Representing Integers by a Sum of Two Coprime Square-Free Numbers

Sun Xuegong[1,2], Liu Wei[1]

(1. School of Mathematics and Computer Science, Nanjing Normal University, Nanjing 210097, China)
(2. Department of Mathematics and Science, Huaihai Institute of Technology, Lianyungang 222005, China)

Abstract Let $Q_1(n)=\{a\mid 1\leqslant a\leqslant n,\ (a,n)=1$ and a is square-free$\}$. An asympotic formula of $|Q_1(n)|$ is given and applied to linear Diophantine equation of two variables to prove that if $n\geqslant 10^{11}$, then there exist two coprime square-free numbers a and b such that $n=a+b$.

Key words: integers; square-free number; Möbius function

CLC number O156.1 **Document code** A **Article ID** 1001-4616(2008)04-0037-03

表整数为两个互素的无平方因子数的和

孙学功[1,2], 刘 炜[1]

(1. 南京师范大学数学与计算机科学学院, 江苏 南京 210097)
(2. 淮海工学院数理系, 江苏 连云港 222005)

[摘要] 设 n 为正整数, 并且 $Q_1(n)=\{a\mid 1\leqslant a\leqslant n,\ (a,n)=1,\ a$ 为无平方因子数$\}$. 给出了 $|Q_1(n)|$ 的渐进公式, 并将其应用于二元一次方程中, 证明了: 当 $n\geqslant 10^{11}$ 时, 存在互素的无平方因子数 a 和 b, 使得 $n=a+b$.

[关键词] 整数, 无平方因子数, Möbius函数

A positive integer q is called square-free if it is the product of distinct prime numbers or $q=1$. Let x be a positive real number. We write

$$Q(x)=|\{n\mid n\in\mathbf{Z},\ n\leqslant x \text{ and } n \text{ is square-free}\}|.$$

Gegenbauer[1] proved that

$$Q(x)=\frac{6}{\pi^2}x+O(\sqrt{x}).$$

Let $k,\ l$ be two positive integers with $(k,\ l)=1$. Then we give the following notations:

$$q(k,\ l)=\min\{kn+l\mid n\in\mathbf{Z},\ kn+l>0 \text{ and } kn+l \text{ is square-free}\}$$

and

$$Q(x,\ k,\ l)=|\{kn+l\mid n\in\mathbf{Z},\ 0<kn+l\leqslant x \text{ and } kn+l \text{ is square-free}\}|.$$

Wien[2] proved the following results:

(1) For any given number $\varepsilon>0$,

$$Q(x,\ k,\ l)=A_k\frac{x}{k}+O(x^{0.5}k^{-0.25+\varepsilon}+k^{0.5+\varepsilon}),$$

where

Received date: 2008-03-12.
Foundation item: Supported by the National Natural Science Foundation of China (10771103).
Corresponding author: Sun Xuegong, doctor, lecturer, majored in number theory. E-mail: xgsunlyg@163.com

$$A_k = \frac{6}{\pi^2}\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)^{-1}.$$

(2) For any given number $\varepsilon > 0$,

$$q(k, l) = O(k^{1.5+\varepsilon}).$$

In 2002, Dai, Sun and Chen[3] improved the results and got

$$Q(x, k, l) = A_k\frac{x}{k} + R(x, k, l)$$

and

$$q(k, l) \leqslant 80\times 2^{v(k)}k^{1.5},$$

where

$$|R(x, k, l)| \leqslant \inf_{y\geqslant 1}\left\{y + \frac{x}{ky} + 2^{v(k)+2}\frac{1}{ky} + 2^{v(k)+1}\frac{x}{y^2}\right\}$$

and $v(k)$ be the number of the distinct prime divisors of integer k.

Now let n be positive integer, then we write

$$Q_1(n) = \{a \mid 1\leqslant a\leqslant n,\ (a, n) = 1 \text{ and } a \text{ is square-free}\}.$$

In the present paper, we prove the following results.

Theorem 1 For any positive integer n, we have

$$|Q_1(n)| = \frac{6}{\pi^2}\phi(n)\prod_{p\mid n}\left(1-\frac{1}{p^2}\right),$$

where

$$|R(n)| \leqslant \left(\frac{1}{\sqrt{n}} + \frac{1}{n}\right)\phi(n) + 2^{v(n)}Q(\sqrt{n}).$$

Theorem 2 If $n\geqslant 10^{11}$, the equation

$$n = a + b,\ a, b\in Q_1(n)$$

can be solved.

Remark Theorem 2 shows that if $n\geqslant 10^{11}$ be an integer, then n can be represented as a sum of two coprime square-free numbers.

1 Proof of Theorem 1

To prove Theorem 2, we should prove Theorem 1 firstly.

Proof of Theorem 1 By the definition of $Q_1(n)$, we have

$$|Q_1(n)| = \sum_{1\leqslant a\leqslant n,\ (a, n)=1,\ a \text{ is square-free}} 1 = \sum_{a=1}^{n}\left(\sum_{d\mid a,\ d\mid n}\mu(d)\right)\left(\sum_{m^2\mid a,\ (m, n)=1}\mu(m)\right) =$$

$$\sum_{d\mid n}\sum_{1\leqslant m\leqslant\sqrt{n},\ (m, n)=1}\mu(d)\mu(m)\sum_{1\leqslant a\leqslant n,\ m^2\mid a,\ d\mid a} 1 = \sum_{d\mid n}\sum_{1\leqslant m\leqslant\sqrt{n},\ (m, n)=1}\mu(d)\mu(m)\left[\frac{n}{m^2 d}\right] =$$

$$\sum_{d\mid n}\sum_{1\leqslant m\leqslant\sqrt{n},\ (m, n)=1}\frac{n\mu(d)\mu(m)}{m^2 d} - \sum_{d\mid n}\sum_{1\leqslant m\leqslant\sqrt{n},\ (m, n)=1}\mu(d)\mu(m)\left\{\frac{n}{m^2 d}\right\} =$$

$$n\sum_{d\mid n}\sum_{m\geqslant 1,\ (m, n)=1}\frac{\mu(d)\mu(m)}{m^2 d} + R(n) = \sum_{d\mid n}\frac{n\mu(d)}{d}\sum_{m\geqslant 1,\ (m, n)=1}\frac{\mu(m)}{m^2} + R(n) =$$

$$\frac{6}{\pi^2}\phi(n)\prod_{p\mid n}\left(1-\frac{1}{p^2}\right)^{-1} + R(n),$$

where

$$R(n) = -n\sum_{d\mid n}\sum_{m>\sqrt{n},\ (m, n)=1}\frac{\mu(d)\mu(m)}{m^2 d} - \sum_{d\mid n}\sum_{1\leqslant m\leqslant\sqrt{n},\ (m, n)=1}\mu(d)\mu(m)\left\{\frac{n}{m^2 d}\right\}.$$

To give the upper bound of the error term, we can get

$$|R(n)| \leqslant \left|\left(\sum_{d\mid n}\frac{\mu(d)n}{d}\right)\left(\sum_{m>\sqrt{n},\ (m, n)=1}\frac{\mu(m)}{m^2}\right)\right| + \sum_{d\mid n}\sum_{1\leqslant m\leqslant\sqrt{n},\ (m, n)=1}|\mu(d)\mu(m)| \leqslant$$

$$\phi(n)\sum_{m>\sqrt{n},\ (m,\ n)=1}\frac{1}{m^{2}}+\left(\sum_{d\mid n}\mid\mu(d)\mid\right)\left(\sum_{1\leqslant m\leqslant\sqrt{n},\ (m,\ n)=1}\mid\mu(m)\mid\right)\leqslant\left(\frac{1}{\sqrt{n}}+\frac{1}{n}\right)\phi(n)+2^{v(n)}Q(\sqrt{n}),$$

where $v(n)$ is the number of the distinct prime divisors of integer n. This completes the proof of Theorem 1.

2 Proof of Theorem 2

Now, we can obtain a proof of Theorem 2 by Theorem 1.

Proof of Theorem 2 We define two sets as follows

$$A:=\{n-x\mid x\in Q_1(n)\},\ B:=\{y\mid y\in Q_1(n)\}.$$

By Theorem 1, when $n\geqslant 4$, we can obtain

$$\begin{aligned}&\mid A\cap B\mid=\mid A\mid+\mid B\mid-\mid A\cup B\mid\geqslant 2\mid Q_1(n)\mid-\phi(n)\geqslant\\&\frac{12}{\pi^{2}}\phi(n)\prod_{p\mid n}\left(1-\frac{1}{p^{2}}\right)^{-1}+2R(n)-\phi(n)\geqslant\\&\frac{12}{\pi^{2}}\phi(n)\prod_{p\mid n}\left(1-\frac{1}{p^{2}}\right)^{-1}-2\left(\frac{1}{\sqrt{n}}+\frac{1}{n}\right)\phi(n)-2^{v(n)+1}Q(\sqrt{n})-\phi(n)=\\&\frac{12}{\pi^{2}}n\prod_{p\mid n}\left(1+\frac{1}{p}\right)^{-1}-2\left(\frac{1}{\sqrt{n}}+\frac{1}{n}\right)\phi(n)-2^{v(n)+1}Q(\sqrt{n})-\phi(n)\geqslant\\&\left(\frac{12}{\pi^{2}}-1\right)n\prod_{p\mid n}\left(1+\frac{1}{p}\right)^{-1}-(2\sqrt{n}+2+2^{v(n)+1}\sqrt{n})\geqslant\\&\left(\frac{12}{\pi^{2}}-1\right)n\prod_{p\mid n}\left(1+\frac{1}{p}\right)^{-1}-3.5\times 2^{v(n)}\sqrt{n}=\\&\sqrt{n}\prod_{p\mid n}\left(1+\frac{1}{p}\right)^{-1}\left(\left(\frac{12}{\pi^{2}}-1\right)\sqrt{n}-3.5\times 2^{v(n)}\prod_{p\mid n}\left(1+\frac{1}{p}\right)\right)=\\&\sqrt{n}\prod_{p\mid n}\left(1+\frac{1}{p}\right)^{-1}\left(\left(\frac{12}{\pi^{2}}-1\right)\sqrt{n}-3.5\prod_{p\mid n}\left(2+\frac{2}{p}\right)\right).\end{aligned}$$

If prime $p\geqslant 13$, we have

$$2+\frac{2}{p}<p^{0.3}.$$

For any integer $n>1$, we can get

$$\prod_{p\mid n}\left(2+\frac{2}{p}\right)\leqslant\prod_{i=1}^{5}\left(2+\frac{2}{p_i}\right)\prod_{i=1}^{5}p_i^{-0.3}\prod_{p\mid n}p^{0.3}\leqslant 9.376\,88n^{0.3},$$

where $p_i\ (i=1,\ \cdots,\ 5)$ is the $i-$ th prime

Since $n\geqslant 10^{11}$, we can obtain

$$\left(\frac{12}{\pi^{2}}-1\right)\sqrt{n}-3.5\prod_{p\mid n}\left(2+\frac{2}{p}\right)\geqslant 0.215\,85n^{0.5}-32.819\,1n^{0.3}>0$$

Hence

$$\mid A\cap B\mid>0,$$

this completes the proof of Theorem 2

[References]

[1] Gegenbauer L. Asymptotische gesetze der zahlentheorie[J]. Denkshcriften Akad Wien, 1885, 49: 37-80

[2] Wien K P. Über die Kleinste quadratfreie Zahl einer arithmetischen Reihe[J]. Monatsh Math, 1958, 62: 173-176.

[3] Dai L X, Sun X G, Chen Y G. The squarefree numbers in an arithmetic progression[J]. Journal of Nanjing Normal University: Natural Science Edition, 2002, 25(4): 5-9.

[责任编辑: 丁 蓉]