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Optimal AOR for Rank Deficient Least Squares Problem

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Abstract: This paper studied the optimal parameters and asymptotical semiconvergence factor of AOR methods for rank deficient linear least squares problem and presented the explicit expressions of these factors. Finally, two numerical examples are given to illustrate our results.

Key words: AOR methods, optimal parameters, 2-cyclic, asymptotical semiconvergence factor, rank deficient linear least squares problem

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亏秩最小二乘问题的最优 AOR 方法

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[摘要] 主要研究了求解亏秩线性最小二乘问题的 AOR 方法的最优参数、渐近半收敛因子及其明晰的表达形式. 并给出了两个数值例子阐明结论.

[关键词] AOR 方法,最优参数,2-循环,渐近半收敛因子,亏秩线性最小二乘问题

Numerous techniques have been proposed for obtaining the least squares solution of the overdetermined system

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{1}$$

where $\boldsymbol{A}$ is a complex rectangular $m \times n$ matrix and $\boldsymbol{b}$ is a vector of size m. Among these direct or indirect methods, iterative methods are often utilized. In the case that $\boldsymbol{A}$ is of full column rank, Chen [1] augmented $(\boldsymbol{A}, \boldsymbol{b})$ to a block 3-cyclic matrix and suggested a combined direct-iterative method. Niethammar, de Pillis and Varga [2] then overcame the problems encountered by Chen in applying the SOR method to the augmented system. Markham, Neumann and Plemmons [3] proved that "2-block SOR" is superior to "3-block SOR". Yiannis G Saridakis [4] determined the optimal values when pertaining extrapolated iterative schemes to least squares problems. In the case that $\boldsymbol{A}$ is rank deficient, Miller and Neumann [5] essentially developed the theory of "2-block SOR" and Tian [6] applied AOR method for finding the least squares solution.

The outline of this paper is as follows. In section 1 we devote necessary notation and preliminaries. In section 2, we examine the optimal parameters and asymptotical semiconvergence factor of AOR methods for rank de-

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ficient linear least squares problem and present the explicit expressions of these factors. These work are based on known results from the optimal 2-cyclic AOR [7] and are further results to the previous researchers in the past twenty years. Finally, we give the numerical examples.

1 Preliminaries

Throughout the paper, $\mathbf{C}^{m\times n}$ denotes the space of $m\times n$ complex matrices and $\mathbf{C}_r^{m\times n}$ if matrices are of rank r. For $\boldsymbol{A}\in\mathbf{C}^{m\times n}$, $\boldsymbol{A}^*$, $\sigma(\boldsymbol{A})$, $\rho(\boldsymbol{A})$ denote the conjugate transpose, the spectrum of $\boldsymbol{A}$ and the spectral radius of $\boldsymbol{A}$, respectively. Moreover, $\delta(\boldsymbol{A})=\max\{|\lambda|:\lambda\in\sigma(\boldsymbol{A}),\lambda\neq 1\}$ and $\|\boldsymbol{A}\|_2=\sqrt{\rho(\boldsymbol{A}^*\boldsymbol{A})}$.

Consider the augmented systems of (1)

$$\begin{bmatrix}\boldsymbol{A} & \boldsymbol{I}\\ \boldsymbol{0} & \boldsymbol{A}^*\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{r}\end{bmatrix}=\begin{bmatrix}\boldsymbol{b}\\ \boldsymbol{0}\end{bmatrix},\boldsymbol{r}=\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x}, \tag{2}$$

where $\boldsymbol{r}$ is named as the residual vector, it is well known that the solution $\hat{\boldsymbol{x}}\in\mathbf{C}^m$ of (2) is a least squares solution to (1), that is

$$\|\boldsymbol{b}-\boldsymbol{A}\hat{\boldsymbol{x}}\|=\min_{x\in\mathbf{C}^m}\|\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x}\|.$$

Suppose $\boldsymbol{A}\in\mathbf{C}_r^{m\times n}$ and its partitioned block form is written as

$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{bmatrix},$$

where $\boldsymbol{A}_{11}\in\mathbf{C}_r^{r\times r}$. Set $\boldsymbol{B}=\boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}$, $\boldsymbol{C}=\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}$ and $\alpha=\|\boldsymbol{B}\|_2$ (i. e., α is the largest singular value of $\sqrt{\boldsymbol{B}^*\boldsymbol{B}}$). Then the augmented system (2) is singular and therefore it can be represented as

$$\hat{\boldsymbol{A}}z=\hat{\boldsymbol{b}},$$

where $z=\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{r}_2\\ \boldsymbol{r}_1\\ \boldsymbol{x}_2\end{bmatrix}$, $\hat{\boldsymbol{b}}=\begin{bmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\\ \boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix}$, $\boldsymbol{x}_1,\boldsymbol{b}_1\in\mathbf{C}^r$ and $\hat{\boldsymbol{A}}=\begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{0} & \boldsymbol{I}_r & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & \boldsymbol{I}_{m-r} & \boldsymbol{0} & \boldsymbol{A}_{22}\\ \boldsymbol{0} & \boldsymbol{A}_{21}^* & \boldsymbol{A}_{11}^* & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A}_{22}^* & \boldsymbol{A}_{12}^* & \boldsymbol{0}\end{bmatrix}$. Split $\hat{\boldsymbol{A}}$ as

$$\hat{\boldsymbol{A}}=\begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{A}_{21} & \boldsymbol{I}_{m-r} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_{11}^* & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{n-r}\end{bmatrix}-\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & -\boldsymbol{A}_{21}^* & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & -\boldsymbol{A}_{22}^* & -\boldsymbol{A}_{12}^* & \boldsymbol{0}\end{bmatrix}-\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_r & -\boldsymbol{A}_{12}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{A}_{22}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{n-r}\end{bmatrix}=\boldsymbol{D}-\boldsymbol{L}-\boldsymbol{U},$$

this kind of decomposition was presented by Miller and Neumann [5]. Then the AOR iteration matrix is defined by

$$\boldsymbol{L}_{\gamma,\omega}=(\boldsymbol{D}-\gamma\boldsymbol{L})^{-1}[(1-\omega)\boldsymbol{D}+(\omega-\gamma)\boldsymbol{L}+\omega\boldsymbol{U}]=$$
$$\begin{bmatrix}(1-\omega)\boldsymbol{I}_r & \boldsymbol{0} & -\omega\boldsymbol{A}_{11}^{-1} & -\omega\boldsymbol{C}\\ \boldsymbol{0} & (1-\omega)\boldsymbol{I}_{m-r} & \omega\boldsymbol{B} & \boldsymbol{0}\\ \boldsymbol{0} & \omega(\gamma-1)\boldsymbol{B}^* & (1-\omega)\boldsymbol{I}_r-\omega\gamma\boldsymbol{B}^*\boldsymbol{B} & \boldsymbol{0}\\ \boldsymbol{0} & -\omega(\gamma-1)^2\boldsymbol{A}_{22}^* & \omega(\gamma-1)\boldsymbol{A}_{12}^*(\boldsymbol{I}+\gamma\boldsymbol{B}^*\boldsymbol{B}) & \boldsymbol{I}_{n-r}\end{bmatrix}, \tag{3}$$

and the AOR iterative scheme is

$$z_{k+1}=\boldsymbol{L}_{\gamma,\omega}z_k+\omega(\boldsymbol{D}-\gamma\boldsymbol{L})^{-1}\hat{\boldsymbol{b}}.$$

Set

$$\boldsymbol{T}_{\gamma,\omega}=\begin{bmatrix}(1-\omega)\boldsymbol{I}_{m-r} & \omega\boldsymbol{B}\\ \omega(\gamma-1)\boldsymbol{B}^* & (1-\omega)\boldsymbol{I}_r-\omega\gamma\boldsymbol{B}^*\boldsymbol{B}\end{bmatrix},$$

by interchanging of rows and columns 2 and 4, 3 and 4 of $\boldsymbol{L}_{\gamma,\omega}$ respectively, we obtain its similar matrix

$$\hat{L}_{\gamma,\omega} = \begin{bmatrix} (1-\omega) I_r & -\omega C & * & * \\ 0 & I_{n-r} & * & \\ 0 & 0 & T_{\gamma,\omega} & \\ 0 & 0 & & \end{bmatrix}. \tag{4}$$

From (3) and (4), we have

$$\sigma(L_{\gamma,\omega}) = \sigma(\hat{L}_{\gamma,\omega}) = \{1-\omega, 1\} \cup \sigma(T_{\gamma,\omega}). \tag{5}$$

Lemma 1 $\lambda \in \sigma(T_{\gamma,\omega})$ and $\mu_j^2 \in \sigma(B^* B)$ satisfy the relationship

$$\lambda^2 + [2(\omega-1) + \omega\gamma\mu_j^2]\lambda + (\omega-1)^2 + \omega(\omega-\gamma)\mu_j^2 = 0. \tag{6}$$

Proof Let

$$S = \begin{bmatrix} I_{m-r} & -B \\ B^* & I_r \end{bmatrix} = \begin{bmatrix} I_{m-r} & 0 \\ 0 & I_r \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -B^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = I - \tilde{L} - \tilde{U},$$

then its associated Jacobi matrix is

$$J_s = \tilde{L} + \tilde{U} = \begin{bmatrix} 0 & B \\ -B^* & 0 \end{bmatrix},$$

and J_s^2 has the nonpositive spectrum. Thus J_s is weakly cyclic of index 2 and S is 2-cyclic [8,9]. In addition, the AOR iteration matrix of S precisely is $T_{\gamma,\omega}$, i. e.,

$$T_{\gamma,\omega} = (I - \gamma\tilde{L})^{-1} [(1-\omega) I + (\omega-\gamma)\tilde{L} + \omega\tilde{U}].$$

Since $t \in \sigma(T_{\gamma,1})$ and $\xi \in \sigma(J_s)$ satisfy the relationship $t^2 = (1-\gamma+\gamma t)\xi^2$ [10] and $T_{\gamma,\omega} = (1-\omega) I + \omega T_{\gamma,1}$, we immediately have (6).

Theorem 1 The semiconvergence region D (see Fig. 1) of AOR method is

$$\begin{cases} 0 < \omega < \dfrac{2}{\sqrt{1+\alpha^2}} = \omega_0, \\ g(\omega) = \omega + \dfrac{w-2}{\alpha^2} < \gamma < \dfrac{1}{2}\left[\omega + \dfrac{(2-\omega)^2}{\omega\alpha^2}\right] = f(\omega), \gamma \neq 0, \end{cases} \tag{7}$$

where two numbers γ_b and $\tilde{\gamma}_b$ appeared in Fig. 1 are defined by $\gamma_b = \dfrac{2}{1+\sqrt{1+\alpha^2}}$, $\tilde{\gamma}_b = \dfrac{2}{1+\sqrt{1+\beta^2}}$ with $\beta^2 = \min\{\mu_j^2 : \mu_j^2 \in \sigma(B^* B)\}$.

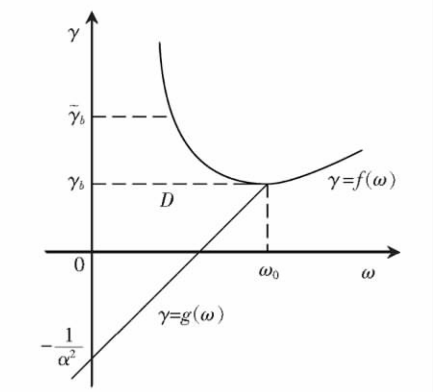

Fig.1 Semiconvergence region according to $L_{\gamma,\omega}$ of AOR method

Proof Replace "$\alpha = \|B\|_2$" by "$\alpha = \rho(J_s)$" in Theorem 2. 1 in [7], then (7) follows.

Theorem 2 Set

$$\rho_1 = \frac{1}{2}\omega\gamma\alpha^2 + \omega - 1 + \frac{1}{2}\omega\alpha\sqrt{\gamma^2\alpha^2 + 4\gamma - 4},$$

$$\rho_2 = 1 - \omega - \frac{1}{2}\omega\gamma\alpha^2 + \frac{1}{2}\omega\alpha\sqrt{\gamma^2\alpha^2 + 4\gamma - 4},$$

$$\rho_3 = \begin{cases} \sqrt{(1-\omega)^2 - \omega(\gamma-\omega)\beta^2}\ (\gamma \geqslant \omega), & \text{if } \beta^2 > 0, \\ 1 - \omega(\gamma \geqslant \omega), & \text{if } \beta^2 = 0, \end{cases}$$

$$\rho_4 = \sqrt{(1-\omega)^2 - \omega(\gamma-\omega)\alpha^2}\ (\gamma \leqslant \omega).$$

Then $\rho(T_{\gamma,\omega})$ has its expression on the region D as follows:

$$\rho(T_{\gamma,\omega}) = \rho_i, \text{ on the subregion } i,\ i = 1,2,3,4,$$

where in the case that $\beta^2 > 0$, the subregions 1,2,3,4 are as shown in Fig. 2, and in the case that $\beta^2 = 0$, the subregions 1,2,3,4 are as shown in Fig. 3. Moreover, curves are denoted as follows:

in Fig. 2:

$$C_1 : \omega = \frac{2}{2+\gamma\alpha^2},$$

$$C_2: \rho_1 = \rho_3,$$
$$\tilde{C}_2: \rho_2 = \rho_3,$$

in Fig. 3:

$$C_3: \omega = \frac{4}{4 + \gamma\alpha^2 + \alpha\sqrt{\gamma^2\alpha^2 + 4\gamma - 4}}.$$

Proof By results obtained in Theorem 2.2 – 2.5 of [7], we can directly get the above expressions of $\rho(\boldsymbol{T}_{\gamma,\omega})$.

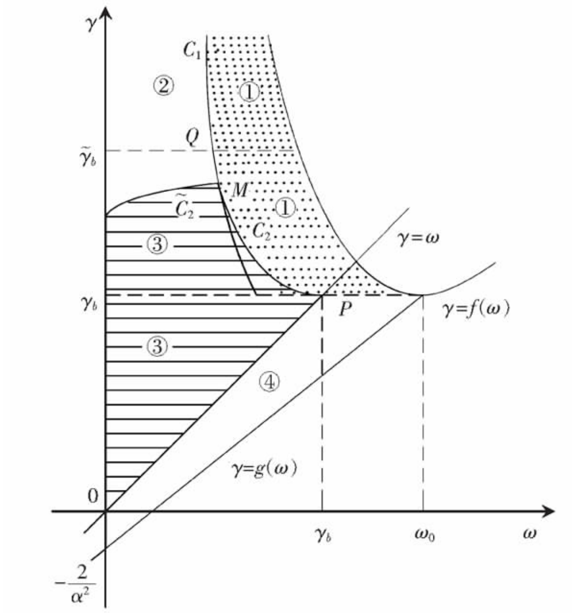

Fig.2 Distribution regions of $\rho(T_{\gamma,\omega})$ when B^*B is nonsingular

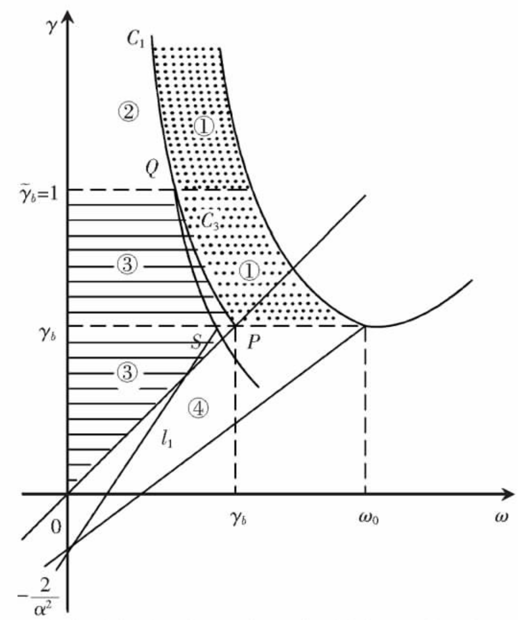

Fig.3 Distribution regions of $\rho(T_{\gamma,\omega})$ and $\delta(L_{\gamma,\omega})$ when B^*B is singular

2 Expression of $\delta(\boldsymbol{L}_{\gamma,\omega})$, Optimal Parameters and Asymptotical Semiconvergence Factor of $\boldsymbol{L}_{\gamma,\omega}$

From (4) or (5) we have

$$\delta(\boldsymbol{L}_{\gamma,\omega}) = \max\{|1-\omega|, \rho(\boldsymbol{T}_{\gamma,\omega})\},$$

so we compare $\rho(\boldsymbol{T}_{\gamma,\omega})$ with $|1-\omega|$ on the subregion $i(i=1,2,3,4)$ to get the expression of $\delta(\boldsymbol{L}_{\gamma,\omega})$. First, we state a critical result in the following theorem.

Theorem 3 Set

$$\rho_1 = \frac{1}{2}\omega\gamma\alpha^2 + \omega - 1 + \frac{1}{2}\omega\alpha\sqrt{\gamma^2\alpha^2 + 4\gamma - 4},$$
$$\rho_2 = 1 - \omega - \frac{1}{2}\omega\gamma\alpha^2 + \frac{1}{2}\omega\alpha\sqrt{\gamma^2\alpha^2 + 4\gamma - 4},$$
$$\rho_3 = 1 - \omega \ (\gamma \geqslant \omega),$$
$$\rho_4 = \sqrt{(1-\omega)^2 - \omega(\gamma-\omega)\alpha^2} \ (\gamma \leqslant \omega).$$

Then $\delta(\boldsymbol{L}_{\gamma,\omega})$ has its expression on the region D as follows:

$$\delta(\boldsymbol{L}_{\gamma,\omega}) = \begin{cases} \rho_1, \text{on the subregion } 1, \\ \rho_2, \text{on the subregion } 2, \\ \rho_3, \text{on the subregion } 3, \\ \rho_4, \text{on the subregion } 4, \end{cases}$$

where in the case that $\beta^2 = 0$, the subregions 1, 2, 3, 4 are as shown in Fig. 3, and in the case that $\beta^2 > 0$, the subregions 1, 2, 3, 4 are as shown in Fig. 4.

Proof First, we compare $\rho_1=\frac{1}{2}\omega\gamma\alpha^2+\omega-1+\frac{1}{2}\omega\alpha\sqrt{\gamma^2\alpha^2+4\gamma-4}$ with $|1-\omega|$ on the subregion 1. If $\omega\geqslant 1$, then $\rho_1>\omega-1$, it shows that $\delta(\boldsymbol{L}_{\gamma,\omega})=\rho_1$. If $\omega\leqslant 1$, then $\rho_1\geqslant 1-\omega$ if and only if $\frac{1}{2}\omega\gamma\alpha^2+\omega-1+\frac{1}{2}\omega\alpha\sqrt{\gamma^2\alpha^2+4\gamma-4}\geqslant 0$, that is to say $\omega\geqslant\frac{4}{4+\gamma\alpha^2+\alpha\sqrt{\gamma^2\alpha^2+4\gamma-4}}$. When $\gamma_b\leqslant\gamma<1$, it is easy to verify that $\frac{4}{4+\gamma\alpha^2+\alpha\sqrt{\gamma^2\alpha^2+4\gamma-4}}\geqslant\frac{2}{2+\gamma\alpha^2}$. Thus the curve C_3 is located in the right side of the curve C_1 and it connects the point $R\left(\frac{2}{2+\gamma\alpha^2},1\right)$ with $P(\gamma_b,\gamma_b)$. The point R is located on the curve C_1. The curve C_3 separates the subregion 1 into two parts, the left and the right sides of C_3. By simple analysis we have

$$\delta(\boldsymbol{L}_{\gamma,\omega})=\begin{cases}\rho_1, & \text{on the right side of } C_3,\\ 1-\omega, & \text{on the left side of } C_3,\\ \rho_1, & \text{on the upper of the line } \gamma=1.\end{cases}$$

Then we compare ρ_2 with $|1-\omega|$ on the subregion 2. Since $\omega\leqslant\frac{2}{2+\gamma\alpha^2}$ (subregion 2 is on the left side of C_1), it follows that $1-\omega\geqslant\frac{1}{2}\omega\gamma\alpha^2>0$. Thus $0<\omega<1$ and $|1-\omega|=1-\omega$. In addition, $\sqrt{\gamma^2+4\gamma-4}\geqslant\gamma\alpha$ if and only if $\gamma\geqslant 1$ and $\sqrt{\gamma^2+4\gamma-4}=\alpha$ if $\gamma=1$. Therefore we have

$$\rho_2\begin{cases}\leqslant 1-\omega, & \text{if } \gamma\leqslant 1,\\ \geqslant 1-\omega, & \text{if } \gamma\geqslant 1.\end{cases}$$

So that the subregion 2 is separated into two parts by the line $\gamma=1$, then comes the expression of $\delta(\boldsymbol{L}_{\gamma,\omega})$.

$$\delta(\boldsymbol{L}_{\gamma,\omega})=\begin{cases}\rho_2, & \text{on the upper of the line } \gamma=1,\\ 1-\omega, & \text{on the lower of the line } \gamma=1.\end{cases}$$

Lastly, we consider cases on the subregion 3 and 4. When $\omega\leqslant\gamma\leqslant 1$, it holds $\rho_3\leqslant|1-\omega|=1-\omega$ and we have $\delta(\boldsymbol{L}_{\gamma,\omega})=1-\omega$ on the subregion 3. When $\omega\geqslant\gamma$, we have $\rho_4\geqslant|1-\omega|$, hence $\delta(\boldsymbol{L}_{\gamma,\omega})=\rho_4$ on the subregion 4.

Summarizing the above discussion we know that in the case $\beta^2=0$, the above Fig. 3 is also suitable for the distribution of $\delta(\boldsymbol{L}_{\gamma,\omega})$, but in the case that $\beta^2>0$, we must repartition the subregions 1, 2, 3 and 4 shown in Fig. 3 to get Fig. 4.

In the coming theorem, we give the fundamental result of this paper.

Theorem 4 The optimal parameters of the AOR methods for solving the rank deficient least squares problem are

$$\omega=\gamma=\gamma_b=\frac{2}{1+\sqrt{1+\alpha^2}},$$

and the asymptotical semiconvergence factor is

$$\delta(\boldsymbol{L}_{\gamma_b,\gamma_b})=\min_{(\omega,\gamma)\in D}\delta(\boldsymbol{L}_{\gamma,\omega})=1-\gamma_b=\frac{\alpha^2}{(1+\sqrt{1+\alpha^2})^2}.$$

Proof We'll discuss this problem by distinguishing two cases according as $\beta^2=0$ and $\beta^2>0$.

(1) In the case that $\beta^2=0$ (see Fig. 3):

Noting that ρ_2 is decreasing on ω and increasing on γ, ρ_2 on the subregion 2 has the minimization point $Q\left(\frac{2}{2+\alpha^2},1\right)$, and the minimum $\frac{\alpha^2}{2+\alpha^2}$. $\rho_3=1-\omega$ on the subregion 3 has the minimization point $P(\gamma_b,\gamma_b)$ and the minimum $1-\gamma_b$. For finding the minimization point of ρ_1 on the subregion 1, we need only to observe magnitude on the curves C_1 and C_3. Based on the above analysis on ρ_2 and ρ_3, we can know that Q and P are the minimization points of ρ_1, the minimums are $\frac{\alpha^2}{2+\alpha^2}$ and $1-\gamma_b$. Since $\rho_4 = \sqrt{(1+\alpha^2)\omega^2-(2+\gamma\alpha^2)\omega+1}$ is decreasing on γ, and is decreading on ω in the left side of the line l_1, where l_1: $w = \frac{2+\gamma\alpha^2}{2(1+\alpha^2)}$, which connects the point $\left(0,-\frac{2}{\alpha^2}\right)$ with the point S, the intersecting point of C_1 and $\gamma=\gamma_b$, ρ_4 on the subregion 4 has the minimization point P and the minimum $1-\gamma_b$.

(2) In the case that $\beta^2>0$ (see Fig. 4):

ρ_2 on the subregion 2 has the minimization point $R\left(\frac{2}{2+\alpha^2},1\right)$ and the minimum $\frac{\alpha^2}{2+\alpha^2}$.

ρ_3 on the subregion 3 and ρ_4 on the subregion 4 have the same minimization point $P(\gamma_b,\gamma_b)$, and the common minimum $1-\gamma_b$.

ρ_1 on the subregion 1 has the minimization points R and P, the corresponding minimums are $\frac{\alpha^2}{2+\alpha^2}$ and $1-\gamma_b$.

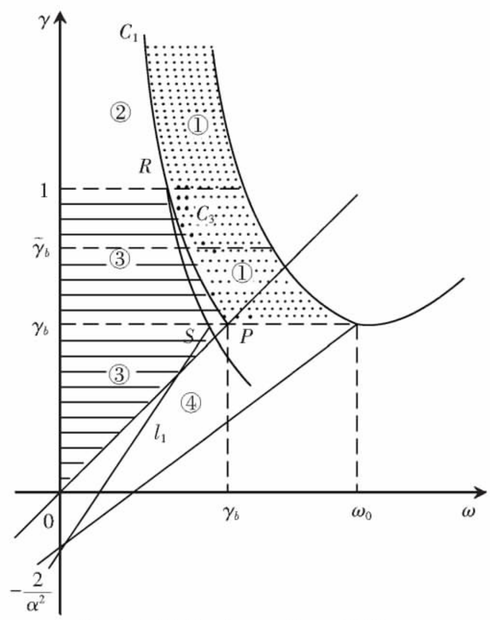

Fig.4 Distribution regions of $\delta(L_{\gamma,\omega})$ when B^*B is nonsingular

In summary, regardless of whether $\beta^2=0$ or $\beta^2>0$, $\delta(L_{\gamma,\omega})$ on the region D has two local minimums, $1-\gamma_b = \frac{\alpha^2}{(1+\sqrt{1+\alpha^2})^2}$ and $\frac{\alpha^2}{2+\alpha^2}$. It is easy to observe that $\frac{\alpha^2}{(1+\sqrt{1+\alpha^2})^2} < \frac{\alpha^2}{2+\alpha^2}$. Thus, Theorem 4 follows.

Remark 1 If $\beta^2>0$ (i. e., B^*B is nonsingular), we have $\delta(L_{\gamma,\omega}) \neq \rho(T_{\gamma,\omega})$ on the region D; If $\beta^2=0$ (i. e., B^*B is singular), we have $\delta(L_{\gamma,\omega}) = \rho(T_{\gamma,\omega})$ on the region D.

3 Numerical Examples

In this section we present two numerical examples to illustrate our theory. The first example is in the case that $\beta^2=0$, and the second one is in the case that $\beta^2>0$. Moreover, Fig. 3 ($\beta^2=0$) is suitable for both rank deficient and full column rank least squares problems.

Set

$$\rho_1 = \frac{1}{2}\omega\gamma\alpha^2+\omega-1+\frac{1}{2}\omega\alpha\sqrt{\gamma^2\alpha^2+4\gamma-4},$$

$$\rho_2 = 1-\omega-\frac{1}{2}\omega\gamma\alpha^2+\frac{1}{2}\omega\alpha\sqrt{\gamma^2\alpha^2+4\gamma-4},$$

$$\rho_3 = 1-\omega \ (\gamma\geqslant\omega),$$

$$\rho_4 = \sqrt{(1-\omega)^2-\omega(\gamma-\omega)\alpha^2} \ (\gamma\leqslant\omega).$$

Example 1 Let $A=\begin{bmatrix}1&0&0\\0&1&1\\0&1&1\end{bmatrix}$, or $A=\begin{bmatrix}1&0\\0&1\\0&1\end{bmatrix}$, then rank$A=2$, $B=[0,1]$, $\alpha=\|B\|_2=1$, $\sigma(B^*B)=\{1,0\}$, $\alpha^2=1$, $\beta^2=0$, $\omega_0=1.4142$, $\gamma_b=0.8284$, $\tilde{\gamma}_b=1$.

The optimal results of AOR of this example are shown in Table 1 and the four subregions 1, 2, 3, 4 shown in Fig. 3 are as follows:

Subregion 1

$$\begin{cases}\gamma\geqslant 1,\ \dfrac{2}{2+\gamma}\leqslant\omega\leqslant\dfrac{1}{2}\left(2+\gamma-\sqrt{\gamma^2+4\gamma-4}\right);\\ 0.828\,4\leqslant\gamma\leqslant 1,\ \dfrac{4}{4+\gamma+\sqrt{\gamma^2+4\gamma-4}}\leqslant\omega\leqslant\dfrac{1}{2}\left(2+\gamma-\sqrt{\gamma^2+4\gamma-4}\right).\end{cases}$$

Subregion 2

$$\gamma\geqslant 1,\ 0<\omega\leqslant\frac{2}{2+\gamma}.$$

Subregion 3

$$\begin{cases}0.828\,4\leqslant\gamma\leqslant 1,\ 0<\omega\leqslant\dfrac{4}{4+\gamma+\sqrt{\gamma^2+4\gamma-4}};\\ 0<\gamma\leqslant 0.828\,4,\ 0<\omega\leqslant\gamma.\end{cases}$$

Subregion 4

$$\begin{cases}0<\gamma\leqslant 0.828\,4,\ \gamma\leqslant\omega<\dfrac{2+\gamma}{2};\\ -2<\gamma<0,\ 0<\omega<\dfrac{2+\gamma}{2}.\end{cases}$$

Table 1 Optimal results of AOR when B^*B is singular

point(ω,γ)	associated subregion i	expression of $\delta(L_{\gamma,\omega})$	value of $\delta(L_{\gamma,\omega})$
(0.5, 2)	1	ρ_1	0.707 1
(0.7, 1)	1	ρ_1	0.400 0
(0.2, 5)	2	ρ_2	0.940 3
(0.5, 1)	2, 3	ρ_2,ρ_3	0.500 0
(0.7, 0.7)	3, 4	ρ_3,ρ_4	0.300 0
(0.8, 0.8)	3, 4	ρ_3,ρ_4	0.200 0
* (0.828 4, 0.828 4)	1, 3, 4	ρ_1,ρ_3,ρ_4	0.171 6
(0.9, 0.9)	1	ρ_1	0.593 1
(1, 0.5)	4	ρ_4	0.707 1
(1, 0.828 4)	1, 4	ρ_1,ρ_4	0.414 2
(1.4, 0.82)	4	ρ_4	0.985 9
(0.2, −1)	4	ρ_4	0.938 1

Example 2 Let $A=\begin{bmatrix}1&0&2&0\\0&1&0&2\\0&1&0&2\\2&0&4&0\end{bmatrix}$, then $\operatorname{rank}A=2$, $B=\begin{bmatrix}0&1\\2&0\end{bmatrix}$, $\alpha=\|B\|_2=2$, $\sigma(B^*B)=\{4,1\}$, $\alpha^2=4$, $\beta^2=1$, $\omega_0=0.894\,4$, $\gamma_b=0.618\,0$, $\tilde{\gamma}_b=0.828\,4$.

The optimal results of AOR of this example are shown in Table 2 and the four subregions 1, 2, 3, 4 shown in Fig. 4 are as follows:

Subregion 1

$$\begin{cases}\gamma\geqslant 1,\ \dfrac{2}{2+4\gamma}\leqslant\omega<\dfrac{1}{5}\left(2+4\gamma-2\sqrt{4\gamma^2+4\gamma-4}\right);\\ 0.618\,0\leqslant\gamma\leqslant 1,\ \dfrac{4}{4+4\gamma+2\sqrt{4\gamma^2+4\gamma-4}}\leqslant\omega<\dfrac{1}{5}\left(2+4\gamma-2\sqrt{4\gamma^2+4\gamma-4}\right).\end{cases}$$

Subregion 2

$$\gamma\geqslant 1,\ 0<\omega\leqslant\frac{2}{2+4\gamma}.$$

Subregion 3

$$\begin{cases} 0.618\,0 \leqslant \gamma \leqslant 1, \ 0 < \omega \leqslant \dfrac{4}{4+4\gamma+2\sqrt{4\gamma^2+4\gamma-4}}; \\ 0 < \gamma \leqslant 0.618\,0, \ 0 < \omega \leqslant \gamma. \end{cases}$$

Subregion 4

$$\begin{cases} 0 < \gamma \leqslant 0.618\,0, \ \gamma \leqslant \omega < \dfrac{2+4\gamma}{5}; \\ -0.5 < \gamma < 0, \ 0 < \omega < \dfrac{2+4\gamma}{5}. \end{cases}$$

Table 2　Optimal results of AOR when $B^* B$ is nonsingular

point(ω,γ)	associated subregion i	expression of $\delta(L_{\gamma,\omega})$	value of $\delta(L_{\gamma,\omega})$
(0.2, 2)	1	ρ_1	0.894 4
(0.146, 3)	1	ρ_1	0.970 5
(0.2, 1.5)	2	ρ_2	0.863 3
(0.3, 1)	2, 3	ρ_2, ρ_3	0.700 0
(0.4, 0.4)	3, 4	ρ_3, ρ_4	0.600 0
(0.6, 0.6)	3, 4	ρ_3, ρ_4	0.400 0
* (0.618 0, 0.618 0)	1, 3, 4	ρ_1, ρ_3, ρ_4	0.382 0
(0.65, 0.65)	1	ρ_1	0.845 0
(0.6, 0.3)	4	ρ_4	0.938 1
(0.7, 0.6)	4	ρ_4	0.608 3
(0.8, 0.6)	4	ρ_4	0.824 6

Remark 2　The optimal parameters (ω,γ) in Table 1 (in the case $B^* B$ is singular) and Table 2 (in the case $B^* B$ is nonsingular) are (0.828 4, 0.828 4) and (0.618 0, 0.618 0) respectively, which testify the results in Theorem 4 that the optimal parameters of the AOR methods for ranking deficient least squares problem are $\omega = \gamma = \gamma_b$.

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