|^{-6\epsilon}\mathrm{d}\omega\leqslant K_1<\infty.$$

The other two summands can be bounded similarly by constants K_2 and K_3, respectively. The result follows by choosing $M=\max\{K_1,K_2,K_3\}$.

Theorem 2 (Asymptotic normality). The Bayesian estimate ϕ_n, satisfies the following limiting distribution as $n\to\infty$:

$$\sqrt{n}(\phi_n-\phi_0)\xrightarrow{D}N(0,T(\phi_0)^{-1}),$$

$$T(\phi_0)=\frac{1}{4\pi}\int_{-\pi}^{\pi}[\nabla\ln f_{\phi_0}(\omega)][\nabla\ln f_{\phi_0}(\omega)]'\mathrm{d}\omega. \tag{5}$$

Proof It sufficies to show that

(1) $$\sqrt{n}\nabla\ln\pi(\phi_0)\xrightarrow{D}N(0,T(\phi_0));$$

(2) $$\phi_n\xrightarrow{P}\phi_0 \text{ implies } |\nabla^2\ln\pi(\phi_n)-\nabla^2\ln\pi(\phi_0)|\xrightarrow{P}0;$$

(3) $$\nabla^2\ln\pi(\phi_0)\xrightarrow{P}T(\phi_0).$$

(1) Part(1) follows with the cumulant method. From the proof of Theorem 1, we have $E(\sqrt{n}\nabla\ln\pi(\phi_0))\to 0$ as $n\to\infty$, By using the product theorem for cumulants[8]:

$$n\,\mathrm{cov}(\nabla\ln\pi(\phi_0),\nabla\ln\pi(\phi_0))=\frac{1}{2n}\mathrm{tr}\{\Omega(\phi_0)^{-1}\Omega(\nabla\phi_0)\Omega(\phi_0)^{-1}\Omega(\nabla\phi_0)\}.$$

By Lemma 5[6], this term converges to $T(\phi_0)$. Similarly, we get

$$n^{\frac{p}{2}}\mathrm{cum}\{\nabla\ln\pi(\phi_0)_{i_1},\cdots,\nabla\ln\pi(\phi_0)_{i_p}\}=\frac{1}{2}n^{-\frac{p}{2}}(-1)^p\sum_{\substack{(j_1,\cdots,j_p)\\ \text{permutation of }(i_1,\cdots,i_p)}}\mathrm{tr}\left[\prod_{k=1}^{p}\left\{\Omega(f_{\phi_0})^{-1}\Omega\left(\frac{\partial f_{\phi_0}}{\partial\phi_{j_k}}\right)\right\}\right]$$

which converges to zero by Lemma 5[6].

(2) Result (2) is a consequence of the equicontinuity of the quadratic form $Z_n^{(i)}$ (see Lemma 7[6]). We have

$$R_n(\phi)=\frac{1}{2n}\mathrm{tr}\{\Omega(f_\phi)^{-1}\Omega(\nabla f_\phi)\Omega(f_\phi)^{-1}\Omega(\nabla f_\phi)\},$$

$$\nabla^2\ln\pi(\phi)=Z_n^{(1)}(\phi)-\frac{1}{2}Z_n^{(2)}(\phi)+R_n(\phi),$$

with $Z_n^{(1)}(\phi)$ and $Z_n^{(2)}(\phi)$ as in Lemma 7[6]. Due to the equicontinuity of the quadratic form $Z_n^{(i)}$ (see Lemma 7[6]), it is sufficient to prove that $R_n(\phi_n)-R_n(\phi_0)\xrightarrow{P}0$.

Let η be given with $0<\eta<\frac{1}{12}$. Choose $\epsilon>0$ for all ϕ with $|\phi-\phi_0|<\epsilon$. We have

$$|R_n(\phi_n)-R_n(\phi_0)|\leqslant|\phi_n-\phi_0|\,[\,|\frac{1}{n}\mathrm{tr}\{(\Omega(f_{\phi_1})^{-1}\Omega(\nabla f_{\phi_1}))^3\}|+$$
$$|\frac{1}{n}\mathrm{tr}\{\Omega(f_{\phi_1})^{-1}\Omega(\nabla^2 f_{\phi_1})\Omega(f_{\phi_1})^{-1}\Omega(\nabla f_{\phi_1})\}|\,], \tag{6}$$

with a mean value ϕ_1 with $|\phi_1-\phi_0|<\epsilon$. We prove that the term in the square brackets is uniformly bounded in ϕ_1 by a deterministic constant which gives the result. Let $\frac{\partial f_{\phi_1}}{\partial\phi_i}=g^+-g^-$, with $g^+,g^-\geqslant 0$. We have for all $\delta>0$,

$$\Omega(g^+)\leqslant\Omega\left(\left|\frac{\partial f_{\phi_1}}{\partial\phi_i}\right|\right)\leqslant\Omega(K|\omega|^{-\alpha-\eta-\delta}).$$

Let $A=\Omega(f_{\phi_1})^{-\frac{1}{2}}\Omega(g^{+})\Omega(f_{\phi_1})^{-\frac{1}{2}}$ and $B=\Omega(f_{\phi_1})^{-\frac{1}{2}}\Omega(K|\omega|^{-\alpha-\eta-\delta})\Omega(f_{\phi_1})^{-\frac{1}{2}}$. Since $0\leqslant A\leqslant B$ we obtain

$$\frac{1}{n}\operatorname{tr}\{A^3\}\leqslant\frac{1}{n}\operatorname{tr}\{A^2B\}=\frac{1}{n}\operatorname{tr}\{A^{\frac{1}{2}}BA^{\frac{1}{2}}A\}\leqslant\frac{1}{n}\operatorname{tr}\{A^{\frac{1}{2}}BA^{\frac{1}{2}}B\}\leqslant\frac{1}{n}\operatorname{tr}\{AB^2\}\leqslant\frac{1}{n}\operatorname{tr}\{B^3\},$$

by Theorem 9.1.19 and 12.2.3 of Ref [9]. Furthermore, we get

$$\Omega(f_{\phi_1})^{-1}\leqslant\Omega(K|\omega|^{-\alpha+\eta+\delta})^{-1},$$

by Theorem 12.2.14(2) of Ref [9], and the same arguments now give

$$\frac{1}{n}\operatorname{tr}[\{\Omega(f_{\phi_1})^{-1}\Omega(g^{+})\}^3]\leqslant\frac{1}{n}\operatorname{tr}[\{\Omega(K|\omega|^{-\alpha+\eta+\delta})^{-1}\Omega(K|\omega|^{-\alpha-\eta-\delta})\}^3],$$

which tends to $K\int_{-\pi}^{\pi}|\omega|^{-6\eta-6\delta}\mathrm{d}\omega$ by Lemma 5[6]. All other terms occurring in (6) can be treated similarly and part (2) therefore is proved.

(3) Finally, part (3) follows from the proof of Theorem 1 part(2), since $E[\nabla^2\ln\pi(\phi_0)]\to T(\phi_0)$ and $\operatorname{var}\nabla^2\ln\pi(\phi_0)$ tends to zero as $n\to\infty$.

Theorem 3 (Efficiency) The Bayesian estimate $\hat{\phi}_n$ is asymptotically an efficient estimate of ϕ_0.

Proof Assuming that $I_n(\phi_0)$ denotes the Fisher information matrix $T(\phi_0)^{-1}$ is the variance of the estimate. It suffices to prove that the Fisher information matrix $nI_n(\phi_0)$ converges to the $T(\phi_0)$, as $n\to\infty$. But, this follows directly from part (2) of the proof of Theorem 1, since

$$nI_n(\phi_0)=nE\{[\nabla\ln\pi(\phi_0)][\nabla\ln\pi(\phi_0)]'\}=n\operatorname{var}[\nabla\ln\pi(\phi_0)]\to$$

$$n\cdot\frac{1}{2n^2}\operatorname{tr}\{\Omega(f_{\phi_0})^{-1}\Omega(\nabla f_{\phi_0})\Omega(f_{\phi_0})^{-1}\Omega(\nabla f_{\phi_0})\}=$$

$$\frac{1}{2n}\operatorname{tr}\{\Omega(f_{\phi_0})^{-1}\Omega(\nabla f_{\phi_0})\Omega(f_{\phi_0})^{-1}\Omega(\nabla f_{\phi_0})\}.$$

By Lemma 5[6] we have

$$nI_n(\phi_0)\to\frac{1}{2n}\operatorname{tr}\{\Omega(f_{\phi_0})^{-1}\Omega(\nabla f_{\phi_0})\Omega(f_{\phi_0})^{-1}\Omega(\nabla f_{\phi_0})\}\to$$

$$\frac{1}{4\pi}\int_{-\pi}^{\pi}[\nabla\ln f_{\phi_0}(\omega)][\nabla\ln f_{\phi_0}(\omega)]'\mathrm{d}\omega=T(\phi_0).$$

So $nI_n(\phi_0)$ tends to $T(\phi_0)$, as $n\to\infty$.

3 Simulation

This section reports the results from a small simulation study carried out to examine how the asymptotic theory derived above performs in finite samples. We choose $(1-B)^d x_t=\epsilon_t$ the simple class of ARFIMA models to examine the finite sample behavior of the estimates. We consider models with parameter sets $(d,\sigma_\epsilon^2)=(0.15,1)$, $(0.25,1)$, $(0.35,1)$, and $(0.45,1)$. First, we generate a white noise process $\{\epsilon_t\}$ from normal distribution $N(0,1)$ and then compute the observed series $\{x_t\}$.

Following the conclusion of Ref [5]:

$$\gamma_k=\frac{(-1)^k\Gamma(1-2d)}{\Gamma(k-d+1)\Gamma(1-k-d)}\sigma^2,\quad k=0,1,2,\cdots.$$

We can calculate $L=-\frac{1}{2}\ln|\Omega_n|-\frac{n}{2}\ln(X_n'\Omega_n^{-1}X_n)$ with 50 different values of d selected uniformly from $(-0.5,0.5)$. And we choose $\hat{d}$ which maximizes L as the estimator of d.

Table 1 shows the results from simulations for ARFIMA$(0,d,0)$ process for different sample size n. In the table, the estimated biases and the corresponding empirical root mean squared errors ($\sqrt{\mathrm{MSE}}$) of the estimates are given.

From Table 1, we see that the biases are generally small, especially for $d=0.25$, 0.35 and 0.45. As sample size increases, all biases, empirical root mean squared errors decrease. Thus, overall, the experimental evidence presented above is supportive of our theoretical results.

Table 1 Estimated Parameter Bias and Square Root of the Mean Squared Error for the ARFIMA (0, d, 0) Model

d	N	Bias	$\sqrt{MSE}$	d	N	Bias	$\sqrt{MSE}$
0.15	200	0.12	0.1392	0.15	400	0.11	0.2032
0.25	200	0.06	0.0223	0.25	400	0.02	0.0224
0.35	200	0.03	0.0442	0.35	400	0.02	0.0524
0.45	200	−0.05	0.104	0.45	400	−0.03	0.0559
0.15	300	0.13	0.2094	0.15	500	0.1	0.2236
0.25	300	0.04	0.0296	0.25	500	−0.02	0.0158
0.35	300	0.01	0.0707	0.35	500	−0.01	0.0442
0.45	300	−0.06	0.0669	0.45	500	0.01	0.04

[References]

[1] Granger C W J, Joyeux R. An introduction to long-memory time series models and fractional differencing[J]. Journal of Time Series Analysis, 1980, 1(1): 15-39.

[2] Chan N H. Estimation of long-memory time series: a survey of likelihood-based methods[J]. Advances in Econometrics, 2006, 20(2): 89-121.

[3] Jeffrey S Pai, Nalini R. Bayesian analysis of autoregressive fractionally integrated moving-average processes[J]. Journal of Time Series Analysis, 1995, 19(1): 99-111.

[4] Hannan E J. The asymptotic theory of linear time-series models[J]. Journal of Applied Probability, 1973, 10(4): 913.

[5] Chang Xuejiang, Chen Ming, Wang Mingsheng. Time Series Analysis[M]. Beijing: China Higher Education Press, 1993.

[6] Palma W, Chan N H. Efficient estimation of seasonal long-dependent processes[J]. Journal of Time Series Analysis, 2005, 26(6): 864-891.

[7] Dahlhaus R. Efficient parameter estimation for self-similar processes[J]. The Annals of Statistics, 1989, 17(4): 1749-1766.

[8] Taniguchi M, Kakizawa Y. Asymptotic Theory of Statistical Inference for Time Series[M]. New York: Springer-Verlag, 2000.

[9] Graybiu F A. Matrices with Applications in Statistics[M]. 2nd ed. Belmont, CA: Wads Worth, 1983.

[责任编辑: 丁　蓉]