# Star Extremality of a Class of Incidence Graphs

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**Abstract** Incidence graph $I(C_n)$ of $C_n$ had been proved to be a circulant graph. It was shown that these incidence graphs $I(C_n)$ of all $C_n$ and some graphs related to $I(C_n)$ were star extremal. The circular chromatic number and the fractional chromatic number of these graphs were obtained with isomorphic.

**Key words** incidence graph, circulant graph, star extremal
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# 一类关联图的 Star Extremal性质

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[摘要] 证明了 $C_n$ 的关联图 $I(C_n)$是循环图, 还证明了所有 $C_n$ 的关联图 $I(C_n)$及一些与 $I(C_n)$有关的图是 star extremal的. 并用一种同构的方法得到了它们的圆色数和分色数.

[关键词] 关联图, 循环图, star extremal

We consider only finite, undirected and simple graph in this paper unless stated otherwise. Let $G$ be a graph and let $V(G)$, $E(G)$, $\alpha(G)$ and $\omega(G)$ be the vertex set, edge set, independence number and clique number of $G$, respectively. The definition of incidence graph $I(G)$ of $G$ given in [1] is as follows:

$$V(I(G)) = \{(v, e) \in V(G) \times E(G): v \text{ is incident with } e\}$$

and two vertices $(u, e)$ and $(v, f)$ are adjacent if one of the following holds: (1) $u = v$; (2) $e = f$; (3) $uv = e$ or $f$.

A proper coloring $f: V(G) \to \{c_1, c_2, \cdots, c_k\}$ of $G$ is an assignment of colors to the vertices of $G$ such that $f(u) \neq f(v)$ for all adjacent vertices $u$ and $v$. The chromatic number $\chi(G)$ of $G$ is the minimum number of colors necessary to color $G$ properly, i.e. there exists a proper coloring of $G$.

So far several significant variations of chromatic number have been introduced. One of them is circular chromatic number of a graph, which was introduced first by Vince[2] under the name as the "star chromatic number" of a graph. Suppose $p$ and $q$ are positive integers such that $p \geqslant 2q$. A $(p, q)$-coloring of a graph $G = (V, E)$ is a mapping $c$ from $V$ to $\{0, 1, 2, \cdots, p-1\}$ such that $\| c(x) - c(y) \|_p \geqslant q$ for any edge $xy \in E$, where $\| a \|_p = \min\{a, p-a\}$. The circular chromatic number $\chi_c(G)$ of $G$ is the infimum of the ratios $p/q$ for which there exist $(p, q)$-coloring of $G$. The fractional chromatic number of a graph is another variation of the chromatic number. As a refinement of chromatic number, it was shown in [2] $\chi(G) = \lceil \chi_c(G) \rceil$ for any graph $G$. A fractional colo-

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ring of a graph $G$ is a mapping $c$ from $S(G)$, the set of all independent sets of $G$, to the interval $[0, 1]$ such that $\sum_{x\in S, S\in S(G)} c(S) \geqslant 1$ for all vertices $x$ in $G$. The fractional chromatic number $\chi_f(G)$ of $G$ is the infimum of the value $\sum_{S\in S(G)} c(S)$ over all fractional colorings of $G$. Moreover[3], for all graphs,

$$\max\left\{\omega(G), \frac{|V(G)|}{\alpha(G)}\right\} \leqslant \chi_f(G) \leqslant \chi_c(G) \leqslant \chi(G). \qquad (*)$$

Circulant graph is formulated as follows: $G = G(p, S)$ with $V(G) = \{0, 1, 2, \cdots, p-1\}$, $E(G) = \{uv \mid \|u - v\|_p \in S\}$, where $p \in \mathbf{Z}^+$ and $S \subseteq \{1, 2, \cdots, \lfloor p/2 \rfloor\}$, $\|x\|_p = \min\{|x|, p - |x|\}$. The so-called distance graph denoted by $G(Z, D)$ is a graph, which is closely related to circulant graph, with the set $\mathbf{Z}$ of integers as vertex set and with an edge joining two vertices $u$ and $v$ if and only if $|u - v| \in D$.

According to inequality $(*)$, we have $\chi_f(G) \leqslant \chi_c(G)$ for any graph $G$. The graph is called star extremal when $\chi_f(G) = \chi_c(G)$. The star extremal graph has many interesting properties[3]. Some star extremal distance graphs and circulant graphs can be found in [3] and [4], respectively.

In [5], the author obtained the circular chromatic number of $I(C_n)$ of cycle $C_n$ for $n = 3m$ and $n = 3m + 2$, and gave a bound for $n = 3m + 1$ $(m \geqslant 1)$, by discussing the structure of these graphs.

In this paper we show that the incidence graph $I(C_n)$ of cycle $C_n$ is isomorphic to circulant graph $G(2n, S)$ with $S = \{1, 2\}$. After the proof of star extremality of these circulant graphs, we determine the circular chromatic number and the fractional chromatic number of these graphs with a different method from that used in [5], which can be viewed as a complement of the result in [5]. In the last, we obtain the star extremality of some graphs related to incidence graph $I(C_n)$.

## 1 Main Results

In [5], by discussing the structure of incidence graph $I(C_n)$, the author obtained the following theorems as an incomplete determination of its circular chromatic number.

**Theorem 1** If $n = 3m$ with any positive integer $m$, then $\chi_c(I(C_n)) = 3$.

**Theorem 2** If $n = 3m + 2$ with any positive integer $m$, then $\chi_c(I(C_n)) = 3 + \frac{1}{2m + 1}$.

**Theorem 3** If $n = 3m + 1$ with any positive integer $m$, then $\chi_c(I(C_n)) = 4$ for $n = 4$; $3 + \frac{1}{8m + 1} \leqslant \chi_c(I(C_n)) \leqslant 4$ for $n = 6m + 1$; and $3 + \frac{1}{8m + 5} \leqslant \chi_c(I(C_n)) \leqslant 3 + \frac{1}{2m + 1}$ for $n = 6m + 4$.

Note that in Theorem 3, the case for $n = 3m + 1$ is subdivided into three subcases: $n = 4$, $n = 6m + 1$, and $n = 6m + 4$. However, to our disappointment, the circular chromatic number of incidence graph $I(C_n)$ was not completely determined except for a bound when $n = 3m + 1$.

It is well known and easy to prove that if a graph $G$ is vertex transitive, then $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$. Therefore for any circulant graph $G(n, S)$, we have $\chi_f(G) = \frac{n}{\alpha(G)}$ because that all circulant graphs are vertex transitive. Thus, to prove that a circulant graph $G(n, S)$ is star extremal, it is sufficient to prove that $\chi_c(G) = \frac{|V(G)|}{\alpha(G)}$. From now on, the circular chromatic number $\chi_c(G(n, S))$ and the fractional chromatic number $\chi_f(G(n, S))$ of circulant graph $G(n, S)$ will be written as $\chi_c(G)$ and $\chi_f(G)$ for short, respectively.

To obtain the main theorem, the following several lemmas are important and useful as the preliminaries.

**Lemma 1**[3] Suppose $G = G(p, S)$ is a circulant graph and $|S| = 2$.

(1) If $S = \{1, k\}$, $k$ is odd, and $p > (k(k - 3) + 2)r/2$, where $r$ is the unique number $0 \leqslant r < k$ satisfying $r \equiv p \pmod k$, then $G$ is star extremal.

(2) If $S = \{1, k\}$, $k$ is even, and $p > k(k-1)$, then $G$ is star extremal.

**Lemma 2** If $C_n = v_1 v_2 \cdots v_n$ denotes a cycle on $n$ vertices, the edge set of $C_n$ is written as $\{v_n v_1\} \cup \{e_i = v_i v_{i+1}: i = 1, 2, \cdots, n-1\}$, $I(C_n)$ is the incidence graph of $C_n$, $G(2n, S)$ is a circulant graph with $S = \{1, 2\}$, then $I(C_n)$ is isomorphic to $G(2n, S)$.

**Proof** We define a mapping $f$ from $V(I(C_n))$ to $\{1, 2, \cdots, 2n-1, 2n\}$ such that

$$f(v_i e_j) = \begin{cases} 2j-1, & \text{if } i = j, \\ 2j, & \text{if } i = j+1, \end{cases}$$

where $i, j \in \{1, 2, \cdots, n-1, n\}$. According to the definition of incidence graph and the incident relation of vertices and edges in $C_n$, we can easily obtain that $(v_i e_j)(v_p e_q) \in E(I(C_n))$ if and only if $\| f(v_i e_j) - f(v_p e_q) \|_{2n} = 1$ or 2, which is equivalent to $\| f(v_i e_j) - f(v_p e_q) \|_{2n} \in S$. Considering the definition of circulant graph, the lemma follows.

By Lemma 2, to determine the circular chromatic number and the fractional chromatic number of incidence graph $I(C_n)$, it suffices to obtain the above two chromatic numbers of circulant graph $G(2n, S)$ with $S = \{1, 2\}$.

**Lemma 3** If $S = \{1, 2\}$, and $n > 1$, $m \geqslant 1$ are integers, $G(2n, S)$ is a circulant graph, it follows that $G(2n, S)$ is star extremal. Furthermore,

$$\chi_c(G) = \chi_f(G) = \begin{cases} 3, & \text{if } n = 3m, \\ 3 + \dfrac{1}{m}, & \text{if } n = 3m+1, \\ 3 + \dfrac{1}{2m+1}, & \text{if } n = 3m+2. \end{cases}$$

**Proof** For the special case of Lemma 1, we can immediately get the following conclusion: circulant graph $G(2n, S)$ with $S = \{1, 2\}$ and $n > 1$ is star extremal. Moreover, combining the fact that circulant graph $G(2n, S)$ with $S = \{1, 2\}$ and $n > 1$ is star extremal and all circulant graphs are vertex transitive, our focus can be diverted to gain the independence number $\alpha(G)$ of circulant graph $G(2n, S)$.

In the next step, we will show that $\alpha(G) = 2m$ when $n = 3m+1$, the proof for $\alpha(G) = 2m+1$ when $n = 3m+2$ and $\alpha(G) = 2m$ when $n = 3m$ is similar but easier and so be omitted. Set $S_0 = \{1, 4, 7, \cdots, 3(2m-1)+1\}$, it can be easily verified that $S_0$ is an independence set of circulant graph $G(2n, S)$, therefore $\alpha(G) \geqslant 2m$. Assume that $\alpha(G) > 2m$, suppose $S' = \{i_1, i_2, \cdots, i_{2m+1}\}$ be a subset of size $2m+1$ with $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{2m+1}$ such that $S'$ is an independence set of circulant graph $G(2n, S)$, then $i_1 \leqslant 1$, $i_2 \leqslant i_1 + 3$, $i_3 \leqslant i_2 + 3 \leqslant i_1 + 2 \times 3$, $\cdots$, $i_{2m+1} \geqslant i_{2m} + 3 \geqslant \cdots \geqslant i_1 + 2m \times 3 \geqslant 6m+1$. By the pigeonhole principle, there exist $i_p, i_q \in S'$ such that $|i_p - i_q| < 3$ or $6m + 2 - |i_p - i_q| < 3$. This implies that vertex $i_p$ and vertex $i_q$ are adjacent by the definition of circulant graph. Therefore $S'$ is not independent, this is a contradiction. Thus $\alpha(G) \leqslant 2m$. And this lemma holds immediately.

Combining Lemma 2 and Lemma 3, the following theorem is obvious.

**Theorem 4** If $C_n = v_1 v_2 \cdots v_n$ denotes a cycle on $n$ vertices, then incidence graph $I(C_n)$ is star extremal, and

$$\chi_c(I(C_n)) = \chi_f(I(C_n)) = \begin{cases} 3, & \text{if } n = 3m, \\ 3 + \dfrac{1}{m}, & \text{if } n = 3m+1, \\ 3 + \dfrac{1}{2m+1}, & \text{if } n = 3m+2. \end{cases}$$

Moreover,

$$\chi(I(C_n)) = \begin{cases} 3, & \text{if } n = 3m, \\ 4, & \text{otherwise}. \end{cases}$$

## 2 Star Extremality of Some Graphs Related to $I(C_n)$

In the next, we will discuss the star extremality of the complement of $I(C_n)$ and the square of $I(C_n)$. To determine whether these graphs be star extremal, we first recall these relevant definitions and lemmas.

The complement of graph $G$ denoted by $\overline{G}$ is a graph with a same vertex set as $G$ in which $v_i$ and $v_j$ are adjacent if and only if $v_i$ and $v_j$ are not adjacent in $G$. The following lemma gives a class of circulant graphs which are star extremal.

**Lemma 4**[3] Suppose that $k' = k + l \leqslant \frac{p}{2}$, $S = \{k, k+1, \cdots, k'\}$ and $G = G(p, S)$ is circulant graph. If $p - 2k' < \min\{k, l\}$, then $G$ is star extremal.

**Theorem 5** The complement of incidence graph $I(C_n)$ is star extremal. Moreover, $\chi_c(\overline{I(C_n)}) = \chi_f(\overline{I(C_n)}) = \frac{2n}{3}$.

**Proof** In view of Lemma 2 and the definition of complement of a graph, the graph $\overline{I(C_n)}$ is isomorphic to circulant graph $G(2n, S)$ with $S = \{3, 4, \cdots, n\}$. With application of Lemma 4 with $k = 3$, $k' = n$ and $p = 2n$, it is easily obtained that $G(2n, S)$ is star extremal. It can be calculated that $\alpha(G(2n, S)) = 3$ without difficulty, thus $\chi_f(G(2n, S)) = \frac{2n}{3}$ because of the vertex-transitivity of circulant graph. According to the definition of star extremal graph, this theorem holds certainly.

The square of a graph $G$, written by $G^2$, has the same vertex set as $G$ and has an edge between two vertices if the distance between them in $G$ is at most 2. The square of the incidence graph $I(C_n)$ will be denoted by $I^2(C_n)$ for short. Next we will concentrate on the star extremality of $I^2(C_n)$.

**Lemma 5**[3] If $S = \{1, 2, \cdots, k-1\}$, then the circulant graph $G(n, S)$ is star extremal.

**Theorem 6** The square graph $I^2(C_n)$ is star extremal, and

$$\chi_c(I^2(C_n)) = \chi_f(I^2(C_n)) = \frac{2n}{\lfloor \frac{n}{2} \rfloor} = \begin{cases} 4, & \text{if } n = 2m, \\ 4 + \frac{2}{m}, & \text{if } n = 2m+1. \end{cases}$$

**Proof** By the definition of the square of graph and Lemma 2, the graph $I^2(C_n)$ is isomorphic to circulant graph $G(2n, S)$ with $S = \{1, 2, 3\}$. With a similar method to that in proof of Lemma 3, we can obtain that $\alpha(G(2n, S)) = \lfloor \frac{2n}{4} \rfloor = \lfloor \frac{n}{2} \rfloor$. Combining Lemma 5 and the fact that $G(2n, S)$ is vertex transitive, we obtain this theorem immediately, by considering the two subcases $n = 2m$ and $n = 2m + 1$.

From this theorem, the chromatic numbers of these graphs $I^2(C_n)$ can be easily determined.

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