L vy模型下亚式期权的等价关系

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[摘要] 证明了泊松随机测度在指数鞅测度变换下仍是泊松随机测度, 并利用该结论及勾舍诺夫定理证明了当风险资产价格 S_t 满足方程 $dS_t = S_{t-}[\mu dt + \sigma dB_t + \int_{R_0} K(x)N(dt, dx)]$ 时浮动执行价与固定执行价的亚式期权之间的等价关系.

[关键词] 亚式期权, L vy过程, 随机测度, 等价关系
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Equivalence of Asian Options in L vy Model

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Abstract It was proved that Possion random measures are also Poisson random measures under a change of exponential martingale measure. And by using the theorem and Girsanov theorem, a symmetry relationship between floating strike and fixed strike Asian options was proved as the risk asset S_t was denoted by the solution of $dS_t = S_{t-}[\mu dt + \sigma dB_t + \int_{R_0} K(x)N(dt, dx)]$.

Key words Asian options, L vy process, random measure, equivalence

亚式期权是一种强路径依赖期权, 其到期日的收益不仅取决于标的资产到期日的价格, 还与整个期权有效期 $[0, T]$ 内某段时间标的资产所经历的价格平均值有关. 这种平均值可以是一段时间内的连续平均, 也可以是若干点的离散平均; 可以是算术平均也可以是几何平均. 每一个确定的平均类型都对应着两种亚式期权的形式, 即固定执行价格和浮动执行价格, 它们都具有只能在到期日执行的欧式期权的风格, 不同的是前者的损益函数是在欧式期权的损益函数中用资产价格的平均值取代资产本身的价格, 而后者的损益函数是在欧式期权的损益函数中用资产价格的平均值取代合约的敲定价格 (见表 1). 相对于欧式期权, 亚式期权具有可以避免人为操纵期权价格的优点, 受到广大投资者的欢迎. 许多学者对亚式期权进行了深刻的研究, 例如文 [1] ~ [4].

表 1 亚式期权损益函数的类型

Table 1 Types of payoffs for Asian

期权类型	损益函数
固定执行价的亚式看涨期权	$(\sum(T_1, T) - K)^+$
固定执行价的亚式看跌期权	$(K - \sum(T_1, T))^+$
浮动执行价的亚式看涨期权	$(\theta S_T - \sum(T_1, T))^+$
浮动执行价的亚式看跌期权	$(\sum(T_1, T) - \theta S_T)^+$

其中 $\theta \in \mathbf{R}^+$, $x^+ = \max(x, 0)$.

算术平均: $\sum(T_1, T) \triangleq \frac{1}{T - T_1}\int_{T_1}^{T} S(u)\,du$.

几何平均: $\sum(T_1, T) \triangleq \exp\{\frac{1}{T - T_1}\int_{T_1}^{T} \ln S(u)\,du\}$.

目前有许多论文研究 L vy模型下的期权定价问题. 文 [5] 研究了 L vy模型下欧式期权的定价和套期保值问题. 文 [1] 证明了 L vy模型下浮动执行价与固定执行价的亚式期权之间的等价关系. 本文的目的是对文 [1] 中的结论作进一步的推广.

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1　基本模型

设风险资产价格 S_t 满足方程

$$dS_t = S_{t-}[\mu dt + \sigma dB_t + \int_{R_0} K(x)\tilde{N}(dt,dx)],\ 0 \leqslant t \leqslant T, \tag{1}$$

其中$(B_t, t \geqslant 0)$ 是一标准布朗运动,$\tilde{N}(t,\cdot)$ 是一 Lévy 测度为 $v(\cdot)$ 的校正泊松随机测度,$(B_t, t \geqslant 0)$,$(\tilde{N}(t,\cdot), t \geqslant 0)$ 相互独立,$K(x)$ 是一确定性函数且 $K(x) > -1$,$R_0 = R - \{0\}$.

无风险资产价格 $A_t = e^{rt}$, 其中 $r \geqslant 0$ 为常数,表示无风险利率.

由 Itô 公式知,风险资产的折现值 $\tilde{S}_t = e^{-rt}S_t$ 满足方程

$$d\tilde{S}_t = e^{-rt}(-r)S_t dt + e^{-rt}dS_t = \tilde{S}_{t-}[(\mu - r)dt + \sigma dB_t + \int_{R_0} K(x)\tilde{N}(dt,dx)],$$

从而有

$$\begin{aligned} d\ln\tilde{S}_t =& \frac{1}{\widetilde{S_{t-}}}\widetilde{S_{t-}}[(\mu - r)dt + \sigma dB_t] - \frac{1}{2\widetilde{S_{t-}^2}}\widetilde{S_{t-}^2}\sigma^2 dt + \\ & \int_{R_0}\ln(1 + K(x))\tilde{N}(dt,dx) + \int_{R_0}[\ln(1 + K(x)) - K(x)]v(dx)dt = \\ & \{\mu - r - \frac{1}{2}\sigma^2 + \int_{R_0}[\ln(1 + K(x)) - K(x)]v(dx)\}dt + \sigma dB_t + \\ & \int_{R_0}\ln(1 + K(x))\tilde{N}(dt,dx), \end{aligned}$$

$$\begin{aligned} \tilde{S}_t =& S_0\exp\{\int_0^t[\mu - r - \frac{1}{2}\sigma^2 + \int_{R_0}[\ln(1 + K(x)) - K(x)]v(dx)]du + \\ & \sigma B_t + \int_{R_0}\ln(1 + K(x))\tilde{N}(t,dx)\}. \end{aligned} \tag{2}$$

定理 1　设 $F(t)$ 是任一平方可积的确定性函数,$H(x)$ 是任一 $L_2(R_0,\mathscr{B}(R_0),v)$ 确定性函数,$A \in \mathscr{B}(R_0)$ 且下有界,令

$$Y_t = \int_0^t F(u)dB_u - \frac{1}{2}\int_0^t F^2(u)du + \int_A H(x)N(t,dx) - t\int_A(e^{H(x)} - 1)v(dx),$$

$$v_Q(dx) = e^{H(x)}v(dx),\quad dQ = e^{Y_T}dP,$$

则在该测度变换下,测度 $Q \sim P$, 且$(N(t,A), 0 \leqslant t \leqslant T)$ 在 Q 下是强度为 $v_Q(A)$ 的泊松过程.

证明

$$\begin{aligned} E_Q(\exp(iuN(t,A))) &= E(e^{Y_T + iuN(t,A)}) = E(e^{Y_t + iuN(t,A)}) = \\ E\{\exp[\int_A H(x)N(t,dx) &- t\int_A(e^{H(x)} - 1)v(dx) + iu\int_A N(t,dx)]\} = \\ \exp(-t\int_A(e^{H(x)} - 1)v(dx))&E[\exp(iu\int_A(1 + \frac{H(x)}{iu})N(t,dx))] = \\ \exp\{-t\int_A(e^{H(x)} - 1)v(dx)\}&\exp\{t\int_A[e^{iu(1 + \frac{H(x)}{iu})} - 1]v(dx)\} = \\ \exp(-t\int_A e^{H(x)}v(dx))&\exp\{t\int_A e^{iu + H(x)}v(dx)\} = \\ \exp((e^{iu} - 1)t\int_A e^{H(x)}v(dx)) &= \exp(tv_Q(A)(e^{iu} - 1)). \end{aligned}$$

其中第 4 个等号根据文献[6] 中定理 2.3.8. 上式右边正是强度为 $v_Q(A)$ 的泊松过程的特征函数. 证毕.

令

$$W_t = B_t - \int_0^t F(u)du,$$

$$\widetilde{N_Q}(t,R_0) = \tilde{N}(t,R_0) - t\int_{R_0}(e^{H(x)} - 1)v(dx) = N(t,R_0) - t\int_{R_0}e^{H(x)}v(dx) = N(t,R_0) - tv_Q(R_0),$$

则(2) 式进一步整理得:

$$\tilde{S}_t = S_0\exp\{\sigma W_t - \frac{1}{2}\sigma^2 t + \int_{R_0}\ln(1+K(x))\widetilde{N_Q}(t,\mathrm{d}x) - t\int_{R_0}[K(x) - \ln(1+K(x))]v_Q(\mathrm{d}x)\}\cdot$$
$$\exp\{(\mu - r)t + \sigma\int_0^t F(u)\mathrm{d}u + t\int_{R_0}K(x)(\mathrm{e}^{H(x)} - 1)v(\mathrm{d}x)\}, \tag{3}$$

由勾舍诺夫定理及指数鞅的性质知,(3) 式右边乘积第一项为指数鞅. 因此,在测度 Q 下,$\tilde{S}_t$ 是鞅的充要条件是对几乎所有的 $t \geqslant 0$ 有

$$\mu - r + \sigma F(t) + \int_{R_0}K(x)(\mathrm{e}^{H(x)} - 1)v(\mathrm{d}x) = 0 \quad a.s. \quad Q \text{ 或 } P \tag{4}$$

选择满足条件(4) 式的 $F(t)$ 和 $H(x)$,作上述变换,则$(\tilde{S}_t, t \geqslant 0)$ 在 Q 下是鞅,且

$$\tilde{S}_t = S_0\exp\{\sigma W_t - \frac{1}{2}\sigma^2 t + \int_{R_0}\ln(1+K(x))\widetilde{N_Q}(t,\mathrm{d}x) - t\int_{R_0}[K(x) - \ln(1+K(x))]v_Q(\mathrm{d}x)\}. \tag{5}$$

2 亚式期权间的等价关系

由(5) 式得:

$$S_t = S_0\exp\{\sigma W_t - \frac{1}{2}\sigma^2 t + rt + \int_{R_0}\ln(1+K(x))\widetilde{N_Q}(t,\mathrm{d}x) - t\int_{R_0}[K(x) - \ln(1+K(x))]v_Q(\mathrm{d}x)\} \triangleq$$
$$S_0\exp\{bt + \sigma W_t + \int_{R_0}\ln(1+K(x))\widetilde{N_Q}(t,\mathrm{d}x)\}.$$

其中 $b = r - \frac{1}{2}\sigma^2 - \int_{R_0}(K(x) - \ln(1+K(x)))v_Q(\mathrm{d}x)$.

记

$$V_{flc}(\theta S_T, \Sigma, r, b, \sigma, v, K(x), T_1, T) = E[\mathrm{e}^{-rT}(\theta S_T - \Sigma S_{T_i})]^+,$$
$$V_{flp}(\theta S_T, \Sigma, r, b, \sigma, v, K(x), T_1, T) = E[\mathrm{e}^{-rT}(\Sigma S_{T_i} - \theta S_T)]^+,$$
$$V_{fxp}(K, \Sigma, r, b, \sigma, v, K(x), T_1, T) = E[\mathrm{e}^{-rT}(K - \Sigma S_{T_i})]^+,$$
$$V_{fxc}(K, \Sigma, r, b, \sigma, v, K(x), T_1, T) = E[\mathrm{e}^{-rT}(\Sigma S_{T_i} - K)]^+,$$

其中 E 表示在测度 Q 下取期望值.

定理 2

$$V_{flc}(\theta S_t, \Sigma, r, b, \sigma, v, K(x), T_1, T) = \mathrm{e}^{rT}V_{fxp}(\theta S_0, \Sigma, r, b^*, \sigma, v^*, h(x), 0, T - T_1),$$
$$V_{flp}(\theta S_t, \Sigma, r, b, \sigma, v, K(x), T_1, T) = \mathrm{e}^{rT}V_{fxc}(\theta S_0, \Sigma, r, b^*, \sigma, v^*, h(x), 0, T - T_1),$$

其中

$$b^* = -[r + \frac{1}{2}\sigma^2 + \int_{R_0}(h(x) - \ln(1+h(x)))v_{Q^*}(\mathrm{d}x)],$$
$$v^*(\mathrm{d}x) = (1+K(x))v(\mathrm{d}x),$$
$$h(x) = -\frac{K(x)}{1+K(x)}.$$

证明 由于在测度 Q 下,$(\frac{\tilde{S}_t}{S_0}, t \geqslant 0)$ 是指数鞅,且 $E_Q(\frac{\tilde{S}_t}{S_0}) = 1$,故令$\frac{\mathrm{d}Q^*}{\mathrm{d}Q} = \frac{\tilde{S}_T}{S_0}$,则 $Q^* \sim Q$. 令

$$W_t^* = W_t - \sigma t,$$
$$\widetilde{N_{Q^*}}(t,A) = \widetilde{N_Q}(t,A) - t\int_A(\mathrm{e}^{\ln(1+K(x))} - 1)v_Q(\mathrm{d}x) =$$
$$\widetilde{N_Q}(t,A) - t\int_A K(x)v_Q(\mathrm{d}x) = N_Q(t,A) - tv_{Q^*}(A),$$

其中 $v_{Q^*}(A) = \int_A(1+K(x))v_Q(\mathrm{d}x), \forall A \in \mathscr{B}(R_0)$.

由勾舍诺夫定理及定理1 知,在 Q^* 下 W_t^* 是标准布朗运动. $N_{Q^*}(t,A) = \widetilde{N_Q^*}(t,A) + tv_{Q^*}(A)$ 是强度为 $v_{Q^*}(A)$ 的泊松过程,且 W_t^* 与 $N_{Q^*}(t,A)$ 独立. 在上述变换下有

$$S_t = S_0\exp\{\sigma W_t - \frac{1}{2}\sigma^2 t + rt + \int_{R_0}\ln(1+K(x))\widetilde{N_Q}(t,\mathrm{d}x) - t\int_{R_0}[K(x) - \ln(1+K(x))]v_Q(\mathrm{d}x)\} =$$
$$S_0\exp\{(r+\frac{1}{2}\sigma^2)t + \sigma W_t^* + \int_{R_0}\ln(1+K(x))\widetilde{N_Q^*}(t,\mathrm{d}x) + t\int_{R_0}[\ln(1+K(x)) - \frac{K(x)}{1+K(x)}]v_{Q^*}(\mathrm{d}x)\}.$$

令 $-\ln(1+K(x)) = \ln(1+h(x))$,则由 $K(x) > -1$ 知 $h(x) > -1$. 记 $S_t^* = \dfrac{S_0 S_t}{S_T}$,则

$$S_t^* = S_0\exp\{[r+\frac{1}{2}\sigma^2 + \int_{R_0}[\ln(1+K(x)) - \frac{K(x)}{1+K(x)}]v_{Q^*}(\mathrm{d}x)](t-T) + \sigma(W_t^* - W_T^*) + \int_{R_0}\ln(1+K(x))(\widetilde{N_{Q^*}}(t,\mathrm{d}x) - \widetilde{N_{Q^*}}(T,\mathrm{d}x))\} \overset{d}{=}$$
$$S_0\exp[b^*(T-t) - \sigma W_{T-t}^* + \int_{R_0}\ln(1+h(x))\widetilde{N_Q^*}(T-t,\mathrm{d}x)].$$

其中 $\overset{d}{=}$ 表示前后同分布,

$$b^* = -[r+\frac{1}{2}\sigma^2 + \int_{R_0}[\ln(1+K(x)) - \frac{K(x)}{1+K(x)}]v_{Q^*}(\mathrm{d}x)] = -[r+\frac{1}{2}\sigma^2 + \int_{R_0}(h(x) - \ln(1+h(x)))v_{Q^*}(\mathrm{d}x)].$$

由于 $-W_t^* \overset{d}{=} W_t^*$,故

$$S_t^* \overset{d}{=} S_0\exp\{b^*(T-t) + \sigma W_{T-t}^* + \int_{R_0}\ln(1+h(x))N_{Q^*}(T-t,\mathrm{d}x)\}.$$

在测度 Q^* 下,$(W_t^*, t\geqslant 0)$ 是标准布朗运动,$(N_{Q^*}(t,R_0), t\geqslant 0)$ 是强度为 $v_{Q^*}(R_0)$ 的泊松过程.

$$V_{flc}(\theta S_T,\Sigma,r,b,\sigma,v,K(x),T_1,T) = E[\mathrm{e}^{-rT}(\theta S_T - \Sigma S_{T_i})^+] =$$
$$E\Big[\frac{S_T\mathrm{e}^{rT}}{S_0}(\frac{\theta S_T S_0}{S_T} - \Sigma\frac{S_{T_i}S_0}{S_T})^+\Big] = E\Big[\frac{\widetilde{S_T}}{S_0}(\theta S_0 - \Sigma S_{T_i}^*)^+\Big] =$$
$$E^*[(\theta S_0 - \Sigma S_{T_i}^*)^+] = \mathrm{e}^{rT}E^*[\mathrm{e}^{-rT}(\theta S_0 - \Sigma S_{T_i}^*)^+] =$$
$$\mathrm{e}^{rT}V_{fxp}(\theta S_0,\Sigma,r,b^*,\sigma,v^*,h(x),0,T-T_1),$$

其中 E 表示在测度 Q 下取数学期望,E^* 表示在测度 Q^* 下取数学期望. 类似方法可证明第二等式. 证毕.

注　取 $K(x) = \mathrm{e}^x - 1$,则模型(1)式即为文[1]中的指数 Lévy 过程模型. 因此本文从模型上推广了文[1]中的结论.

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