# B-S推广模型的亚式期权定价

刘国祥[1], 陈 波[2], 翁 琴[3]

(1. 南京师范大学数学科学学院, 江苏 南京 210046)
(2 江苏教育学院数学与信息技术学院, 江苏 南京 210013)
(3 招商银行, 江苏 江阴 214431)

[摘要] 分别假设金融资产为有连续红利支付和波动率是随机的股票, 得到相应的亚式看涨期权的定价公式和算术平均亚式期权价格的上界.

[关键词] Black-Scholes, 亚式期权定价, 测度变换
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## On Asian Option Pricing for Generalized Model of Black-Scholes Model

Liu Guoxiang[1], Chen Bo[2], Weng Qin[3]

(1. School of Mathematical Sciences, Nanjing Normal University, Nanjing 210046, China)
(2. School of Mathematics and Information Technology, Jiangsu Institute of Education, Nanjing 210013, China)
(3. China Merchants Bank, Jiangyin 214431, China)

**Abstract**: Assumed that the financial asset is stock with consecutive capital bonus rate or stochastic volatility, the paper gets the corresponding pricing of the Asian call option and the upper bound of arthmetic average Asian option prices.
**Key words**: Black-Scholes, Asian option pricing, measure transforms

亚式期权是根据合约期 $(0, T)$ 内股票价格的平均值 $J_T$ 的高低来决定是否执行期权的合约, 这里所谓的平均值有两种含义: 算术平均和几何平均. 平均值 $J_t$ 被称为路径变量, 它表示从起始时刻到 $t$ 时刻的平均值, $J_t = \frac{1}{t}\int_0^t S_\tau \mathrm{d}\tau$ (算术平均) 或 $J_t = e^{\frac{1}{t}\int_0^t \ln S_\tau \mathrm{d}\tau}$ (几何平均). 在 $T$ 时刻收益函数 $V_T = (J_T - K)^+$, 由鞅方法可得在 $t$ 时刻亚式看涨期权价格 $C(K, t) = E^Q[\overline{V_t} \mid \mathscr{F}_t]$.

传统的 B-S模型对 $(S_t)_{t\geqslant 0}$ 要求苛刻, 随后人们对其假设条件作不断的改进, 例如文[1-4]. 以下重点讨论两种推广情形下亚式看涨期权的定价, 而看跌期权可通过期权平价公式得到.

## 1 *B-S* 模型的推广

一般地, 证券市场可以描述为一个带 σ-域流的概率空间 $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, P)$, 其中 $\{\mathscr{F}_t, 0\leqslant t\leqslant T\}$ 为标准布朗运动 $\{B_t, 0\leqslant t\leqslant T\}$ 产生的自然 σ-域流.

### 1.1 推广一

假设市场无套利、无摩擦且完备; 在期权有效期内金融资产有连续红利支付, 股票平均回报率 $\mu_t$、红利率 $q_t$、波动率 $\sigma_t$、无风险利率 $r_t$ 均为时间 $t$ 的确定性函数且可积; 市场中只含债券和股票两种资产, 价格过程分别如下

$$\mathrm{d}S_t^0 = r_t S_t^0 \mathrm{d}t, \quad (S_0^0 = 1) \tag{1}$$

$$\mathrm{d}S_t = (\mu_t - q_t)S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}B_t. \tag{2}$$

**命题 1** 若市场无套利且完备, 则存在惟一的与 $P$ 等价的测度 $Q$, 使得支付红利的股票的贴现价格

收稿日期: 2009-03-28
通讯联系人: 刘国祥, 副教授, 研究方向: 权证定价理论及实证. E-mail: gxliu63@163.com

$(\overline{S_t})_{t\geqslant 0}$ 在 $Q$ 下是鞅, 其中 $\overline{S_t} = S_t e^{\int_0^t q_s ds}$, 称这样的测度 $Q$ 为风险中性概率测度.

证明　由 $S_t = e^{-\int_0^t r_s ds} S_t$, 得 $\overline{S_t} = e^{-\int_0^t (r_s - q_s) ds} S_t$, 由 (2) 式及 Itô公式, 有:

$$d\overline{S_t} = d\left(e^{-\int_0^t (r_s - q_s)ds} S_t\right) = e^{\int_0^t (r_s - q_s)ds} dS_t - (r_t - q_t) e^{\int_0^t (r_s - q_s)ds} S_t dt =$$
$$e^{\int_0^t (r_s - q_s)ds}[(\mu_t - q_t)S_t dt + \sigma_t S_t dB_t - (r_t - q_t)S_t dt] =$$
$$e^{\int_0^t (r_s - q_s)ds}[(\mu_t - r_t)S_t dt + \sigma_t S_t dB_t] = e^{\int_0^t (r_s - q_s)ds}\sigma_t S_t dW_t = \overline{S_t}\sigma_t dW_t,$$

其中 $W_t = B_t + \int_0^t \theta_s ds$, 此处 $\theta_t = \dfrac{\mu_t - r_t}{\sigma_t}$.

令 $e^{Y_t} = \exp\left\{-\int_0^t \theta_s dB_s - \dfrac{1}{2}\int_0^t \theta_s^2 ds\right\}$, 定义测度 $Q$, 使得 $\dfrac{dQ_t}{dP_t} = \exp\left\{-\int_0^t \theta_s dB_s - \dfrac{1}{2}\int_0^t \theta_s^2 ds\right\}$, 由 Girsanov 定理知, $Q$ 是 $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0})$ 空间上与 $P$ 等价的一个概率测度, 而 $(W_t)_{t\geqslant 0}$ 是 $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, Q)$ 上的一个标准布朗运动, 由鞅表示定理可知, $(\overline{S_t})_{t\geqslant 0}$ 在 $Q$ 下是鞅.

下面在概率空间 $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, Q)$ 上考虑期权的定价问题. 此时, (2) 式变成

$$dS_t = (\mu_t - q_t)S_t dt + \sigma_t S_t(dW_t - \theta_t dt) = (\mu_t - q_t - (\mu_t - r_t))S_t dt + \sigma_t S_t dW_t =$$
$$(r_t - q_t)S_t dt + \sigma_t S_t dW_t. \tag{3}$$

由于算术平均亚式期权定价中, 股票平均价格 $J_T$ 的分布不再服从对数正态分布, 要求其分布很困难, 不少学者通过特定的方法推得了一些结论, 例如文 [5-7]. 以下考察几何平均 $J_T = e^{\frac{1}{T}\int_0^T \ln S_t dt}$ 下 $\ln J_T$ 关于 $\mathcal{F}_t$ 的条件分布.

**引理 1**　$A_t = \dfrac{1}{T}\int_t^T \sigma_s (T - s) dW_s \sim N(0, \overline{\sigma_t}^2)$, 其中, $\overline{\sigma_t}^2 = \dfrac{1}{T^2}\int_t^T (T - s)^2 \sigma_s^2 ds$

证明　设 $t = S_0 < S_1 < \cdots < S_j < \cdots < S_m = \tau_i$, $t = \tau_0 < \tau_1 < \cdots < \tau_i < \cdots < \tau_n = T$,
根据随机积分的定义, 记

$$A_t^{m,n} = \frac{1}{T}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sigma_{S_j}(W_{S_{j+1}} - W_{S_j})(\tau_{i+1} - \tau_i),$$
$$A_t = \frac{1}{T}\int_t^T\int_t^\tau \sigma_s dW_s d\tau = \frac{1}{T}\int_t^T \sigma_s (T - s) dW_s.$$

由于 $A_t^{m,n}$ 为一列正态随机变量, 且均方收敛到 $A_t$, 所以 $A_t^{m,n}$ 依概率收敛到 $A_t$, 而 $A_t$ 为正态随机变量. 于是可得 $A_t$ 的均值和方差:

$$E^Q[A_t] = \frac{1}{T}\int_t^T E^Q\left(\int_t^\tau \sigma_s dW_s\right) d\tau = \frac{1}{T}\int_t^T 0 d\tau = 0,$$
$$\overline{\sigma_t}^2 = Var_Q[A_t] = \frac{1}{T^2}E^Q\left[\int_t^T\int_t^\tau \sigma_s dW_s d\tau\right]^2 = \frac{1}{T^2}E^Q\left[\int_t^T\int_s^T \sigma_s d\tau dW_s\right]^2 =$$
$$\frac{1}{T^2}E^Q\left[\int_t^T \sigma_s (T - s) dW_s\right]^2 = \frac{1}{T^2}\int_t^T (T - s)^2 \sigma_s^2 ds \tag{4}$$

**定理 1**　$\ln J_T \sim N(\ln S_t + \mu_t, \overline{\sigma_t}^2)$, 其中

$$\mu_t = \frac{1}{T}\left[\int_t^T\left(r_s - q_s - \frac{1}{2}\sigma_s^2\right)(T - s) ds - \int_0^t\left(r_s - q_s - \frac{1}{2}\sigma_s^2\right) s ds - \int_0^t \sigma_s s dW_s\right].$$

证明　由于 $d\overline{S_t} = \sigma_t \overline{S_t} dW_t$, 再利用 Itô公式:

$$d\ln\overline{S_t} = \frac{1}{\overline{S_t}}d\overline{S_t} + \frac{-1}{2\overline{S_t}^2}(\sigma_t\overline{S_t})^2 dt = \frac{1}{\overline{S_t}}\sigma_t\overline{S_t}dW_t - \frac{1}{2}\sigma_t^2 dt = \sigma_t dW_t - \frac{1}{2}\sigma_t^2 dt,$$

因此

$$\ln\overline{S_t} = \ln S_0 - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s,$$
$$\ln S_t = \ln\left(e^{\int_0^t (r_s - q_s)ds}\overline{S_t}\right) = \int_0^t (r_s - q_s) ds + \ln\overline{S_t} = \ln S_0 + \int_0^t\left(r_s - q_s - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dW_s,$$

由此可得

$$\ln J_T=\frac{1}{T}\int_0^T\ln S_\tau\,\mathrm{d}\tau=\frac{1}{T}\left(\int_0^t\ln S_\tau\,\mathrm{d}\tau+\int_t^T\ln S_\tau\,\mathrm{d}\tau\right).$$

当 $0<\tau<t$时,由(4)式有:

$$\ln S_t=\ln S_0+\int_0^t\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)\mathrm{d}s+\int_0^t\sigma_s\,\mathrm{d}W_s=\ln S_\tau+\int_\tau^t\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)\mathrm{d}s+\int_\tau^t\sigma_s\,\mathrm{d}W_s,$$

故

$$\ln S_\tau=\ln S_t-\int_\tau^t\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)\mathrm{d}s-\int_\tau^t\sigma_s\,\mathrm{d}W_s,$$

$$\int_0^t\ln S_\tau\,\mathrm{d}\tau=\int_0^t\left[\ln S_t-\int_\tau^t\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)\mathrm{d}s-\int_\tau^t\sigma_s\,\mathrm{d}W_s\right]\mathrm{d}\tau=t\ln S_t-\int_0^t\int_\tau^t\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)\mathrm{d}s\,\mathrm{d}\tau-$$

$$\int_0^t\int_\tau^t\sigma_s\,\mathrm{d}W_s\,\mathrm{d}\tau=t\ln S_t-\int_0^t\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)s\,\mathrm{d}s-\int_0^t\sigma_s s\,\mathrm{d}W_s;\tag{5}$$

当 $t<\tau<T$时,由(4)式有

$$\ln S_\tau=\ln S_t+\int_t^\tau\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)\mathrm{d}s+\int_t^\tau\sigma_s\,\mathrm{d}W_s,$$

因此

$$\int_t^T\ln S_\tau\,\mathrm{d}\tau=\int_t^T\left[\ln S_t+\int_t^\tau\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)\mathrm{d}s+\int_t^\tau\sigma_s\,\mathrm{d}W_s\right]\mathrm{d}\tau=$$

$$(T-t)\ln S_t+\int_t^T\int_t^\tau\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)\mathrm{d}s\,\mathrm{d}\tau+\int_t^T\int_t^\tau\sigma_s\,\mathrm{d}W_s\,\mathrm{d}\tau=$$

$$(T-t)\ln S_t+\int_t^T\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)(T-s)\,\mathrm{d}s+\int_t^T\sigma_s(T-s)\,\mathrm{d}W_s,\tag{6}$$

由(5)式和(6)式,

$$\ln J_T=\frac{1}{T}\left(\int_0^t\ln S_\tau\,\mathrm{d}\tau+\int_t^T\ln S_\tau\,\mathrm{d}\tau\right)=\frac{1}{T}\left[t\ln S_t-\int_0^t\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)s\,\mathrm{d}s-\int_0^t\sigma_s s\,\mathrm{d}W_s+\right.$$

$$\left.(T-t)\ln S_t+\int_t^T\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)(T-s)\,\mathrm{d}s+\int_t^T\sigma_s(T-s)\,\mathrm{d}W_s\right]=$$

$$\ln S_t+\frac{1}{T}\left[\int_t^T\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)(T-s)\,\mathrm{d}s-\int_0^t\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)s\,\mathrm{d}s-\right.$$

$$\left.\int_0^t\sigma_s s\,\mathrm{d}W_s\right]+\frac{1}{T}\int_t^T\sigma_s(T-s)\,\mathrm{d}W_s=\ln S_t+\mu_t+A_t.$$

显然,由于 $r_s$、$q_s$、$\sigma_s$ 均为关于时间的确定性函数,故 $\int_t^T\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)(T-s)\,\mathrm{d}s$, $\int_0^t\left(r_s-q_s-\frac{1}{2}\sigma_s^2\right)s\,\mathrm{d}s$, $\int_0^t\sigma_s\,\mathrm{d}W_s$ 均关于 $\mathscr{F}_t$可测,故 $\mu_t$ 关于 $\mathscr{F}_t$可测,而由布朗运动的独立增量性以及随机积分的性质可知:$A_t$关于 $\mathscr{F}_t$独立.

## 1.2 推广二

一般地,股票价格波动率是变化的,近期,Takaoka鉴于此对 B-S模型进行了推广,设风险资产价格 $S_t$ 满足:

$$S_t=S_0\,\mathrm{e}^{rt}\int_0^\infty\exp\left\{\sigma(W_t+Ct)-\frac{\sigma^2}{2}t\right\}\lambda(\mathrm{d}\sigma),\ \forall\,t\geqslant 0,\tag{7}$$

其中

(1) $W$ 为流概率空间 $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant 0},P)$ 上的布朗运动,$(\mathscr{F}_t)_{t\geqslant 0}$是由 $W$ 生成的;

(2) $\lambda$是 $(R_{++},B(R_{++}))$ 上给定的测度,满足:$\lambda(R_{++})=1$, $\int_0^\infty\sigma\lambda(\mathrm{d}\sigma)<\infty$;

(3) $r$, $C$ 为常数.

从模型中可以看出随机过程 $S$为几何布朗运动的加权平均,条件 $\int_0^\infty\sigma\lambda(\mathrm{d}\sigma)<\infty$保证了随机 Fubini

定理的应用.

引理 2　存在惟一的等价鞅测度 $Q$, $\frac{dQ}{dP}\Big|\mathscr{F}_t = \exp\left\{-CW_t - \frac{C^2}{2}t\right\}$, 在此测度下 $W_t^Q = W_t + Ct$为 $Q$ – 标准布朗运动, 股票折现值 $\tilde{S}_t = S_t e^{-rt}$为 $Q$ – 鞅, 且

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \frac{\int_0^\infty \sigma \exp\left[\sigma W_t^Q - \frac{1}{2}\sigma^2 t\right]\lambda(d\sigma)}{\int_0^\infty \exp\left[\sigma W_t^Q - \frac{1}{2}\sigma^2 t\right]\lambda(d\sigma)}.$$

从上式可以看出波动率为 $\int_0^\infty \sigma\lambda_t(d\sigma)$, 其中

$$\lambda_t(d\sigma) = \frac{\exp\left[\sigma W_t^Q - \frac{1}{2}\sigma^2 t\right]\lambda(d\sigma)}{\int_0^\infty \exp\left[\sigma W_t^Q - \frac{1}{2}\sigma^2 t\right]\lambda(d\sigma)}.$$

性质 1　对于 $\forall\, 0 \leqslant s < t \leqslant T$,

$$S_t = S_s e^{r(t-s)}\int_0^\infty \exp\left\{\sigma(W_t^Q - W_s^Q) - \frac{1}{2}\sigma^2(t-s)\right\}\lambda_s(d\sigma),\quad a.s$$

证明

$$S_t = S_0 e^{rt}\int_0^\infty \exp\left\{\sigma B_t^Q - \frac{\sigma^2}{2}t\right\}\lambda(d\sigma) =$$

$$\frac{S_s S_0 e^{rt}\int_0^\infty \exp\left\{\sigma B_t^Q - \frac{\sigma^2}{2}t\right\}\lambda(d\sigma)}{S_s} = \frac{S_s e^{rt}\int_0^\infty \exp\left\{\sigma B_t^Q - \frac{\sigma^2}{2}t\right\}\lambda(d\sigma)}{e^{rs}\int_0^\infty \exp\left\{\sigma B_s^Q - \frac{\sigma^2}{2}s\right\}\lambda(d\sigma)} =$$

$$S_s e^{r(t-s)}\int_0^\infty \exp\left\{\sigma(B_t^Q - B_s^Q) - \frac{1}{2}\sigma^2(t-s)\right\}\cdot\frac{\exp\left\{\sigma B_s^Q - \frac{1}{2}\sigma^2 s\right\}}{\int_0^\infty \exp\left\{\sigma B_s^Q - \frac{1}{2}\sigma^2 s\right\}\lambda(d\sigma)}\lambda(d\sigma) =$$

$$S_s e^{r(t-s)}\int_0^\infty \exp\left\{\sigma(B_t^Q - B_s^Q) - \frac{1}{2}\sigma^2(t-s)\right\}\lambda_s(d\sigma).$$

## 2　推广模型下亚式看涨期权的定价公式

### 2.1　推广一的定价公式

定理 2　敲定价格为 $K$ 的几何平均亚式看涨期权的定价公式为:

$$C(K, t) = e^{-\int_t^T (r_s - q_s)ds}\left[S_t e^{\mu_t + \frac{1}{2}\bar{\sigma}_t^2}N(d_1) - KN(d_2)\right],$$

其中 $d_1 = d_2 + \bar{\sigma}_t$, $d_2 = \dfrac{\ln\frac{S_t}{K} + \mu_t}{\bar{\sigma}_t}$, $N(d) = \int_{-\infty}^{d}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$, $\mu_t$, $\bar{\sigma}_t$如上定义.

证明

$$C(K, t) = e^{-\int_t^T (r_s - q_s)ds}E^Q[(e^{\ln S_t + \mu_t + A_t} - K)^+ \mid \mathscr{F}_t] = e^{-\int_t^T (r_s - q_s)ds}E^Q[(e^{x + A_t} - K)^+ \mid \mathscr{F}_t]_{x = \ln S_t + \mu_t} =$$

$$e^{-\int_t^T (r_s - q_s)ds}\int_{\ln K - \mu_t - \ln S_t}^{+\infty}(e^{\ln S_t + \mu_t + y} - K)\frac{1}{\sqrt{2\pi}\bar{\sigma}_t}e^{-\frac{y^2}{2\bar{\sigma}_t^2}}dy =$$

$$e^{-\int_t^T (r_s - q_s)ds}\int_{\ln K - \mu_t - \ln S_t}^{+\infty}\left[e^{\ln S_t + \mu_t + \frac{1}{2}\bar{\sigma}_t^2}\frac{1}{\sqrt{2\pi}\bar{\sigma}_t}e^{-\frac{(y - \bar{\sigma}_t^2)^2}{2\bar{\sigma}_t^2}} - K\frac{1}{\sqrt{2\pi}\bar{\sigma}_t}e^{-\frac{y^2}{2\bar{\sigma}_t^2}}\right]dy =$$

$$e^{-\int_t^T (r_s - q_s)ds}\left[e^{\ln S_t + \mu_t + \frac{1}{2}\bar{\sigma}_t^2}\int_{\frac{\ln K - \mu_t - \ln S_t - \bar{\sigma}_t^2}{\bar{\sigma}_t}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy - K\int_{\frac{\ln K - \mu_t - \ln S_t}{\bar{\sigma}_t}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy\right] =$$

$$e^{-\int_t^T (r_s - q_s)ds}\left[e^{\ln S_t + \mu_t + \frac{1}{2}\bar{\sigma}_t^2}N\left(-\frac{\ln K - \mu_t - \ln S_t - \bar{\sigma}_t^2}{\bar{\sigma}_t}\right) - KN\left(-\frac{\ln K - \mu_t - \ln S_t}{\bar{\sigma}_t}\right)\right] =$$

$$e^{-\int_t^T (r_s-q_s)\,ds}\left[S_t e^{\mu_t+\frac{1}{2}\bar{\sigma}_t^2}N\left(\frac{\ln\frac{S_t}{K}+\mu_t+\bar{\sigma}_t^2}{\sigma_t}\right)-KN\left(\frac{\ln\frac{S_t}{K}+\mu_t}{\sigma_t}\right)\right]=$$

$$e^{-\int_t^T (r_s-q_s)\,ds}\left[S_t e^{\mu_t+\frac{1}{2}\bar{\sigma}_t^2}N(d_1)-KN(d_2)\right].$$

## 2.2 推广二的定价公式

这里我们只讨论算术平均这一更为复杂的情形.

**引理 3** 记 $q=\dfrac{KT-\int_0^t S_u\,dt}{S_u}$, 则

$$C(K,t)=\frac{1}{T}e^{-r(T-t)}S_tE^Q\left(\int_0^{T-t}e^{ru}\int_0^\infty \exp\left\{\sigma W_u^Q-\frac{1}{2}\sigma^2u\right\}\lambda_t(d\sigma)\,du-q\right)_+,$$

其中 $C(K,t)$ 为算术平均亚式看涨期权的价格.

证明

$$C(K,t)=e^{-r(T-t)}E^Q\left[\left(\frac{1}{T}\int_0^T S_u\,du-K\right)_+\Big|\mathscr{F}_t\right]=\frac{1}{T}e^{-r(T-t)}E^Q\left[\left(\int_0^T S_u\,du-TK\right)_+\Big|\mathscr{F}_t\right]=$$

$$\frac{1}{T}e^{-r(T-t)}E^Q\left[\left(\int_0^t S_u\,du+\int_t^T S_u\,du-TK\right)_+\Big|\mathscr{F}_t\right]=\frac{1}{T}e^{-r(T-t)}E^Q\left[\left(\int_t^T S_u\,du-\left(TK-\int_0^t S_u\,du\right)\right)_+\Big|\mathscr{F}_t\right],$$

代入(7)式, 并注意到 $B^Q$ 的独立增量性和平稳增量性, 以及 $\lambda_t(d\sigma)$ 关于 $\mathscr{F}_t$ 可测, 于是得

$$C(K,t)=\frac{1}{T}e^{-r(T-t)}E^Q\left(\int_t^T S_t e^{r(u-t)}\int_0^\infty \exp\left\{\sigma(B_u^Q-B_t^Q)-\frac{1}{2}\sigma^2(u-t)\right\}\cdot\lambda_t(d\sigma)\,du-\right.$$

$$\left.\left(TK-\int_0^t S_u\,du\right)\right)_+=\frac{1}{T}e^{-r(T-t)}S_tE^Q\left(\int_0^{T-t}e^{ru}\int_0^\infty \exp\left\{\sigma B_u^Q-\frac{1}{2}\sigma^2u\right\}\lambda_t(d\sigma)\,du-\right.$$

$$\left.\frac{\left(TK-\int_0^t S_u\,du\right)}{S_t}\right)_+=\frac{1}{T}e^{-r(T-t)}S_tE^Q\left(\int_0^{T-t}e^{ru}\int_0^\infty \exp\left\{\sigma B_u^Q-\frac{1}{2}\sigma^2u\right\}\lambda_t(d\sigma)\,du-q\right)_+.$$

一般地, 我们难以求出 $\int_0^{T-t}e^{ru}\int_0^\infty e^{\sigma B_u^Q-\frac{1}{2}\sigma^2u}\lambda_t(d\sigma)\,du$ 的分布, 即使在 $\sigma$ 的测度集中于一点而退化为标准 B-S模型的情形时, 我们至今还没有得到 $C(K,t)$ 的显式表达式. 但是, 因为 $q=\dfrac{KT-\int_0^t S_u\,du}{S_t}$ 关于 $\mathscr{F}_t$ 可测, 所以在 $t$ 时刻, 若 $q\leqslant 0$, 我们便可以精确地计算出 $C(K,t)$; 而若 $q>0$, 我们可以得到 $C(K,t)$ 的一个上界.

**定理 3** 在 $t$ 时刻, 若 $q\leqslant 0$, 则

$$C(K,t)=\frac{S_t}{rT}-\left[\frac{S_t}{rT}-K+\frac{1}{T}\int_0^t S_u\,du\right]e^{-r(T-t)}=A_1;$$

若 $q>0$, 则

$$C(K,t)\leqslant\frac{1}{2}A_1+\frac{S_t}{2T}e^{-r(T-t)}\left\{\int_0^\infty\frac{1}{2r+\sigma^2}\left[e^{(2r+\sigma^2)(T-t)}-1\right]\lambda_t(d\sigma)-\frac{2q}{r(T-t)}\left[e^{r(T-t)}-1\right]+\frac{q^2}{T-t}\right\}^{\frac{1}{2}}.$$

证明 在 $t$ 时刻, 若 $q\leqslant 0$, 即

$$\int_0^{T-t}e^{ru}\int_0^\infty \exp\left\{\sigma W_u^Q-\frac{1}{2}\sigma^2u\right\}\lambda_t(d\sigma)\,du-q\geqslant 0,$$

$$C(K,t)=\frac{1}{T}e^{-r(T-t)}S_tE^Q\left[\int_0^{T-t}e^{ru}\int_0^\infty \exp\left\{\sigma W_u^Q-\frac{1}{2}\sigma^2u\right\}\lambda_t(d\sigma)\,du-q\right]=$$

$$\frac{1}{T}e^{-r(T-t)}S_t\left[\int_0^{T-t}e^{ru}\int_0^\infty E^Q\exp\left\{\sigma W_u^Q-\frac{1}{2}\sigma^2u\right\}\lambda_t(d\sigma)\,du-q\right]=\frac{1}{T}e^{-r(T-t)}S_t\left[\int_0^{T-t}e^{ru}\int_0^\infty\lambda_t(d\sigma)\,du-q\right]=$$

$$\frac{1}{T}e^{-r(T-t)}S_t\left[\frac{1}{r}\left(e^{r(T-t)}-1\right)-\frac{KT-\int_0^t S_u\,dt}{S_u}\right]=\frac{S_t}{rT}-\left[\frac{S_t}{rT}-K+\frac{1}{T}\int_0^t S_u\,du\right]e^{-r(T-t)};$$

若 $q > 0$, 则

$$C(K,t) = \frac{S_t}{T}e^{-r(T-t)}E^Q\left[\left(\int_0^{T-t}e^{ru}\int_0^{\infty}\exp\left\{\sigma W_u^Q - \frac{1}{2}\sigma^2 u\right\}\lambda_t(\mathrm{d}\sigma)\mathrm{d}u - q\right)_+\right] \leqslant$$

$$\frac{S_t}{2T}e^{-r(T-t)}E^Q\left[\int_0^{T-t}\int_0^{\infty}\exp\left\{ru + \sigma W_u^Q - \frac{1}{2}\sigma^2 u\right\} - \frac{q}{T-t}\lambda_t(\mathrm{d}\sigma)\mathrm{d}u\right] +$$

$$\frac{S_t}{2T}e^{-r(T-t)}E^Q\left[\int_0^{T-t}\int_0^{\infty}\left|\exp\left\{ru + \sigma W_u^Q - \frac{1}{2}\sigma^2 u\right\} - \frac{q}{T-t}\right|\lambda_t(\mathrm{d}\sigma)\mathrm{d}u\right] \leqslant$$

$$\frac{1}{2}A_1 + \frac{S_t}{2T}e^{-r(T-t)}\left\{\int_0^{T-t}\int_0^{\infty}E^Q\left(\exp\left\{ru + \sigma W_u^Q - \frac{1}{2}\sigma^2 u\right\} - \right.\right.$$

$$\left.\left.\frac{q}{T-t}\right)^2\lambda_t(\mathrm{d}\sigma)\mathrm{d}u\right\}^{\frac{1}{2}} = \frac{1}{2}A_1 + \frac{S_t}{2T}e^{-r(T-t)}\left\{\int_0^{T-t}\int_0^{\infty}E^Q\left(\exp\{2ru + 2\sigma W_u^Q - \sigma^2 u\} - \right.\right.$$

$$\left.\left.\frac{2q}{T-t}\exp\left\{ru + \sigma W_u^Q - \frac{1}{2}\sigma^2 u\right\} + \left(\frac{q}{T-t}\right)^2\right)\lambda_t(\mathrm{d}\sigma)\mathrm{d}u\right\}^{\frac{1}{2}},$$

其中

$$E^Q\left[\exp\{2ru + 2\sigma W_u^Q - \sigma^2 u\} - \frac{2q}{T-t}\exp\left\{ru + \sigma W_u^Q - \frac{1}{2}\sigma^2 u\right\} + \left(\frac{q}{T-t}\right)^2\right] =$$

$$E^Q[\exp\{2ru + 2\sigma W_u^Q - \sigma^2 u\}] - \frac{2q}{T-t}E^Q\left[\exp\left\{ru + \sigma W_u^Q - \frac{\sigma^2 u}{2}\right\}\right] + \left(\frac{q}{T-t}\right)^2 =$$

$$\exp\{2ru + \sigma^2 u\} - \frac{2q}{T-t}\exp\{ru\} + \left(\frac{q}{T-t}\right)^2.$$

所以有

$$C(K,t) \leqslant \frac{1}{2}A_1 + \frac{S_t}{2T}e^{-r(T-t)}\left\{\int_0^{T-t}\int_0^{\infty}\left[\exp(2ru + \sigma^2 u) - \frac{2q}{T-t}\exp(ru) + \left(\frac{q}{T-t}\right)^2\right]\lambda_t(\mathrm{d}\sigma)\mathrm{d}u\right\}^{\frac{1}{2}} =$$

$$\frac{1}{2}A_1 + \frac{S_t}{2T}e^{-r(T-t)}\left\{\int_0^{\infty}\int_0^{T-t}\exp(2ru + \sigma^2 u)\mathrm{d}u\lambda_t(\mathrm{d}\sigma) - \frac{2q}{T-t}\int_0^{\infty}\int_0^{T-t}\exp(ru)\mathrm{d}u\lambda_t(\mathrm{d}\sigma) + \frac{q^2}{T-t}\right\}^{\frac{1}{2}} =$$

$$\frac{1}{2}A_1 + \frac{S_t}{2T}e^{-r(T-t)}\left\{\int_0^{\infty}\frac{1}{2r+\sigma^2}[\exp(2r+\sigma^2)(T-t) - 1]\lambda_t(\mathrm{d}\sigma) - \frac{2q}{r(T-t)}[e^{r(T-t)} - 1] + \frac{q^2}{T-t}\right\}^{\frac{1}{2}}.$$

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