Generalization of a Normality Criteria

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Abstract A normality criteria for families of meromorphic functions that concern the exceptional functions of derivatives is obtained, which improve and generalize related result of Xu and Pang. Let $(\quad 0)$ be a function holomorphic in a domain G C, and k N. Let $\mathscr{F}$ be a family of meromorphic functions defined in G, all of whose poles are multiple and whose zeros all have multiplicity at least $k+2$. If, for every function f $\mathscr{F}$, $f^{(k)}(z)+a_1(z)f^{(k-1)}(z)+ \quad + a_k(z)f(z) \quad (z)$, and $a_1(z)$, $a_2(z)$, , $a_k(z)$ be holomorphic functions in domain G, then it is proved that $\mathscr{F}$ is normal in G.

Key words: meromorphic function, normal family, exceptional function

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一个正规定则的推广

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[摘要] 研究了亚纯函数的正规性, 推广了徐焱和庞学成的正规定则. 得到: 设 $(\quad 0)$是 G C上的一列全纯函数族, 且 k N. 设 $\mathscr{F}$是 G上的一列亚纯函数族, 且零点的级数为 2, 极点的级数至少为 $k+2$. 对于任意的 f $\mathscr{F}$, 都有 $f^{(k)}(z)+a_1(z)f^{(k-1)}(z)+ \quad + a_k(z)f(z) \quad (z)$, 这里的 $a_1(z)$, $a_2(z)$, , $a_k(z)$是 G上的全纯函数, 则 $\mathscr{F}$在 G正规.

[关键词] 亚纯函数, 正规族, 例外函数

Let G be a domain in **C**, and $\mathscr{F}$ be a family of meromorphic functions defined in G, $\mathscr{F}$ is said to be normal in G, in the sense of Montel, if for any sequence f_n $\mathscr{F}$ there exists a subsequence f_{n_j}, such that f_{n_j} converges spherically locally uniformly in G, to a meromorphic function or (see[1-3]).

In 2002, Xu[4] proved the following normality criterion, which improve and generalize related results of Gu, Yang, Schwick, Wang-Fang and Pang-Zalcman (see[5-9]).

Theorem A Let $(\quad 0)$ be a function holomorphic in a domain G C, and k N. Let $\mathscr{F}$ be a family of meromorphic functions defined in G, all of whose poles are multiple and whose zeros all have multiplicity at least $k+2$. If, for every function f $\mathscr{F}$, $f^{(k)}(z) \quad (z)$, then $\mathscr{F}$ is normal in G.

In 2004, Wang and Pang[10] obtained the following result:

Theorem B Let $\mathscr{F}$ be a family of meromorphic functions on domain D, all of whose zeros are of multiplicity at least $m+3$. Let h 0, a_0, a_1, , a_{m-1} be holomorphic functions on D. If for any f $\mathscr{F}$, $f^{(m)}(z)+a_m(z)f^{(m-1)}(z)+ \quad + a_0(z)f(z) \quad h(z)$, then $\mathscr{F}$ is normal in D.

It is natural to consider to weak the condition $m+3$ in Theorem B. In this paper, we shall prove the following result:

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Theorem 1 Let $(\quad 0)$ be a function holomorphic in a domain $G \quad \mathbf{C}$, and $k \quad \mathbf{N}$. Let $\mathscr{F}$ be a family of meromorphic functions defined in G, all of whose poles are multiple and whose zeros all have multiplicity at least $k+2$. If, for every function $f \quad \mathscr{F}$, $f^{(k)}(z)+a_1(z)f^{(k-1)}(z)+ \quad +a_k(z)f(z) \quad (z)$, and $a_1(z), a_2(z), \quad , a_k(z)$ be holomorphic functions in domain G, then $\mathscr{F}$ is normal in G.

Remark 1 The following example shows that Theorem 1 cannot be extended to the case where $a_1(z), \quad , a_k(z)$ are meromorphic functions in the $k=1$ case.

Example 1 Let $G=\{z \quad |z|<1\}$, $a_1(z)=(1/z^3-3/z)$, $\quad (z)=z$ and $\mathscr{F}=\{\quad nz^3 \; |z \quad G, \; n=1,2, \quad \}$. Then for $k=1$, we have $f\ (z)+a_1(z)f(z)=n \quad z$. However, $\mathscr{F}$ is not normal in G.

1 Some Lemmas

To prove our result, we need the following lemmas.

Lemma 1[8] Let f be a meromorphic function of finite order in the plane, k be a positive integer. If all zeros of f are of order at least $k+2$ and $f^{(k)}(z) \quad 1$, then $f(z)$ is a constant.

Lemma 2[11] Let k be a positive integer and let $\mathscr{F}$ be a family of functions meromorphic in a domain D, such that each function $f \quad \mathscr{F}$ has only zeros of multiplicity at least k, and suppose that there exists $A \quad 1$ such that $|f^{(k)}(z)| \quad A$ whenever $f(z)=0$. If $\mathscr{F}$ is not normal at $z_0 \quad D$, then, for each $0 \quad \quad k$, there exist a sequence of points $z_n \quad D$, $z_n \quad z_0$, a sequence of positive numbers $_n \quad 0$, and a sequence of functions $f_n \quad \mathscr{F}$, such that

$$g_n(\quad)=\frac{f_n(z_n+\quad_n\quad)}{_n} \qquad g(\quad)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbf{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\quad) \quad g^{\#}(0)=kA+1$. Moreover, g has order at most 2.

Lemma 3 Let $(\quad 0)$ be a function holomorphic in a domain $G \quad \mathbf{C}$, and $k \quad \mathbf{N}$. Let $\mathscr{F}$ be a family of meromorphic functions defined in G, all of whose zeros all have multiplicity at least $k+2$. If, for every function $f \quad \mathscr{F}$, $f^{(k)}(z)+a_1(z)f^{(k-1)}(z)+ \quad +a_k(z)f(z) \quad (z)$, and $a_1(z), a_2(z), \quad , a_k(z)$ be holomorphic functions in domain G, then $\mathscr{F}$ is normal in G.

Proof Suppose that $\mathscr{F}$ is not normal at $z_0 \quad G$. then, by Lemma 2, there exists a sequence $f_n \quad \mathscr{F}$, a sequence of complex numbers $z_n \quad z_0$ and a sequence of positive number $_n \quad 0$, such that

$$g_n(\quad)=\frac{f_n(z_n+\quad_n\quad)}{^k_n} \qquad g(\quad)$$

converges spherically uniformly on compact subsets of $\mathbf{C}$, $g(\quad)$ is a nonconstant meromorphic function on $\mathbf{C}$, all zeros of $g(\quad)$ have multiplicity at least $k+2$, and moreover $g(\quad)$ is of order at most 2. Obviously, the function $g^{(k)}(\quad)- \quad (z_0)$ is the uniform limit of

$$g_n^{(k)}(\quad)+\quad_n a_1(z_n+\quad_n\quad)\,g_n^{(k-1)}(\quad)+\quad+\quad^k_n a_k(z_n+\quad_n\quad)\,g_n(\quad)-\quad(z_n+\quad_n\quad)=$$

$$f_n^{(k)}(z_n+\quad_n\quad)+a_1(z_n+\quad_n\quad)f_n^{(k-1)}(z_n+\quad_n\quad)+\quad+a_k(z_n+\quad_n\quad)f_n(z_n+\quad_n\quad)-\quad(z_n+\quad_n\quad).$$

Since $f_n^{(k)}(z)+a_1(z)f_n^{(k-1)}(z)+ \quad +a_k(z)f_n(z) \quad (z)$, we can get $g^{(k)}(\quad)- \quad (z_0) \quad 0$ or $g^{(k)}(\quad) - \quad (z_0) \quad 0$ by Hurwitz s theorem.

If $g^{(k)}(\quad)- \quad (z_0) \quad 0$, then $g(\quad)$ is a polynomial, and its degree is k, which contradicts with that the zeros of $g(\quad)$ are of multiplicity $\quad k+2$.

If $g^{(k)}(\quad)- \quad (z_0) \quad 0$, without loss of generality, we assume $g^{(k)}(\quad) \quad 1$. By Lemma 1, we can get $g(\quad)$ is a constant, which contradicts with $g(\quad)$ is a nonconstant meromorphic function. Lemma 3 is proved.

Lemma 4[12] Let f be a transcendental meromorphic function of finite order and let $b(z)$ be a polynomial which does not vanish identically. If f has only zeros of order at least 2, then $f-b(z)$ has infinitely many zeros.

Lemma 5[4] Let f be a transcendental meromorphic function, $k(\quad 2)$, l be positive integers. If f has only zeros of order at least 3, then $f^{(k)}-z^l$ has infinitely many zeros.

Lemma 6[4] Let k be a positive integer, and let $Q(z)$ be a rational function all of whose zeros are of order at least $k+2$ and all of whose poles are multiple with the possible exception of $z=0$. Then, for each positive integer l, $Q^{(k)}(z)=z^{l}$ has a solution in $\mathbf{C}$.

2 Proof of Theorem

Proof of Theorem 1 Since normality is a local property, without loss of generality, we may assume $G=D$ $=\{z\ \ |z|<1\}$, and

$$(z)=z^{l}+a_{l+1}z^{l+1}+\quad =z^{l}\quad (z),\quad z\quad D$$

where $l\quad 1$, $\ (0)=1$, $\ (z)\quad 0$ for $0<|z|<1$, and it is enough to show that $\mathscr{F}$ is normal at each $z\quad D$. By Lemma 3, we only need to prove that $\mathscr{F}$ is normal at $z=0$.

Consider the family $\mathscr{G}=\{g(z)=f(z)/\ (z):f\quad \mathscr{F}, z\quad D\}$. If $f\quad \mathscr{F}$, then $f^{(k)}(0)+a_1(0)f^{(k-1)}(0)+$ $+a_k(0)f(0)\quad\ (0)=0$, so that $f(0)\quad 0$. Otherwise if $f(0)=0$, since all zeros of f have multiplicity at least $k+2$, then $f\ (0)=\quad =f^{(k)}(0)=0$, which contradicts with $f^{(k)}(0)+a_1(0)f^{(k-1)}(0)+\quad +a_k(0)f(0)$ 0. Thus, for any $g\quad \mathscr{G}$, $\{g(0)=f(0)/\ (0)=\quad\}$.

We first prove that $\mathscr{G}$ is normal in D. Suppose, on the contrary, that $\mathscr{G}$ is not normal at $z_0\quad D$. Then by Lemma 2, there exist a sequence of functions $g_n\quad \mathscr{G}$, a sequence of complex numbers $z_n\quad z_0$ and a sequence of positive numbers $\ _n\quad 0$, such that

$$G_n(\ )=\frac{g_n(z_n+\ _n\ )}{\ _n^{k}}\quad G(\ )$$

converges spherically uniformly on compact subsets of $\mathbf{C}$. $G(\ )$ is a nonconstant meromorphic function on $\mathbf{C}$, all zeros of $G(\ )$ have multiplicity at least $k+2$, and moreover $G(\ )$ is of order at most 2.

We distinguish two cases.

Case 1 $z_n/\ _n\qquad$. By simple calculation, we have

$$g_n^{(k)}(z)=\frac{f_n^{(k)}(z)}{\ (z)}-C_n^{1}g_n^{(k-1)}(z)\frac{\ (z)}{\ (z)}-C_n^{2}g_n^{(k-2)}(z)\frac{\ (z)}{\ (z)}-\quad -g_n(z)\frac{\ ^{(k)}(z)}{\ (z)}.$$

Thus, using notation $\hat{z}_n=z_n+\ _n\ $ for brevity, we have

$$G_n^{(k)}(\ )=g_n^{(k)}(\hat{z}_n)=\frac{f_n^{(k)}(\hat{z}_n)}{\ (\hat{z}_n)}-C_n^{1}g_n^{(k-1)}(\hat{z}_n)\frac{\ (\hat{z}_n)}{\ (\hat{z}_n)}-\quad -g_n(\hat{z}_n)\frac{\ ^{(k)}(\hat{z}_n)}{\ (\hat{z}_n)}=$$

$$\frac{f_n^{(k)}(\hat{z}_n)}{\ (\hat{z}_n)}-C_n^{1}g_n^{(k-1)}(\hat{z}_n)\left(\frac{l}{\hat{z}_n}+\frac{\ (\hat{z}_n)}{\ (\hat{z}_n)}\right)-\quad -g_n(\hat{z}_n)\left(\frac{l!}{(l-k)!\ \ (\hat{z}_n)^{k}}+\right.$$

$$\left.C_k^{1}\frac{l!}{(l-k+1)!\ \ (\hat{z}_n)^{k-1}}\frac{\ (\hat{z}_n)}{\ (\hat{z}_n)}+\quad +\frac{\ ^{(k)}(\hat{z}_n)}{\ (\hat{z}_n)}\right)=$$

$$\frac{f_n^{(k)}(\hat{z}_n)}{\ (\hat{z}_n)}-C_n^{1}\frac{g_n^{(k-1)}(\hat{z}_n)}{\ _n}\left(\frac{l\ _n}{\hat{z}_n}+\frac{\ _n\ \ (\hat{z}_n)}{\ (\hat{z}_n)}\right)-\quad -\frac{g_n(\hat{z}_n)}{\ _n^{k}}\left(\frac{l!\ \ _n^{k}}{(l-k)!\ \ (\hat{z}_n)^{k}}+\right.$$

$$\left.C_k^{1}\frac{l!\ \ _n^{k-1}}{(l-k+1)!\ \ (\hat{z}_n)^{k-1}}\frac{\ _n\ \ (\hat{z}_n)}{\ (\hat{z}_n)}+\quad +\frac{\ _n^{k}\ \ ^{(k)}(\hat{z}_n)}{\ (\hat{z}_n)}\right).$$

On the other hand, we have $\lim_{n}(\ _n/\hat{z}_n)=0$, and

$$\lim_{n}\frac{\ _n\ \ ^{(i)}(\hat{z}_n)}{\ (\hat{z}_n)}=0\ \ (i=1,2,\quad ,k)$$

uniformly on compact subsets of $\mathbf{C}$. Therefore, on every compact subsets of $\mathbf{C}$ which contains no poles of $G(\ )$, we have

$$\frac{f_n^{(k)}(\hat{z}_n)}{\ (\hat{z}_n)}\quad G^{(k)}(\ ).$$

In a similar way we get

$$_{n}G_{n}^{(k-1)}(\ ) = g_{n}^{(k-1)}(\hat{z}_{n}) = \frac{f_{n}^{(k-1)}(\hat{z}_{n})}{(\hat{z}_{n})} - C_{k-1}^{1} g_{n}^{(k-2)}(\hat{z}_{n}) \frac{(\hat{z}_{n})}{(\hat{z}_{n})} - \quad - g_{n}(\hat{z}_{n}) \frac{^{(k-1)}(\hat{z}_{n})}{(\hat{z}_{n})}.$$

Then $\frac{f_{n}^{(k-1)}(\hat{z}_{n})}{(\hat{z}_{n})}$ 0, and we get $\frac{f_{n}^{(k-2)}(\hat{z}_{n})}{(\hat{z}_{n})}$ 0, , $\frac{f_{n}(\hat{z}_{n})}{(\hat{z}_{n})}$ 0 by using a similar way. Then we get

$$\frac{a_{1}(\hat{z}_{n})f_{n}^{(k-1)}(\hat{z}_{n}) + a_{2}(\hat{z}_{n})f_{n}^{(k-2)}(\hat{z}_{n}) + \quad + a_{k}(\hat{z}_{n})f_{n}(\hat{z}_{n})}{(\hat{z}_{n})} \quad 0$$

and

$$\frac{f_{n}^{(k)}(\hat{z}_{n}) + a_{1}(\hat{z}_{n})f_{n}^{(k-1)}(\hat{z}_{n}) + \quad + a_{k}(\hat{z}_{n})f_{n}(\hat{z}_{n})}{(\hat{z}_{n})} \quad G^{(k)}(\ ).$$

Since $\frac{f_{n}^{(k)}(z) + a_{1}(z)f_{n}^{(k-1)}(z) + \quad + a_{k}(z)f_{n}(z)}{(z)}$ 1, by Hurwitz s theorem, we know that either $G^{(k)}(\ )$ 1 or $G^{(k)}(\ )$ 1 for any **C** that is not a pole of $G(\ )$. Clearly, these also hold for all **C**. If $G^{(k)}(\ )$ 1, then by Lemma 1, $G(\ )$ is a constant, a contradiction. If $G^{(k)}(\ )$ 1, then

$$G(\ ) = \frac{1}{k!}\ ^{k} + C_{k-1}\ ^{k-1} + \quad + C_{0},$$

which contradicts the fact that all zeros of $G(\ )$ have multiplicity at least $k+2$.

Case 2 $z_{n}/_{n}$, a finite complex number. Then

$$\frac{g_{n}(\ _{n}\ )}{_{n}^{k}} = \frac{g_{n}(z_{n} + \ _{n}(\ - z_{n}/_{n}))}{_{n}^{k}} = G_{n}(\ - z_{n}/_{n}) \quad G(\ -\ ) = G(\ )$$

spherically uniformly on compact subsets of **C**. Clearly, all zeros of $G(\ )$ have multiplicity at least $k+2$, and $= 0$ is a pole of $G(\ )$ with multiplicity at least l, and the other poles of $G(\ )$ are multiple.

Set $H_{n}(\ ) = f_{n}(\ _{n}\ )/_{n}^{k+l}$. Then

$$H_{n}(\ ) = \frac{(\ _{n}\ )}{_{n}^{l}}\frac{f_{n}(\ _{n}\ )}{_{n}^{k}(\ _{n}\ )} = \frac{(\ _{n}\ )}{_{n}^{l}}\frac{g_{n}(\ _{n}\ )}{_{n}^{k}}.$$

Note that $\lim_{n} (\ _{n}\ )/_{n}^{l} = \ ^{l}$ uniformly on compact subsets of **C**, thus

$$H_{n}(\ ) \quad ^{l}G(\ ) = H(\ )$$

uniformly on compact subsets of **C**. Obviously, all zeros of $H(\ )$ have multiplicity at least $k+2$, and all nonzero poles of $H(\ )$ are multiple, and $H(0)$ 0 since G has a pole of order at least l at $= 0$. We also have

$$H_{n}^{(k)}(\ ) - \frac{(\ _{n}\ )}{_{n}^{l}} \quad H^{(k)}(\ ) - \ ^{l}$$

uniformly on every compact subset of **C** which contains no pole of G.

Claim $H^{(k)}(\ _{0})\quad ^{l}$.

Otherwise there exists $_{0}$ such that $H^{(k)}(\ _{0}) = \ _{0}^{l}$. Then H is holomorphic at $_{0}$. We consider two subcases.

Case 2.1 $_{0}$ 0. Obviously, the function $H^{(k)}(\ ) - \ ^{l}$ is the uniform limit of

$$H_{n}^{(k)}(\ ) + \ _{n}a_{1}(\ _{n}\ )H_{n}^{(k-1)}(\ _{n}\ ) + \quad + \ _{n}^{k}a_{k}(\ _{n}\ )H_{n}(\ _{n}\ ) - \frac{(\ _{n}\ )}{_{n}^{l}} =$$

$$\frac{f_{n}^{(k)}(\ _{n}\ ) + \ _{n}^{l+1}a_{1}(\ _{n}\ )H_{n}^{(k-1)}(\ _{n}\ ) + \quad + \ _{n}^{k+l}a_{k}(\ _{n}F)H_{n}(Q_{n}F) - \mathrm{U}(Q_{n}F)}{Q_{n}^{l}} =$$

$$\frac{f_{n}^{(k)}(Q_{n}F) + a_{1}(Q_{n}F)f_{n}^{(k-1)}(Q_{n}F) + , \ + a_{k}(Q_{n}F)f_{n}(Q_{n}F) - \mathrm{U}(Q_{n}F)}{Q_{n}^{l}}.$$

Since $f_{n}^{(k)}(z) + a_{1}(z)f_{n}^{(k-1)}(z) + , \ + a_{k}(z)f_{n}(z)\ X\ U(z)$, we can get $H^{(k)}(F) - F^{l}\ s$ 0 or $H^{(k)}(F) - F^{l}$ X 0 by Hurwitz. s Theorem. Since there exists F_{0} such that $H^{(k)}(F_{0}) = F_{0}^{l}$, then we get $H^{(k)}(F) - F^{l}\ s$ 0. Thus

$$H(F) = \frac{F^{k+l}l!}{(l+k)!} + a_{1}F^{k-1} + a_{2}F^{k-2} + , \ + a_{k},$$

where $a_{1}, a_{2}, , \ , a_{k}$ are constants, and

$$H^{(k-1)}(\zeta)=\frac{\zeta^{l+1}}{l+1}+(k-1)!\,a_1.$$

Since all zeros of $H(\zeta)$ have multiplicity at least $k+2$, then $H^{(k-1)}(\zeta)$ must have a zero with multiplicity at least 4. Hence $a_1=0$. Similarly, we can deduce that $a_2=a_3=\cdots=a_k=0$. It follows that $H(\zeta)=\zeta^{k+l}\,l!/(l+k)!$. Then $G(\zeta)=\zeta^{k}\,l!/(l+k)!$, which contradicts the fact that all zeros of $G(\zeta)$ have multiplicity at least $k+2$.

Case 2.2 $\zeta_0=0$. Then $H_n(\zeta)$ is holomorphic and $H_n(\zeta)\to H(\zeta)$ uniformly on a neighbourhood of 0. Indeed, $H(\zeta)$ is holomorphic at 0, so $G(\zeta)$ has a pole of exact order l at 0. On the other hand, for each n, the pole of $g_n(\rho_n\zeta)$ at 0 has also exact order l. Then, $\zeta=0$ is the zero of $1/G(\zeta)$ and $1/g_n(\rho_n\zeta)$ is of order l. Note that since $g_n(\rho_n\zeta)/\rho_n^k\to G(\zeta)$, spherically uniformly on compact subsets of $\mathbf{C}$, there exist a positive integer n_0 and $r>0$ such that

$$\left|\frac{\rho_n^k}{g_n(\rho_n\zeta)}-\frac{1}{G(\zeta)}\right|\le\left|\frac{1}{G(\zeta)}\right|$$

for all $n\ge n_0$ and each $\zeta\in\{\zeta: |\zeta|=r\}$. By Rouche's theorem, $1/g_n(\rho_n\zeta)$ has no zeros in $D'_r=\{\zeta: 0<|\zeta|<r\}$ for $n\ge n_0$, and then $g_n(\rho_n\zeta)$ has no poles in D'_r for $n\ge n_0$. Thus $H_n(\zeta)$ is holomorphic in D'_r, and $H_n(\zeta)\to H(\zeta)$ uniformly on a nerghbourhood of 0. Hence, the same argument as in Case 2.1 also applies for Case 2.2.

Now, we have $H^{(k)}(\zeta)\ne\zeta^l$. By Lemma 4 (for $k=1$) and Lemma 5 (for $k\ge 2$), $H(\zeta)$ must be a rational function. However, Lemma 6 shows that $H^{(k)}(\zeta)=\zeta^l$ has a solution in $\mathbf{C}$, a contradiction. We have proved that $\mathscr{G}$ is normal on D.

It remains to show that $\mathscr{F}$ is normal at $z=0$. Since $\mathscr{G}$ is normal in D, then the family $\mathscr{G}$ is equicontinuous in D with respect to the spherical distance. On the other hand, $g(0)=\infty$ for each $g\in\mathscr{G}$, so there exists $\delta>0$ such that $|g(z)|\ge 1$ for all $g\in\mathscr{G}$ and each $z\in D_\delta=\{z: |z|<\delta\}$. It follows that $f(z)\ne 0$ for all $f\in\mathscr{F}$ and $z\in D_\delta$. Suppose that $\mathscr{F}$ is not normal at $z=0$. Since $\mathscr{F}$ is normal in $0<|z|<1$, the family $1/\mathscr{F}=\{1/f: f\in\mathscr{F}\}$ is holomorphic in D_δ and normal in $D'_\delta=\{z: 0<|z|<\delta\}$, but it is not normal at $z=0$. Thus there exists a sequence $\{1/f_n\}\subset 1/\mathscr{F}$ which converges locally uniformly in D'_δ but not in D_δ. The maximum modulus principle implies that $1/f_n\to\infty$ in D'_δ. Thus $f_n\to 0$ converges locally uniformly in D'_δ, and hence so does $\{g_n\}\subset\mathscr{G}$, where $g_n=f_n/U$. But $|g_n(z)|\ge 1$ for $z\in D_\delta$, a contradiction. This completes the proof of Theorem 1.

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