On Contractive Conditions and Fixed Points in Fuzzy Metric Space

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Abstract It was pointed out that two common fixed point theorems in fuzzy metric space recently given by Pant V contained some mistakes. By modifying the contractive condition, some new common fixed point theorems in fuzzy metric space were established, which improved and generalized the results of Pant.

Key words fuzzy metric space, noncompatible maps, pointwise R-weak commutativity, common fixed point

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关于模糊度量空间中的压缩条件与不动点

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[摘要] 指出 Pant V最近给出的模糊度量空间中的两个公共不动点定理有错误. 通过修改压缩条件, 在模糊度量空间中建立几个新的公共不动点定理, 改进并推广了 Pant的结果.

[关键词] 模糊度量空间, 不相容映射, 点式 R-弱交换性, 公共不动点

1 Basic Definition

The theory of fuzzy sets was introduced by Zadeh in 1965[1]. Since then, many authors have introduced the concept of fuzzy metric space in different ways. In this paper we deal with the fuzzy metric space defined by Kramosil and Michalek[2] and modified by George and Veeramani[3].

Definition 1 The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space (for short, FM-space) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leqslant M(x, z, t+s)$,

(FM-5) $M(x, y, \cdot): (0, \infty) \to [0, 1]$ is continuous,

for all $x, y, z \in X$ and $t, s > 0$.

Remark 1 (FM-2) and (FM-4) imply that $M(x, y, \cdot)$ is nondecreasing for all x, y in X.

Definition 2 Let A and B be maps from a FM-space $(X, M, *)$ into itself.

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(1) A and B are said to be compatible (or asymptotically commuting), if for all $t>0$, $\lim_{n} M(ABx_n, BAx_n, t)=1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Bx_n=z$ for some $z\in X$ [4].

(2) A and B are said to be pointwise R-weakly commuting if there exists $R>0$ such that $M(ABx, BAx, t)\geqslant M(Ax, Bx, t/R)$ for each $x\in X$ and $t>0$ [5].

Remark 2 It is clear from the above definition that A and B will be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Bx_n=z$ for some $z\in X$ but either $\lim_{n\to\infty} M(ABx_n, BAx_n, t)\neq 1$ or the limit does not exist for some $t>0$.

Recently, Pant V[6] gave the following two common fixed point theorems in FM-space under contractive conditions using the notion of noncompatible maps.

Theorem A[6] Let f and g be noncompatible pointwise R-weakly commuting selfmaps of a FM-space $(X, M, *)$ such that

(i) $f(X)\subset g(X)$;

(ii) $M(x, y, t)>\max\{M(gx, gy, th), M(fx, gx, th), M(fy, gy, th), M(fy, gx, th), M(fx, gy, th)\}$, $0\leqslant h<1$, $t>0$.

If the range of f or g is a complete subspace of X, then f and g have a unique common fixed point.

Theorem B[6] Let (A, S) and (B, T) be pointwise R-weakly commuting selfmaps of a FM-space $(X, M, *)$ satisfying the conditions:

(1) $A(X)\subset T(X)$, $B(X)\subset S(X)$;

(2) $M(Ax, By, t)>\max\{M(Sx, Ty, th), M(Ax, Sx, th), M(By, Ty, th), M(Ax, Ty, th), M(By, Sx, th)\}$, $0\leqslant h<1$, $t>0$.

Let (A, S) or (B, T) be a noncompatible pair of mappings. If the range of one of the mappings is a complete subspace of X, then A, B, S and T have a unique common fixed point.

It is easy to see that the conditions of these two theorems can never be satisfied. In fact, if x^* is a common fixed point of f and g (or A, B, S and T), we put $x=y=x^*$ in (ii), which induces a contradiction. The two theorems contain some evident errors, such as "max" and "th" in (ii) should to be "min" and "t/h", respectively. "$M(x, y, t)$" in (ii) of Theorem A should to be "$M(fx, fy, t)$". Moreover, In the proofs of the theorems, also there are some mistakes.

In this paper, by means of modifying the contractive conditions of Theorem A and Theorem B, we establish some new common fixed point theorems in FM-space, which improve and generalize Theorem A and Theorem B.

2 Main Results

In the following, we always assume that $(X, M, *)$ is an FM-space with the following property:

(FM-6) $\lim_{t\to\infty} M(x, y, t)=1$ for all x, y in X.

Actually, in the proof of Theorem A and Theorem B, the author also used the condition (FM-6).

Definition 3 Let A and B be maps from a FM-space $(X, M, *)$ into itself. A and B (or (A, B)) are said to be asymptotically coincident if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Bx_n=z$ for some z in X.

Remark 3 It is easy to see that if A and B are noncompatible, then they are asymptotically coincident for certain.

Definition 4 Let f and g be two functions from $\mathbf{R}^+$ into $[0, 1]$. We define $f>g$, if $f(t)\geqslant g(t)$ for all $t>0$, and there exists at least one $t_0>0$ such that $f(t_0)>g(t_0)$.

Theorem 1 Let (A, S) and (B, T) be pointwise R-weakly commuting selfmaps of a FM-space $(X, M, *)$ satisfying the conditions:

(1) $A(X)\subset T(X)$, $B(X)\subset S(X)$;

(2) $M(Ax, By, t) > \min\{M(Sx, Ty, t), M(By, Sx, t), M(By, Ty, t), M(Ax, Sx, t/h), M(Ax, Ty, t/h)\}$, $0 < h < 1$, $t > 0$, with $Ax \neq By$;

(3) (B, T) is asymptotically coincident;

(4) $B(X)$ or $S(X)$ is a closed subspace of X.

Then A, B, S and T have a unique common fixed point.

Proof Since B and T are asymptotically coincident, there exists a sequence $\{x_n\}$ in X such that $Bx_n \to p$, $Tx_n \to p$ as $n \to \infty$, for some p in X. Also, because $B(X) \subset S(X)$, for each x_n, there exists y_n in X such that $Bx_n = Sy_n$. Thus $Sy_n \to p$.

Ⅰ. Suppose that $S(X)$ is a closed subspace of X, then there exists a point u in X such that $p = Su$. Using condition (2), we get

$$M(Au, Bx_n, t) \geqslant \min\{M(Su, Tx_n, t), M(Bx_n, Su, t), M(Bx_n, Tx_n, t), M(Au, Su, t/h), M(Au, Tx_n, t/h)\}.$$

Letting $n \to \infty$ we have $M(Au, Su, t) \geqslant M(Au, Su, t/h) \geqslant \cdots \geqslant M(Au, Su, t/h^n) \to 1$ as $n \to \infty$, so $Au = Su$.

Since A and S are pointwise R-weak commutating, there exists $R > 0$ such that $M(ASu, SAu, t) \geqslant M(Au, Su, t/R) = 1$ for all $t > 0$, that is $ASu = SAu$, and so $AAu = ASu = SAu = SSu$.

Since $A(X) \subset T(X)$, there exists a point w in X such that $Au = Tw$. We assert that $Tw = Bw$. If not, by condition (2), there exists $t_0 > 0$ such that

$$M(Au, Bw, t_0) > \min\{M(Su, Tw, t_0), M(Bw, Su, t_0), M(Bw, Tw, t_0), M(Au, Su, t_0/h), M(Au, Tw, t_0/h)\} = \min\{1, M(Bw, Su, t_0), M(Bw, Tw, t_0), 1, 1\} = M(Au, Bw, t_0)$$

a contradiction. Hence $Au = Bw = Tw = Su$. Pointwise R-weak commutativity of B and T implies that $BTw = TBw$ and $BBw = BTw = TBw = TTw$. So, we can prove that $Au = AAu$. In fact, if $Au \neq AAu$, then using (2), there exists $t_1 > 0$ such that

$$M(Au, AAu, t_1) = M(AAu, Bw, t_1) > \min\{M(SAu, Tw, t_1), M(Bw, SAu, t_1), M(Bw, Tw, t_1), M(AAu, SAu, t_1/h), M(AAu, Tw, t_1/h)\} = M(Au, AAu, t_1)$$

a contradiction. Thus, $Au = AAu = SAu$ and Au is a common fixed point of A and S. Similarly, we can prove that $Bw = BBw = TBw$, i.e., Bw is a common fixed point of B and T. Since $Au = Bw$, Au is a common fixed point of A, B, S and T.

Ⅱ. Suppose that $B(X)$ is a closed subspace of X. Since $B(X) \subset S(X)$ and $Bx_n \to p$, there exists a point u in X such that $p = Su$, and so the proof is similar to the previous case that $S(X)$ is closed.

It is easy to prove that the common fixed point of A, B, S and T is unique. In fact, if there exists a point ζ in X with $\zeta = A\zeta = B\zeta = S\zeta = T\zeta$ but $\zeta \neq Au$, then by condition (2), there exists $t_* > 0$ such that

$$M(\zeta, Au, t_*) = M(A\zeta, AAu, t_*) > \min\{M(S\zeta, TAu, t_*), M(AAu, S\zeta, t_*), M(AAu, TAu, t_*), M(A\zeta, S\zeta, t_*/h), M(A\zeta, TAu, t_*/h)\} = M(\zeta, Au, t_*)$$

a contradiction. Therefore A, B, S and T have a unique common fixed point.

Exchanging the stations of A, B, S and T, x and y in Theorem 1, respectively, we obtain the following theorem.

Theorem 2 Let (A, S) and (B, T) be pointwise R-weakly commuting selfmaps of a FM-space $(X, M, *)$ satisfying the conditions:

(1) $A(X) \subset T(X)$, $B(X) \subset S(X)$;

(2)′ $M(Ax, By, t) > \min\{M(Sx, Ty, t), M(Ax, Sx, t), M(Ax, Ty, t), M(By, Sx, t/h), M(By, Ty, t/h)\}$, $0 < h < 1$, $t > 0$, with $Ax \neq By$;

(3)′ (A, S) is asymptotically coincident;

(4)′ $A(X)$ or $T(X)$ is a closed subspace of X.

Then, A, B, S and T have a unique common fixed point.

Remark 4 Theorem 1 and Theorem 2 are both correction and generalization of Theorem B.

Setting $A = B = f$ and $S = T = g$ in Theorem 2, we get the following result.

Theorem 3 Let f and g be asymptotically coincident and pointwise R-weakly commuting selfmaps of a fuzzy metric space $(X, M, *)$ such that

(i) $f(X) \subset g(X)$;

(ii) $M(fx, fy, t) > \min\{M(gx, gy, t), M(fx, gx, t), M(fx, gy, t), M(fy, gx, t/h), M(fy, gy, t/h)\}$, $0 \leqslant h < 1$, $t > 0$, $x \neq y$.

If the range of f or g is a closed subspace of X, then f and g have a unique common fixed point.

Remark 5 Theorem 3 is an improvement and generalization of Theorem A.

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