n个无关变元的非线性时滞不等式

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[摘要]　讨论带有时滞的欧阳型非线性积分不等式. 这些结果不仅在本质上改进和推广了已有的相关结果, 而且在研究微分方程定性理论中起着重要作用, 同时也给出了一种新的研究不等式的方法.

[关键词]　n个无关变元, 非线性积分不等式, Ou-Iang型不等式

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One Inequality in n Independent Variables With Retardation

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Abstract Some Ou-Iang type integral inequalities with retardation are established, which not only generalizes and improves some existing results, but also plays an important role in the qualitative theory of certain differential equations. A new way to study inequality is also brought forward.

Key words n independent variables, nonlinear integral inequality, Ou-Iang type inequality

1998年, 杨恩浩在文[1]的基础上给出了一个积分不等式[2]: 设　$C(\mathbf{R}_+, \mathbf{R}_+)$严格单调, $(\ \ ) =$,　$c(\mathbf{R}_+, \mathbf{R}_+)$ 单调非减, c　0为常数. u, f　$C(\mathbf{R}_+, \mathbf{R}_+)$. 如果

$$(u(t)) \quad c + \int_0^t f(s) \quad (u(s))\,ds, \quad t \quad \mathbf{R}_+,$$

则

$$u(t) \quad ^{-1}\{G^{-1}[G(c) + \int_0^t f(s)\,ds]\}, \quad t \quad [0, T),$$

其中 $G(z) = \int_{z_0}^{z} \frac{ds}{^{-1}(s)}$,　$0 < z_0 \quad z \quad ^{-1}$, G^{-1}是　, G的反函数. T　$\mathbf{R}_+$, 正数T满足

$$G(c) + \int_0^t f(s)\,ds \quad \mathrm{Dom}(G^{-1}), \quad t \quad [0, T].$$

文[3, 4]则将上述不等式推广到关于 n个独立变元的非线性积分不等式和离散型不等式. 在本文中, 我们将讨论带有时滞的非线性积分不等式. 所得结论包含了文[1-5]中的定理与推论. 为了讨论的方便, 给出如下规定:

(　) $\int_0^t f(s)\,ds = \int_0^{s_1} \quad \int_0^{s_n} f(s_1, \quad , s_n)\,ds_n \quad ds_1$,

(　) $D_i f(t) = \frac{f(t)}{t_i}$　$(i = 1, 2, \quad , n)$,

(　) $\int_0^t f(s)\,ds = 0$, 当某个 i使得 $t_i = 0$时,

(　) $s \quad t \quad s_i \quad t_i (i = 1, 2, \quad , n)$, 首先给出如下定义

定义 **1**　若连续函数　: $\mathbf{R}_+^n$　$\mathbf{R}_+^n$

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$(\quad)\quad(t) = (\ {}_1(t),\ {}_2(t),\quad,\ {}_n(t)),\quad {}_i(t)\quad [\mathbf{R}_+^n,\ \mathbf{R}_+],\ i = 1, 2,\quad, n,$

$(\quad)\quad(t)\quad t,$

则称　属于函数类 $\mathscr{F}$.

1　主要结果

定理 1　设　$C(\mathbf{R}_+, \mathbf{R}_+)$, 严格单调递增, 可导, $(\quad) = \quad.\quad c(\mathbf{R}_+, \mathbf{R}_+)$, 单调不减, 可导且满足 $x, y\quad [0,\quad),\quad {}^{-1}(xy)\quad {}^{-1}(x)\quad {}^{-1}(y)$. 对于 $h > 0, \dfrac{{}^{-1}(t)}{h}\quad {}^{-1}\left(\dfrac{t}{h}\right)$. $f_1(t)$: $\mathbf{R}_+^n\quad \mathbf{R}_+$, 单调不减. $u(t), g(t), h(t), f_2(t)\quad c(\mathbf{R}_+^n, \mathbf{R}_+)$ 且 $g(t)\quad 1.\quad(t),\quad(t)\quad \mathscr{F}$. 如果

$$(u(t))\quad f_1(t) + \int_0^t f_2(s)\quad(u(\quad(s)))\,\mathrm{d}s + g(t)\int_0^t h(s)\quad(u(\quad(s)))\,\mathrm{d}s,\quad t\quad \mathbf{R}_+^n, \tag{1}$$

则

$$u(t)\quad {}^{-1}\{k(t)g(t)H^{-1}[H(f_1(t)) + \int_0^t f_2(s)\quad {}^{-1}(k(\quad(s)))\,\mathrm{d}s]\}.$$

这里

$$G = \int_{z_0}^{z}\frac{\mathrm{d}s}{{}^{-1}(s)},\quad k(t) = g(t)G^{-1}[G(1) + \int_0^t h(s)g(\quad(s))\,\mathrm{d}s],\ H(z) = \int_{z_0}^{z}\frac{\mathrm{d}s}{(s)}.$$

对于任意 $M\quad \mathbf{R}_+^n$ 满足

$$\int_0^t h(s)g(\quad(s))\,\mathrm{d}s + G(1)\quad \mathrm{Dom}(G^{-1})\ H(f_1(t)) + \int_0^t f_2(s)\quad {}^{-1}(k(\quad(s)))\,\mathrm{d}s\quad \mathrm{Dom}(H^{-1}).$$

证明　令(1)式右端为 $w(t)$. 则由(1)式可得

$$u(t)\quad {}^{-1}(w(t)), \tag{2}$$

且

$$w(t)\quad f_1(t) + \int_0^t f_2(s)\quad {}^{-1}(w(\quad(s)))\,\mathrm{d}s + g(t)\int_0^t h(s)\quad {}^{-1}(w(\quad(s)))\,\mathrm{d}s, \tag{3}$$

对任意的 $T\quad \mathbf{R}_+^n$, 令 $f(t) = f_1(T) + \int_0^t f_2(s)\quad {}^{-1}(w(\quad(s)))\,\mathrm{d}s$.

因 $f(t)$ 非负且不减, $g(t)\quad 1$, 故由(3)可得:

$$\frac{w(t)}{f(t)}\quad g(t)[1 + \int_0^t \frac{h(s)}{f(s)}\quad {}^{-1}(w(\quad(s)))\,\mathrm{d}s],\quad 0\quad t\quad T < M(\quad \mathbf{R}_+^n).$$

令 $z(t) = 1 + \int_0^t \dfrac{h(s)}{f(s)}\quad {}^{-1}(w(\quad(s)))\,\mathrm{d}s$, 则

$$\frac{w(t)}{f(t)}\quad g(t)z(t), \tag{4}$$

又因 $(t)\quad t$, 故由(4)可得:

$$w(\quad(t))\quad f(t)g(\quad(t))z(\quad(t))\quad g(\quad(t))f(t)z(t),$$

且

$$\begin{aligned} D_1D_2\quad D_nz(t) &= \frac{h(t)}{f(t)}\quad {}^{-1}(w(\quad(t))) \\ &\quad \frac{h(t)g(\quad(t))}{f(t)g(\quad(t))}\quad {}^{-1}(g(\quad(t))f(t)z(t)) \\ &\quad h(t)g(\quad(t))\quad {}^{-1}(z(t)), \end{aligned}$$

从而

$$\frac{D_1D_2\quad D_nz(t)}{{}^{-1}(z(t))}\quad h(t)g(\quad(t)).$$

由于

$$D_n\left(\frac{D_1D_2\quad D_{n-1}z(t)}{{}^{-1}z(t)}\right) = \frac{{}^{-1}z(t)D_1D_2\quad D_nz(t) - D_n\quad {}^{-1}z(t)D_1D_2\quad D_{n-1}z(t)}{(\quad {}^{-1}z(t))^2},$$

又由于 $D_n \quad {}^{-1}(z(t)) \quad 0, D_1, D_2, \quad , D_{n-1}z(t) \quad 0$, 所以得:

$$D_n\left(\frac{D_1D_2 \quad D_{n-1}z(t)}{{}^{-1}z(t)}\right) \quad \frac{D_1D_2 \quad D_nz(t)}{{}^{-1}z(t)} \quad h(t)g(\quad (t)), \tag{5}$$

在上式中固定 $t_1, t_2, \quad , t_{n-1}, s_n = t_n$, 对 s_n 从 0到 t_n 积分得到:

$$\frac{D_1D_2 \quad D_{n-1}z(t)}{{}^{-1}z(t)} - \frac{D_1D_2 \quad D_{n-1}z(t_1, t_2, \quad , t_{n-1}, 0)}{{}^{-1}z(t_1, t_2, \quad , t_{n-1}, 0)} \quad {}_0^{t_n}h(t_1, t_2, \quad , t_{n-1}, s_n)g(\quad (t_1, t_2, \quad , t_{n-1}, s_n))\mathrm{d}s_n,$$

即

$$\frac{D_1D_2 \quad D_{n-1}z(t)}{{}^{-1}z(t)} \quad {}_0^{t_n}h(t_1, t_2, \quad , t_{n-1}, s_n)g(\quad (t_1, t_2, \quad , t_{n-1}, s_n))\mathrm{d}s_n,$$

依次类推可得:

$$\frac{D_1z(t)}{{}^{-1}z(t)} \quad {}_0^{t_2} \quad {}_0^{t_n}h(t_1, s_2, \quad , s_{n-1}, s_n)g(\quad (t_1, s_2, \quad , s_{n-1}, s_n))\mathrm{d}s_n \quad \mathrm{d}s_2.$$

由 $G(z)$ 的定义可得: $D_1G(z(t)) = \dfrac{D_1z(t)}{{}^{-1}z(t)}$,

所以

$$D_1G(z(t)) \quad {}_0^{t_2} \quad {}_0^{t_n}h(t_1, s_2, \quad , s_{n-1}, s_n)g(\quad (t_1, s_2, \quad , s_{n-1}, s_n))\mathrm{d}s_n \quad \mathrm{d}s_2. \tag{6}$$

对 (6) 式两边作 0到 t_1 的积分得:

$$G(z(t)) \quad G(z(0, t_2, \quad , t_n)) + {}_0^{t}h(s)g(\quad (s))\mathrm{d}s \quad G(1) + {}_0^{t}h(s)g(\quad (s))\mathrm{d}s,$$

由 (4) 式可得:

$$w(t) \quad f(t)g(t)G^{-1}[G(1) + {}_0^{t}h(s)g(\quad (s))\mathrm{d}s], \tag{7}$$

则有

$${}^{-1}(w(\quad (t))) \quad {}^{-1}(f(t)) \quad {}^{-1}\{g(\quad (t))G^{-1}[G(1) + {}_0^{(t)}h(s)g(\quad (s))\mathrm{d}s]\}.$$

令

$$k(\quad (t)) = g(\quad (t))G^{-1}[{}_0^{(t)}h(s)g(\quad (s))\mathrm{d}s + G(1)],$$

即有

$$\frac{D_1D_2 \quad D_nf(t)}{{}^{-1}(f(t))} = \frac{{}^{-1}(w(\quad (t)))f_2(t)}{{}^{-1}f(t)} \quad f_2(t) \quad {}^{-1}(k(\quad (t))).$$

由于

$$D_n\left(\frac{D_1D_2 \quad D_{n-1}f(t)}{{}^{-1}(f(t))}\right) = \frac{{}^{-1}(f(t))D_1D_2 \quad D_nf(t) - D_n \quad {}^{-1}(f(t))D_1D_2 \quad D_{n-1}f(t)}{(\quad {}^{-1}(f(t)))^2},$$

且 $D_n \quad {}^{-1}(f(t)) \quad 0, D_1, D_2, \quad , D_{n-1} \quad {}^{-1}(f(t)) \quad 0$, 所以得:

$$D_n\left(\frac{D_1D_2 \quad D_{n-1}f(t)}{{}^{-1}f(t)}\right) \quad f_2(t) \quad {}^{-1}(k(\quad (t))),$$

对上式两边作 0到 t_n 的积分得

$$\frac{D_1D_2 \quad D_{n-1}f(t)}{{}^{-1}(f(t))} - \frac{D_1D_2 \quad D_{n-1}f(t_1, t_2, \quad , t_{n-1}, 0)}{{}^{-1}(f(t_1, \quad , t_{n-1}, 0))}$$
$$ {}_0^{t_n}f_2(t_1, t_2, \quad , t_{n-1}, s_n) \quad {}^{-1}(k(\quad (t_1, t_2, \quad , t_{n-1}, s_n)))\mathrm{d}s_n,$$

即有

$$\frac{D_1D_2 \quad D_{n-1}f(t)}{{}^{-1}(f(t))} \quad {}_0^{t_n}f_2(t_1, t_2, \quad , t_{n-1}, s_n) \quad {}^{-1}(k(\quad (t_1, t_2, \quad , t_{n-1}, s_n)))\mathrm{d}s_n,$$

依次类推可得

$$\frac{D_1 f(t)}{{}^{-1}(f(t))} \quad {}_0^{t_2} \quad {}_0^{t_n} f_2(t_1, s_2, \quad , s_{n-1}, s_n) \quad {}^{-1}(k(\ (t_1, s_2, \quad , s_{n-1}, s_n)))\,\mathrm{d}s_n \quad \mathrm{d}s_2.$$

由 $H(z)$ 的定义可得:

$$H(f(t)) \quad H(f(0, t_2, \quad , t_n)) + {}_0^t f_2(s) \quad {}^{-1}(k(\ (s)))\,\mathrm{d}s = H(f_1(T)) + {}_0^t f_2(s) \quad {}^{-1}(k(\ (s)))\,\mathrm{d}s,$$

则有

$$f(t) \quad H^{-1}[H(f_1(T)) + {}_0^t f_2(s) \quad {}^{-1}(k(\ (s)))\,\mathrm{d}s], \quad 0 \quad t \quad T < M(\quad \mathbf{R}_+^n). \tag{8}$$

将(8)式代入(7)式得:

$$w(t) \quad g(t)k(t)H^{-1}[H(f_1(T)) + {}_0^t f_2(s) \quad {}^{-1}(k(\ (s)))\,\mathrm{d}s], \quad 0 \quad t \quad T < M(\quad \mathbf{R}_+^n),$$

令 $t = T$, 可得:

$$w(T) \quad g(T)k(T)H^{-1}[H(f_1(T)) + {}_0^T f_2(s) \quad {}^{-1}(k(\ (s)))\,\mathrm{d}s], \quad 0 \quad T < M(\quad \mathbf{R}_+^n).$$

由 T 的任意性可知, 任给 $t \quad [0, M)$, 有

$$w(t) \quad g(t)k(t)H^{-1}[H(f_1(t)) + {}_0^t f_2(s) \quad {}^{-1}(k(\ (s)))\,\mathrm{d}s].$$

结合上式与(2)式可得定理 1所需结论.

注 1　当 $n = 1$, $(t) = t$, $(t) = t$ 时, 可在本文的基础上得到文[1, 2]中的相关结论.

注 2　当 $h(t) = 0$, $(t) = t$ 时, 即为文[3]中所得定理与推论.

推论 1　在定理 1其它条件不变的情况下, 令 $(x) = x^p$, $(x) = x^q$, 其中 $p \quad q > 0$ 则可得到如下结论:

$$u(t) \quad \begin{cases} \left\{ k(t)g(t)\left(f_1^{1-q}(t) + (1-q) {}_0^t f_2(s)k(\ (s))\,\mathrm{d}s \right)^{q-1} \right\}^{\frac{1}{p}}, & p = q, \\ \left\{ k(t)g(t)\left(f_1^{1-q}(t) + (1-q) {}_0^t f_2(s)(k(\ (s)))^{\frac{q}{p}}\,\mathrm{d}s \right)^{q-1} \right\}^{\frac{1}{p}}, & p > q, \end{cases}$$

其中

$$k(t) = \begin{cases} g(t)\exp\left({}_0^t h(s)g(\ (s))\,\mathrm{d}s \right), & p = q, \\ g(t)\left(\dfrac{p}{p-q} + \dfrac{p}{p-q} {}_0^t h(s)g(\ (s))\,\mathrm{d}s \right)^{\frac{p}{p-q}}, & p > q. \end{cases}$$

注 3　当 $(x) = (x) = a^x (a > 1)$ 或 $(x) = (x) = \log_a^x (a > 1)$ 时, 也满足定理 1需求.

2　结论

从以上证明过程可以看出, 充分利用已有条件所提供的有效信息进行证明, 不仅可以得到满意的结果, 而且克服了以往一些方法的繁杂计算, 使整个证明过程变得更为简单方便, 采用以上的证明方法, 还可以讨论诸多类似的不等式.

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