Existence of Explosive Solutions for a Class of Quasilinear Ordinary Differential Equations

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Abstract By the quadrature method, an explosive solution for a class of quasilinear ordinary differential equations with boundary conditions are obtained.

Key words quasilinear ordinary differential equation, nonlinear boundary conditions, Nagumo condition, explosive solutions

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一类拟线性微分方程爆破解的存在性

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[摘要] 通过积分的方法得到了一类带有边值条件的拟线性微分方程爆破解的存在性.

[关键词] 拟线性微分方程, 非线性边值条件, Nagumo条件, 爆破解

In this paper, we investigate sufficient conditions for the existence of boundary blow-up solutions of the following model problem

$$(\Phi_p(u'))' = \lambda f(u(x)), \tag{1}$$

$$\lim_{x\to 0^+} u(x) = \infty = \lim_{x\to 1^-} u(x), \tag{2}$$

where λ is a positive parameter, $f\in C^1(\mathbf{R})$ or $f\in C([a,\infty))\cap C^1(a,\infty)$ for some $a\in\mathbf{R}$, and $\Phi_p(u) = |u|^{p-2}u$, $p>1$.

Large solutions of the problem

$$u(x) = f(u(x)),\quad x\in\Omega, \tag{3}$$

$$u|_{\partial\Omega} = \infty, \tag{4}$$

where Ω is bounded domain in $\mathbf{R}^N$ $(N\geqslant 1)$ have been extensively studied[1-13]. A problem of this was first considered by Bieberbach[6] in 1916, where $f(u) = -e^u$ and $N=2$. Bieberbach showed that if Ω is a bounded domain in $\mathbf{R}^2$ such that $\partial\Omega$ is a C^2 sub-mainfold of $\mathbf{R}^2$, then there exists a unique $u\in C^2(\Omega)$ such that $-u = -e^u$ in Ω and $|u(x) - \ln(d(x))^{-2}|$ is bounded on Ω. Here $d(x)$ denotes the distance from a point x to $\partial\Omega$. Rademacher[10], using the idea of Bieberbach, extended to smooth bounded domain in $\mathbf{R}^3$. In this case the problem plays an important role, when $N=2$, in the theory of Riemann surfaces of constant negative curvature and in the theory of automorphic functions, and when $N=3$, according to [10], in the study of the electric potential in a glowing

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hollow metal body. Lazer and McKenna[4] extended the results for Ω a bounded domain in $\mathbf{R}^N$ $(N \geqslant 1)$ satisfying a uniform external sphere condition and the nonlinearity $f = f(x, u) = p(x)e^u$, where $p(x)$ is continuous and strictly negative on E. Lazer and McKenna[4] obtained similar results when replaced by the Monge-Ampere operator and Ω is a smooth, strictly convex, bounded domain. Similar results were also obtained for $f = p(x)u^a$ with $a > 1$. Posteraro[9], for $f(u) = -e^u$ and $N \geqslant 2$, proved estimates for the solution $u(x)$ of (3), (4) and for the measure of Ω comparing this problem with a problem of the same type defined in a ball. In particular, when $N = 2$, Posteraro obtained an explicit estimate of the minimum of $u(x)$ in terms of the measure of Ω:

$$\min_{\Omega} u(x) \geqslant \ln(8\pi / |\Omega|).$$

Further, the case replaced by the p-Laplacian has been discussed by Diaz and Letelier[3] when $f(u) = bu^a$ with $a > p-1$ and $p > 2$.

For general nonlinearities $f(u)$ and in one space dimension, Anuradha et al[1], Wang Shin Hwa[2] and Zhang Zhijun[14] proved the existence and multiplicity of large nonnegative solutions basing on building a quadrature method for problem (3), (4). Very recently, Yang H and Yang Z[15] proved the existence and multiplicity of large nonnegative solutions for the problem (1), (2) basing on building a quadrature method. In this section, we further obtain some new existence and non-existence results of large nonnegative solutions to (1), (2) under new conditions by using quadrature method, the extends and complementary to the cases considered in [1, 2, 14, 15].

1 Main Results

First, define

$$F(s) = \int_0^s f(t)\,dt,$$

and

$$I = \{s \in \mathbf{R}: f(s) > 0 \text{ and } F(s) > F(u) \text{ for all } u > s \text{ and } \int^{\infty} \frac{du}{(F(s) - F(u))^{1/p}} < \infty\}.$$

Suppose that u is a nonnegative solution of problem (1), (2).

Let

$$\rho = \inf_{x \in (0,1)} u(x).$$

First, we shall prove that u is symmetric to $x = 1/2$. Let u be a positive solution of (1), (2). Then u has only one minimum point in (0, 1) (there is no local maximum point of u in (0, 1)) and u is the unique solution of the problem

$$v'' = -\frac{f(v)}{(p-1)\,|v'|^{p-2}},$$
$$v(\varepsilon) = v_0,$$
$$v'(\varepsilon) = u'(\varepsilon),$$

in $[\varepsilon, \xi_0)$, ε is given arbitrarily constant and ξ_0 is the minimum point of u by the standard ordinary differential equation theory. (Suppose u has two minimum points in (0, 1), we can reach a contradiction by directly integrating (1). The similar argument implies that there is no local maximum point of u in (0, 1).) Let $y = 1 - x$ for $x \in (\xi_0, 1-\varepsilon]$ and $u(y) = u(1-x)$. Then $u(y)$ satisfies the problem

$$u_{yy} = -\frac{f(u)}{(p-1)\,|u|^{p-2}} \text{ in } [\varepsilon, 1-\xi_0),$$
$$u(\varepsilon) = v_0,$$
$$u(\varepsilon) = u'(\varepsilon).$$

Thus, $u(x)$ and $u(y)$ satisfy the same initial value problem. Let η be the minimum point of u, then $\eta = \xi_0$. Since $u(x) = u(y)$ in (ε, ξ_0) and $u_x(\xi_0) = -u_y(1-\xi_0) = 0$, then $\eta = 1 - \xi_0$. This implies $\xi_0 = 1/2$ and $u(x)$

$= u(1-x)$ for $x \in (\varepsilon, 1/2)$, from ε being arbitrarily, therefore, u is symmetric to $x = 1/2$ and $u' < 0$ in $(0, 1/2)$ and $u' > 0$ in $(1/2, 1)$. That is, $u(x)$ must achieve its minimum at $x = 1/2$.

Multiplying (1) through by $u'(x)$, we obtain

$$(\Phi_p(u'))' u'(x) = \lambda f(u) u'(x),$$

which can be integrated yielding

$$\frac{(p-1)}{p} |u'|^p = \lambda F(u(x)) + C. \tag{5}$$

If $\rho = \inf_{x \in (0,1)} u(x)$, then $u(1/2) = \rho$. Substituting $x = 1/2$ in (5), we have

$$C = -\lambda F(u(1/2)) = -\lambda F(\rho).$$

Thus

$$u'(x) = -\left(\frac{p\lambda}{p-1}\right)^{1/p} (F(u) - F(\rho))^{1/p}, \quad \forall x \in (0, 1/2), \tag{6}$$

and by symmetry,

$$u'(x) = \left(\frac{p\lambda}{p-1}\right)^{1/p} (F(u) - F(\rho))^{1/p}, \quad \forall x \in (1/2, 1).$$

Dividing through by $(F(u) - F(\rho))^{1/p}$ and integrating (6) from 0 to x, we obtain

$$\int_{u(0)}^{u(x)} \frac{\mathrm{d}s}{(F(s) - F(\rho))^{1/p}} (F(u) - F(\rho))^{1/p}, \ \forall x \in (0, 1/2). \tag{7}$$

Substituting $x = 1/2$ in (7), we see that $G(\rho)$ must exist, $G(\rho)$, λ and ρ must satisfy

$$G(\rho) = 2\left(\frac{p-1}{p}\right)^{1/p} \int_{\rho}^{\infty} \frac{\mathrm{d}s}{(F(s) - F(\rho))^{1/p}} = \lambda^{1/p}. \tag{8}$$

In fact we have the following lemma

Lemma 1 Assume that $p > 1$. Then, given $\lambda > 0$, there exists a solution u to (1), (2) with $\inf_{x \in (0,1)} u(x) = \rho \in \mathbf{R}$ if and only if

$$G(\rho) = 2\left(\frac{p-1}{p}\right)^{1/p} \int_{\rho}^{\infty} \frac{\mathrm{d}u}{(F(s) - F(\rho))^{1/p}} = \lambda^{1/p}, \quad \text{for } \rho \in I.$$

Lemma 2 Let $\alpha > 0$, $f \in C^1(\mathbf{R})$ or $f \in C([a, \infty)) \cap C^1((a, \infty))$ for some $a \in \mathbf{R}$ and $m \in I$, $G(m) < \infty$, if and only if

$$H(m) := \int_m^{\infty} \frac{\mathrm{d}s}{(F(s))^{1/p}} < \infty. \tag{9}$$

Proof Note that

$$G(m) := 2\left(\frac{p-1}{p}\right)^{1/p} \int_m^{\infty} \frac{\mathrm{d}s}{(F(s) - F(m))^{1/p}} < \infty, \quad \text{for } m \in I,$$

if and only if

$$\int_m^{m+\delta} \frac{\mathrm{d}s}{(F(s) - F(m))^{1/p}} < \infty, \quad \delta > 0$$

and

$$\int_b^{\infty} \frac{\mathrm{d}s}{(F(s) - F(m))^{1/p}} < \infty, \quad b > m.$$

Let $m \in I$, since I is in $[0, \infty)$, there exists $\delta > 0$, such that $[m, m+\delta] \subset I$, and $f(s) > 0$, $\forall s \in [m, m+\delta]$. Define $c(m, \delta) = \min_{s \in [m, m+\delta]} f(s)$, we see that $c(m, \delta) > 0$ and

$$\int_m^{m+\delta} \frac{\mathrm{d}s}{\left(\int_m^s f(t)\,\mathrm{d}t\right)^{1/p}} \leqslant \int_m^{m+\delta} \frac{\mathrm{d}s}{(c(m, \delta)(s-m))^{1/p}} = \frac{1}{(c(m, \delta))^{1/p}} \int_m^{m+\delta} \frac{\mathrm{d}s}{(s-m)^{1/p}} =$$

$$\frac{1}{(c(m, \delta))^{1/p}} \cdot \frac{\delta^{\frac{p-1}{p}}}{\frac{p-1}{p}} = \frac{p \cdot \delta^{\frac{p-1}{p}}}{(p-1)(c(m, \delta))^{1/p}} < \infty.$$

Obviously,

$$\int_{b}^{\infty} \frac{\mathrm{d}s}{(F(s))^{1/p}} \quad \int_{b}^{\infty} \frac{\mathrm{d}s}{(F(s)-F(m))^{1/p}} < \infty.$$

On the other hand, if

$$\int_{b}^{\infty} \frac{\mathrm{d}s}{(F(s))^{1/p}} < \infty, \quad \text{i.e.} \ \forall \varepsilon > 0, \ \exists B > b,$$

such that

$$\int_{B'}^{B''} \frac{\mathrm{d}s}{(F(s))^{1/p}} < \frac{\varepsilon}{2}, \ B'' > B' > B.$$

Since $F(s)$ is increasing on I, we see that

$$0 < \frac{s}{(F(s))^{1/p}} \leqslant 2\int_{\frac{s}{2}}^{s} \frac{\mathrm{d}t}{(F(t))^{1/p}} < \varepsilon, \ \forall s > 2B,$$

i.e., $\lim\limits_{s\to+\infty}\frac{s}{(F(s))^{1/p}} = 0$. Thus, there exists $B_1 > 2m$ such that $F(s) \geqslant 2F(m)$, $s \geqslant B_1$. Consequently,

$$F(s) - F(m) \geqslant \frac{1}{2}F(s), \ \forall s \geqslant B_1,$$

$$\int_{B_1}^{\infty} \frac{\mathrm{d}s}{(F(s)-F(m))^{1/p}} \leqslant 2^{1/p} \int_{B_1}^{\infty} \frac{\mathrm{d}s}{(F(s))^{1/p}} < \infty.$$

Theorem 1 If there exists $\alpha \leqslant p$, such that

$$\lim_{s\to+\infty} \sup \frac{f(s)}{s^{p-1}(\ln s)^{\alpha}} = L, \ L \in [0, \infty), \tag{10}$$

then, problem (1), (2) has no solution in $C^1(0, 1)$, for any $\lambda > 0$.

Proof By (10), there exist $L > L$ and $B_2 > 0$, such that

$$f(s) \leqslant Lg(s), \ g(s) = L(1+s)^{p-1}[p(\ln(1+s))^{\alpha} + \alpha(\ln(1+s))^{\alpha-1}], \ \forall s \geqslant B_2,$$

which implies

$$F(s) \leqslant L\int_{0}^{s} g(t)\,\mathrm{d}t = L(1+s)^{p}[\ln(1+s)]^{\alpha}, \ \forall s \geqslant B_2,$$

$$\int_{B_2}^{\infty} \frac{\mathrm{d}s}{(F(s))^{1/p}} \geqslant \frac{1}{L}\int_{B_2}^{\infty} \frac{\mathrm{d}s}{(1+s)[\ln(1+s)]^{\alpha/p}} = \begin{cases} \frac{1}{L}\ln(\ln(1+s))\Big|_{B_2}^{\infty} = \infty, & \alpha = p, \\ \frac{p}{L(p-2)}[\ln(1+s)]^{(p-\alpha)/p}\Big|_{B_2}^{\infty} = \infty, & \alpha < p. \end{cases}$$

Hence, $G(m) = \infty$, i.e., problem (1), (2) has no solution, for any $\lambda > 0$.

Theorem 2 If there exists $\alpha > p$, such that

$$\lim_{s\to+\infty} \inf \frac{f(s)}{s^{p-1}(\ln s)^{\alpha}} = L, \ L \in (0, \infty], \tag{11}$$

then, there exists a solution $u \in C^1(0, 1)$ to problem (1), (2) for some $\lambda > 0$. Moreover, $G(m)$ is well defined and continuous, for all $m \in I$.

Proof By (11), there exist $l \in (0, L)$ and $B_3 > 0$, such that

$$f(s) \geqslant lg(s), \ g(s) = (1+s)^{p-1}[p(\ln(1+s))^{\alpha} + \alpha(\ln(1+s))^{\alpha-1}], \ \forall s \geqslant B_2,$$

which implies

$$F(s) \geqslant l\int_{0}^{s} g(s)\,\mathrm{d}s = l(1+s)^{p}[\ln(1+s)]^{\alpha}, \ \forall s \geqslant B_3,$$

$$\int_{B_3}^{\infty} \frac{\mathrm{d}s}{(F(s))^{1/p}} \leqslant \frac{1}{l}\int_{B_3}^{\infty} \frac{\mathrm{d}s}{(1+s)[\ln(1+s)]^{\alpha/p}} = \frac{1}{l}\,\frac{p}{p-\alpha}[\ln(1+B_3)]^{p-\alpha/p} < \infty,$$

for $\alpha > p$. Thus, $G(m) < \infty$. By Lemma 1 and Lemma 2, we obtain that there exists a solution $u \in C^1(0, 1)$ to problem (1), (2) for some $\lambda > 0$. Moreover, $G(m)$ is well defined and continuous, for all $m \in I$.

Corollary 1 If there exists $\alpha \leqslant p$, such that

$$\lim_{s\to+\infty}\sup\frac{f(s)}{s(\ln s)^p(\ln(\ln s))^\alpha}=L,\ L\in[0,\infty].$$

Then problem (1), (2) has no solution in $C^1(0,1)$, for any $\lambda>0$.

Corollary 2 If there exists $\alpha>p$, such that

$$\lim_{s\to+\infty}\inf\frac{f(s)}{s(\ln s)^p(\ln(\ln s))^\alpha}=L,\ L\in(0,\infty),$$

then there exists a solution $u\in C^1(0,1)$ to problem (1), (2) for some $\lambda>0$. Moreover, $G(m)$ is well defined and continuous, for $m\in I$.

Corollary 3 If there exists $\alpha\leqslant p$, such that

$$\lim_{s\to\infty}\sup\left(\frac{f(s)}{s^{p-1}(\underbrace{\ln\cdots\ln}_{n-1}s)^p(\underbrace{\ln\cdots\ln}_{n}s)^\alpha}\right)=L,\ L\in[0,\infty),$$

then problem (1), (2) has no solution in $C^1(0,1)$ for any $\lambda>0$.

Corollary 4 If there exists $\alpha>p$, such that

$$\lim_{s\to\infty}\inf\left(\frac{f(s)}{s^{p-1}(\underbrace{\ln\cdots\ln}_{n-1}s)^p(\underbrace{\ln\cdots\ln}_{n}s)^\alpha}\right)=L,\ L\in(0,\infty],$$

then there exists a solution $u\in C^1(0,1)$ to problem (1), (2) for any $\lambda>0$.

2 Application

The following example illustrates an application of the main results for this paper.

Example Consider the problem

$$\begin{cases}(\Phi_p(u'))'=\lambda e^u,\\ \lim\limits_{x\to0+}u(x)=\infty=\lim\limits_{x\to1-}u(x).\end{cases}$$

This example for which $f(u)=e^u$ demonstrates Lemma 1. Note that $F(u)=e^u$ implies

$$G(\rho)=2\left(\frac{p-1}{p}\right)^{1/p}\int_\rho^\infty\frac{du}{(e^u-e^\rho)^{1/p}}.$$

Letting $w=e^{u/p}$, we obtain

$$G(\rho)=2p\left(\frac{p-1}{p}\right)^{1/p}\int_{e^{\rho/p}}^\infty\frac{dw}{w(w^p-e^\rho)^{1/p}}.$$

Letting $w=e^{\rho/p}\sec^{2/p}\theta$, we obtain

$$G(\rho)=2p\left(\frac{p-1}{p}\right)^{1/p}e^{-\rho/p}\int_0^{\pi/2}\tan^{(p-2)/p}\theta d\theta=p\pi e^{-\rho/p}\left(\frac{p-1}{p}\right)^{1/p}\csc(\pi/p).$$

From $\lim\limits_{\rho\to\infty}G(\rho)=0^+$, $\lim\limits_{\rho\to0+}G(\rho)=\infty$, $G(\rho)$ is strictly decreasing on $(0,\infty)$. These results imply that G is a bijective mapping from $(0,\infty)$ onto $(0,\infty)$. Thus, given $\lambda>0$ there exists a unique $\rho>0$ such that $G(\rho)=\lambda^{1/p}$. Hence, by Lemma 1, there is a nonnegative solution to (1), (2) for each $\lambda>0$. Hence, $\lambda=p^p(\pi)^p e^{-\rho}((p-1)/p)(\csc(\pi/p))^p$.

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