# Pressure Spectrum for Birkhoff Averages

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**Abstract:** The strategy behind the use of Legendre transforms is to shift, from a function with one of its parameters an independent variable, to a new function with its dependence on a new variable. In this paper, we show that pressure spectra may be obtained as Legendre transforms of functions $T$: $\mathbf{R}\to\mathbf{R}$ arising in the thermodynamic formalism. The primary hypothesis we require is that the functions $T$ be continuously differentiable. In this way we make rigorous the general paradigm of reducing questions regarding the multifractal formalism to questions regarding the thermodynamic formalism. These results hold for a broad class of measurable potentials, which includes (but is not limited to) continuous functions.

**Key words:** multifractal, spectrum, Birkhoff, pressure

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# 伯克霍夫平均的压谱

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[摘要] Legendre 变换的作用是将一个独立参变量函数转化为一个新的独立参变量函数. 本文证明了在热力学形式中压谱可以由函数 $T$: $\mathbf{R}\to\mathbf{R}$ 的 Legendre 变换所获得. 所考虑的函数 $T$ 为连续可微. 用这种方法我们将重分形的问题转化为热力学问题,这些结果对更广一类的可测函数仍成立,包含了(而不仅限于)连续函数.

[关键词] 重分形,谱,伯克霍夫,压

Multifractal analysis is a subarea of the dimension theory of dynamical systems. Briefly speaking, it studies the complexity of the level sets of invariant local quantities obtained from a dynamical system. Usually, we consider three local qualities: pointwise dimensions, local entropies, and Brikhoff averages. These functions are usually only measurable and thus their level sets are rarely manifolds. Therefore, to measure the complexity of these sets it is appropriate to use global quantities such as the topological entropy or the Hausdorff dimension. The basic elements of the multifractal formalism were first proposed by Halsey et al in [1], where they considered what they referred to as the dimension spectrum or the $f(\alpha)$-spectrum for dimensions, which characterizes an invariant measure $\mu$ for a dynamical system $f$: $X\to X$ in terms of the level set of the pointwise dimension.

The pointwise dimension of $\mu$ at $x$ is defined as

$$d_\mu(x)=\lim_{\varepsilon\to 0}\frac{\log\mu(B(x,\varepsilon))}{\log\varepsilon},$$

provided the limit exists, and the level sets are denoted

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$$K_\alpha^{\mathscr{D}} = \{ x \in X \mid d_\mu(x) = \alpha \}.$$

Many measures of interest are exact-dimensional, that is, the pointwise dimension is constant $\mu$-almost everywhere. In particular, this is true of hyperbolic measure (those with non-zero Lyapunov exponents almost everywhere)[2]. For an exact-dimensional measure, one of the $K_\alpha^{\mathscr{D}}$ has full measure, and the rest have measure 0, and so we measure the sizes of these sets with the Hausdorff dimension rather than with the measure; in this way we obtain the dimension spectrum for pointwise dimensions, which is given by the function

$$\mathscr{D}(\alpha) = \dim_H K_\alpha^{\mathscr{D}}.$$

One may consider the measure of small balls which are refined dynamically, rather than statically. Rather than $B(x,\varepsilon)$ we consider the Bowen ball of radius $\delta$ and length $n$, given by

$$B(x,n,\delta) = \{ y \in X \mid f^k(y) \in B(f^k(x),\delta), \text{ for } k = 0,1,\cdots,n \}.$$

The local entropy of $\mu$ at $x$ is defined by

$$h_\mu(x) = \lim_{\delta\to 0}\lim_{n\to 0} -\frac{1}{n}\log\mu(B(x,n,\delta)),$$

provided the limit exists. We denote the level sets of the local entropy by

$$K_\alpha^{\varepsilon} = \{ x \in X \mid h_\mu(x) = \alpha \}.$$

It was shown by Brin and Katok[3] that if $\mu$ is ergodic, then one of the level sets $K_\alpha^{\varepsilon}$ has full measures, and the rest have measure 0; thus we must once again quantify them using a (global) dimensional characteristic. It turns out to be more natureal to measure the size of the sets $K_\alpha^{\varepsilon}$ with the topological entropy rather than Hausdorff dimension; because these level sets are in general not compact, we must use the definition of topological entropy in the sense of Bowen[4]. Upon doing so, we obtain the entropy spectrum for local entropies

$$\varepsilon(\alpha) = h_{top}(K_\alpha^{\varepsilon}).$$

For Gibbs measure on conformal repellers, this spectrum was studied in [5]. Takens and Verbitskiy[6] carried out the multifractal analysis in the more general case of expansive maps satisfying a specification property.

In fact, the proofs of the known results for both the dimension and entropy spectra contain (at least implicitly) a similar result for the Birkhoff spectrum. Writing the sum of $\varphi$ along an orbit as $S_n\varphi(x) = \sum_{k=0}^{n-1}\varphi(f^k(x))$, the Birkhoff average of $\varphi$ at $x$ is given by

$$\varphi^+(x) = \lim_{n\to\infty}\frac{1}{n}S_n\varphi(x),$$

provided the limit exists. The level sets of the Birkhoff averages are

$$K_\alpha^{\mathscr{B}} = \{ x \in X \mid \varphi^+(x) = \alpha \},$$

and the Birkhoff ergodic theorem guarantees that for any ergodic measure $\mu$, one of the level sets has full measure, and the rest have measure 0. Thus we once again measure their size in terms of topological entropy, and obtain the entropy spectrum of Brikhoff averages

$$\mathscr{B}(\alpha) = h_{\text{top}}(K_\alpha^{\mathscr{B}}).$$

One important example of a Birkhoff spectrum is worth noting. In the particular case where $f$ is conformal map and $\varphi(x) = -\log \| Df(x) \|$, the Birkhoff averages coincide with the Lyapunov exponents: $\lambda(x) = \varphi^+(x)$. In this case we will also denote the level sets by

$$K_\alpha^{\mathscr{L}} = \{ x \in X \mid \lambda(x) = \alpha \},$$

it turns out that we are able to examine not only the entropy spectrum for Lyapunov exponents

$$\mathscr{L}_E(\alpha) = h_{\text{top}} K_\alpha^{\mathscr{L}},$$

but also the dimension spectrum for Lyapunov exponents

$$\mathscr{L}_D(\alpha) = \dim_H K_\alpha^{\mathscr{L}}$$

by using a generalisation of Bowen's equation to non-compact sets.

The Legendre transform is an important tool in theoretical physics, playing a critical role in classical mechanics, statistical mechanics, and thermodynamics. As long as the thermodynamic functions are given, we can systematically study the thermodynamic properties of the system with Legendre. In [7], Climenhaga gave the Legendre transform of the Birkhoff entropy spectrum. Furthermore, we can define the Birkhoff pressure spectrum: for $g \in \mathscr{C}(X)$,

$$P_g(\alpha) = P_{K_\alpha^{\mathscr{B}}}(g),$$

where $P_{K_\alpha^{\mathscr{B}}}(g)$ denote the topological pressure for $g$ on non-compact set $K_\alpha^{\mathscr{B}}$. According to the variational principles for non-compact set version in [8], we immediately have

$$P_g(\alpha) = P_{K_\alpha^{\mathscr{B}}}(g) = \sup_{\mu \in M(K_\alpha^{\mathscr{B}}, f)} \left\{ h(\mu) + \int g \mathrm{d}\mu \right\}.$$

What we concern is whether the Birkhoff pressure spectrum has the form of Legendre transformation and the dual form. This result is given in Theorem 1, which applies to continuous maps $f$: $X \to X$, function $g$: $X \to \mathbf{R}$ and function $\varphi$: $X \to \mathbf{R}$ which lies in a certain class $\mathscr{A}_f$; this class contains, but is not limited to, the space of all continuous functions. For such maps and functions, we show that $P_g(\alpha)$ is the Legendre transform of the function $T_{\mathscr{B},\varphi,g}$: $q \quad P(q\varphi + g)$, provided $T_{\mathscr{B},\varphi,g}$ is continuously differentiable and equilibrium measures exist. If the hypotheses on $T_{\mathscr{B},\varphi,g}$ only hold for certain values of $q$, we still obtain a partial result on $P_g(\alpha)$ for the corresponding values of $\alpha$.

## 1 Definitions and Results

Let $(X,\rho)$ be a compact metric space with metric $\rho$, $f$: $X \to X$ a continuous map, and $\varphi$: $X \to \mathbf{R}$ a continuous function. Consider a finite open cover $\mathscr{U}$ of $X$ and denote by $S_m(\mathscr{U})$ the set of all strings $U = \{U_{i_0} \cdots U_{i_{m-1}}: U_{i_j} \in \mathscr{U}\}$ of length $m = m(U)$. We put $S = S(\mathscr{U}) = \bigcup_{m \geqslant 0} S_m(\mathscr{U})$.

To a given string $U = \{U_{i_0} \cdots U_{i_{m-1}}\} \in S(\mathscr{U})$ we associate the set

$$X(U) = \{x \in X: f^j(x) \in U_{i_j}, \text{ for } j = 0, \cdots, m(U) - 1\}.$$

Given $Z \subset X$ and $N \in N$, we let $S(Z, \mathscr{U}, N)$ denote the set of all finite or countable collections $\mathscr{G}$ of strings of length at least $N$ which cover $Z$; that is, $\mathscr{G} \subset S(\mathscr{U})$ is in $S(Z, \mathscr{U}, N)$ if and only if

(1) $m(U) \geqslant N$ for all $U \in \mathscr{G}$, and also

(2) $\bigcup_{U \in \mathscr{G}} X(U) \supset Z$.

Then we define a set function by

$$M(Z, \alpha, \varphi, \mathscr{U}, N) = \inf_{S(Z,\mathscr{U},N)} \left\{ \sum_{U \in \mathscr{G}} \exp\left( -\alpha m(U) + \sup_{x \in X(U)} \sum_{k=0}^{m(U)-1} \varphi(f^k(x)) \right) \right\},$$

and the critical value of $m(Z, \alpha, \varphi, \mathscr{U}) = \lim_{N \to \infty} M(Z, \alpha, \varphi, \mathscr{U}, N)$ by

$$P_Z(\varphi, \mathscr{U}) = \inf\{m(Z, \alpha, \varphi, \mathscr{U}) = 0\} = \sup\{m(Z, \alpha, \varphi, \mathscr{U}) = \infty\}.$$

The topological pressure is $P_Z(\varphi) = \lim_{|\mathscr{U}| \to 0} P_Z(\varphi, \mathscr{U})$, where

$$|\mathscr{U}| = \max\{\operatorname{diam} U_i \mid U_i \in \mathscr{U}\}$$

is the diameter of the cover $\mathscr{U}$.

Furthermore, the Carathéodory function $\underline{r}_C(Z, \alpha)$ and $\bar{r}_C(Z, \alpha)$ (where $Z \subset X$ and $\alpha \in \mathbf{R}$) depend on the cover $\mathscr{U}$ and are given by

$$\underline{r}_C(Z, \alpha) = \varliminf_{N \to \infty} R(Z, \alpha, \varphi, \mathscr{U}, N), \bar{r}_C(Z, \alpha) = \varlimsup_{N \to \infty} R(Z, \alpha, \varphi, \mathscr{U}, N),$$

where

$$R(Z, \alpha, \varphi, \mathscr{U}, N) = \inf_{\mathscr{G}} \left\{ \sum_{U \in \mathscr{G}} \exp\left( -\alpha N + \sup_{x \in X(U)} \sum_{k=0}^{N-1} \varphi(f^k(x)) \right) \right\} \tag{1}$$

and the infimum is taken over all finite or countable collections of strings $\mathscr{G} \subset S(\mathscr{U})$ such that $m(U) = N$ for all $U \in \mathscr{G}$ and $\mathscr{G}$ covers $Z$.

The critical value of $\underline{r}_C(Z, \alpha)$ and $\bar{r}_C(Z, \alpha)$ are separately denoted by

$$\underline{CP}_Z(\varphi,\mathscr{U}) = \inf\{\underline{r}_C(Z,\alpha) = 0\} = \sup\{\underline{r}_C(Z,\alpha) = \infty\},$$

and

$$\overline{CP}_Z(\varphi,\mathscr{U}) = \inf\{\bar{r}_C(Z,\alpha) = 0\} = \sup\{\bar{r}_C(Z,\alpha) = \infty\}.$$

Similarly, the lower and upper Carathéodory capacity are separately defined by

$$\underline{CP}_Z(\varphi) = \lim_{|\mathscr{U}|\to 0}\underline{CP}_Z(\varphi,\mathscr{U}),$$

and

$$\overline{CP}_Z(\varphi) = \lim_{|\mathscr{U}|\to 0}\overline{CP}_Z(\varphi,\mathscr{U}).$$

As shown in [8], we have

$$\overline{CP}_Z(\varphi) = \lim_{|\mathscr{U}|\to 0}\overline{\lim_{N\to\infty}}\frac{1}{N}\log\Lambda(Z,\varphi,\mathscr{U},N)$$

and

$$\underline{CP}_Z(\varphi) = \lim_{|\mathscr{U}|\to 0}\underline{\lim_{N\to\infty}}\frac{1}{N}\log\Lambda(Z,\varphi,\mathscr{U},N),$$

where

$$\Lambda(Z,\varphi,\mathscr{U},N) = \inf_{\mathscr{G}}\Big\{\sum_{U\in\mathscr{G}}\exp\Big(\sup_{x\in X(U)}\sum_{k=0}^{N-1}\varphi(f^k(x))\Big)\Big\}$$

and the infimum is taken over all finite or countable collections of strings $\mathscr{G}\subset S(\mathscr{U})$ such that $m(U) = N$ for all $U\in\mathscr{G}$ and $\mathscr{G}$ covers $Z$.

In the following we set

$$\underline{CP}_Z^*(\varphi) = \lim_{\delta\to 0}\underline{\lim_{N\to\infty}}\frac{1}{N}\log\inf_{\substack{E_N \text{ is an } (N,\delta)\\ \text{spanning set for } Z}}\Big(\sum_{y\in E_N}\mathrm{e}^{S_N\varphi(y)}\Big).$$

We got the following lemma.

**Lemma 1** Suppose $\varphi$: $Z\to\mathbf{R}$ is continuous, then we have $\underline{CP}_Z(\varphi) = \underline{CP}_Z^*(\varphi)$.

**Proof** Given $\delta > 0$, let

$$\varepsilon(\delta) = \sup\{|\varphi(x) - \varphi(y)| \mid \rho(x,y) < \delta\}$$

and observe that since $\varphi$ is continuous and $X$ is compact, $\varphi$ is in fact uniformly continuous, hence $\varepsilon(\delta)$ is finite, and $\lim_{\delta\to 0}\varepsilon(\delta) = 0$.

Choose an open cover $\mathscr{U}$ of $Z$ such that $|\mathscr{U}| < \varepsilon(\delta)$ and let $\gamma(\mathscr{U})$ be the Lebesgue number of $\mathscr{U}$. Suppose $E_N = \{x_1,x_2,x_3,\cdots\}$ is an $(N,\gamma(\mathscr{U}))$ spanning set of $Z$. Then for each $x_i$ there exists $U_i\in S_N(\mathscr{U})$ such that $B(x_i,N,\gamma(\mathscr{U}))\subset X(U_i)$; let $\mathscr{G}' = \{U_i\}$, and then

$$\Lambda(Z,\varphi,\mathscr{U},N) = \inf_{\mathscr{G}}\Big\{\sum_{U\in\mathscr{G}}\exp\Big(\sup_{x\in X(U)}\sum_{i=0}^{N-1}\varphi(f^i(x))\Big)\Big\} \leqslant$$

$$\sum_{U_i\in\mathscr{G}'}\exp\Big(\sup_{x\in X(U_i)}\sum_{i=0}^{N-1}\varphi(f^i(x))\Big) = \sum_{U_i\in\mathscr{G}'}\exp(N\varepsilon(\delta) + S_N\varphi(x_i)) =$$

$$\sum_{x\in E_N}\exp(N\varepsilon(\delta) + S_N\varphi(x)) = \exp(N\varepsilon(\delta))\sum_{x\in E_N}\exp(S_N\varphi(x)).$$

Hence we have

$$\underline{CP}_Z(\varphi) \leqslant \underline{CP}_Z^*(\varphi).$$

For the other inequality, fix a cover $\mathscr{U}$ of $X$ with $|\mathscr{U}| < \delta$. Given $\mathscr{G}\in S(Z,\mathscr{U},N)$, we may assume without loss of generality that for every $U\in\mathscr{G}$, we have $X(U)\cap Z\neq\varnothing$ (otherwise we may eliminate some sets from $\mathscr{G}$, which does not increase the sum in (1)). Thus for each such $U$, we choose $x_U\in X(U)\cap Z$; we see that $X(U)\subset B(x_v,N,\delta)$, and so

$$\Lambda(Z,\varphi,\mathscr{U},N) = \inf_{\mathscr{G}}\Big\{\sum_{U\in\mathscr{G}}\exp\Big(\sup_{x\in X(U)}\sum_{i=0}^{N-1}\varphi(f^i(x))\Big)\Big\} \geqslant \inf_{\substack{E_N \text{ is an } (N,\delta)\\ \text{spanning set for } Z}}\sum_{x\in E_N}\exp(S_N\varphi(x)).$$

Thus $\underline{CP}_Z(\varphi,\mathscr{U}) \geqslant \underline{CP}_Z^*(\varphi,\mathscr{U})$, and taking the limit as $\delta\to 0$ gives

$$\underline{CP}_Z(\varphi) \geqslant \underline{CP}_Z^*(\varphi).$$

**Remark 1** If $F_N$ is an $(N,\delta)$ separated subset of $Z$ of maximum cardinality then $F_N$ is an $(N,\delta)$ spanning set for $Z$. Hence, we have

$$P_Z(\varphi) \leqslant \underline{CP}_Z(\varphi) \leqslant \lim_{\delta\to 0}\varliminf_{N\to\infty}\frac{1}{N}\log \sup_{\substack{F_N \text{ is an } (N,\delta)\\ \text{separated set for } Z}} \sum_{x\in F_N}\exp(S_N\varphi(x)).$$

**Definition 1** Given a function $g$: $\mathbf{R}\to[-\infty,+\infty]$, we may refer to either of the following as the Legendre transform of $g$:

$$g^{L_1}(x) = \sup_{y\in\mathbf{R}}(g(y)+xy),$$
$$g^{L_2}(x) = \inf_{y\in\mathbf{R}}(g(y)-xy).$$

If $g$ is concave $(g''<0)$, then the Legendre transform of $g$ most naturally refers to $g^{L_1}$; if $g$ is convex $g''>0$, the most natural meaning is $g^{L_2}$. However, each of the $g^{L_i}$ is defined without reference to concavity or convexity, and so we may consider $g^{L_1}$ and/or $g^{L_2}$ even if $g$ is neither convex nor concave ($g$ need not even be continuous).

Our main result is to give the Legendre transform of the Birkhoff pressure spectrum of the following function:

$$T_{\mathscr{B},\varphi,g}(q) = \sup_{\mu\in M^f(X)}\left\{h(\mu)+\int q\varphi+g\mathrm{d}\mu : g\in\mathscr{C}(X)\right\}.$$

Finally, before stating the general result, we describe the class of functions to which it applies. Given a function $\varphi$: $X\to\mathbf{R}$, let $\mathscr{C}(\varphi)\subset X$ denote the set of points at which $\varphi$ is discontinuous. Then we let $\mathscr{A}_f$ denote the class of function $\varphi$: $X\to\mathbf{R}$ which satisfy the following conditions:

(A) $\varphi$ is bounded (both above and below);

(B) $\mu(\overline{\mathscr{C}(\varphi)})=0$ for all $\mu\in M^f(X)$.

In particular, $\mathscr{A}_f$ includes all continuous functions $f\in\mathscr{C}(X)$. It also includes all bounded measurable functions $\varphi$ for which $\mathscr{C}(\varphi)$ is finite and contains no periodic points, and more generally, all bounded measurable functions for which $\overline{\mathscr{C}(\varphi)}$ is disjoint from all its iterates. We will see later that passing from $\mathscr{C}(X)$ to $\mathscr{A}_f$ dose not change the weak* topology at measure in $M^f(X)$, which is the key to including discontinuous functions in our results.

**Lemma 2** Let $X$ be a compact metric space, $f$: $X\to X$ be continuous, and $\varphi\in\mathscr{A}_f$. Let $\mu\in M^f(X)$ be an invariant measure, and consider a sequence of (not necessarily $f$-invariant) measures $\{\mu_n\}\subset M(X)$ such that $\mu_n\to\mu$ in the weak* topology. Then

$$\lim_{n\to\infty}\int\varphi\mathrm{d}\mu_n = \int\varphi\mathrm{d}\mu.$$

**Proof**

See Theorem 2.1 in [7].

**Lemma 3** Let $X$ be a compact metric space, $f$: $X\to X$ be continuous, and $\eta,\zeta\in\mathscr{A}_f$. Let $\{P_n\}\subset\mathbf{R}^+$ be any positive sequence. Fix $Z\subset X$ and let $\beta_1,\beta_2\in[-\infty,\infty]$ be given by

$$\beta_1 = \inf_{x\in X}\varliminf_{n\to\infty}\frac{1}{n}S_n\eta(x),$$

$$\beta_2 = \sup_{x\in X}\varlimsup_{n\to\infty}\frac{1}{n}S_n\eta(x).$$

Finally, suppose that there exists a constant $\gamma>0$ such that for every $n\in N$ and $\delta>0$, there exists an $(n,\delta)$-separated set $E_n\subset Z$ with

$$\sum_{y\in E_n}\mathrm{e}^{S_n\zeta}(y) \geqslant \gamma P_n.$$

Then there exists $\mu\in M^f(X)$ satisfying the following:

$$\int\eta\mathrm{d}\mu \in [\beta_1,\beta_2],$$

$$h(\mu)+\int\zeta\mathrm{d}\mu\geqslant\overline{\lim_{n\to\infty}}\frac{1}{n}\log P_n.$$

**Proof** See Theorem 2.1 in [7].

**Theorem 1** (The pressure spectrum for Birkhoff averages). Let $X$ be a compact metric space, $f$: $X\to X$ be continuous, and $\varphi\in\mathscr{A}_f$, $g\in\mathscr{C}(X)$. Then $T_{\mathscr{B},\varphi,g}$ is the Legendre transform of the Birkhoff pressure spectrum:

$$T_{\mathscr{B},\varphi,g}(q)=\sup_{\alpha\in R}(P_g(\alpha)+q\alpha)=P_g^{L_1}(q)$$

for every $q\in\mathbf{R}$.

**Proof** By Birkhoff's ergodic theorem, every ergodic measure $\nu$ has $\nu(K_\alpha^B)=1$ for some $\alpha$, and so for $\nu$-almost every $x\in K_\alpha^{\mathscr{B}}$ (in particular, for some $x\in K_\alpha^{\mathscr{B}}$), we have $\int_X\varphi\mathrm{d}\nu=\varphi^+(x)=\alpha$. It follows that

$$T_{\mathscr{B},\varphi,g}(q)=\sup_{\mu\in M^f(X)}\left\{h(\mu)+\int q\varphi\mathrm{d}\mu+\int g\mathrm{d}\mu\right\}\leqslant$$

$$\sup_{\alpha\in\mathbf{R}}\left(\sup_{\nu\in M_E^f(K_\alpha^{\mathscr{B}})}\left\{h(\nu)+\int g\mathrm{d}\nu\right\}+q\alpha\right)\leqslant\sup_{\alpha\in\mathbf{R}}(P_{K_\alpha^{\mathscr{B}}}(g)+q\alpha)=P_g^{L_1}(q),$$

where the inequality $h(\nu)+\int g\mathrm{d}\nu\leqslant P_{K_\alpha^{\mathscr{B}}}(g)$ follows from Theorem A 2.1 in [8].

Now we prove the reverse inequality by showing that $T_{\mathscr{B},\varphi,g}(q)\geqslant P_g(\alpha)+q\alpha$ for all $\alpha\in\mathbf{R}$. To this end, we construct for every $\varepsilon>0$ a measure $\mu\in M^f(X)$ such that

$$h(\mu)+\int g\mathrm{d}\mu+\int q\varphi\mathrm{d}\mu\geqslant P_g(\alpha)+q\alpha-q\varepsilon.$$

To this end, we fix $\varepsilon>0$ and $N\in\mathbf{N}$, and consider the following "approximate level sets":

$$F_\alpha^{\varepsilon,N}(\varphi)=\bigcap_{n\geqslant N}\left\{x\in X\,\middle|\,\left|\frac{1}{n}S_n\varphi(x)-\alpha\right|<\varepsilon\right\},$$
$$F_\alpha^{\varepsilon}(\varphi)=\bigcup_{N\in\mathbf{N}}F_\alpha^{\varepsilon,N}(\varphi).$$

For these we have

$$K_\alpha^{\mathscr{B}}(\varphi)=\bigcap_{\varepsilon>0}F_\alpha^{\varepsilon}(\varphi).$$

For any $g\in\mathscr{C}(X)$,

$$P_{K_\alpha^{\mathscr{B}}}(g)\leqslant\sup_N\lim_{\delta\to0}\underline{\lim_{n\to\infty}}\frac{1}{n}\log\sup_{\substack{F_n\text{ is an }(n,\delta)\\\text{separated set for }F_\alpha^{\varepsilon,N}}}\sum_{y\in F_n}\exp(S_ng(y)).$$

Applying Lemma 3, with $P_n=\sup_{\substack{F_n\text{ is an }(n,\delta)\\\text{separated set for }F_\alpha^{\varepsilon,N}}}\sum_{y\in F_n}\exp(S_ng(y))$, $\zeta=g$, $\eta=\varphi$, $Z=F_\alpha^{\varepsilon,N}$, and $\gamma=1$. Let $F_n$ be the $(n,\delta)$ separated set for $F_\alpha^{\varepsilon,N}$ and satisfying

$$\log\sum_{y\in F_n}\exp(S_ng(y))\geqslant\log P_n-1.$$

We see that the measure $\mu$ which is constructed as a weak* limit of empirical measures on the separated sets $F_n$ satisfies $h(\mu)+\int g\mathrm{d}\mu\geqslant\underline{\lim_{n\to\infty}}\frac{1}{n}\log|P_n|$ and $\int\varphi\mathrm{d}\mu>\alpha-\varepsilon$. It follows that

$$T_{\mathscr{B},\varphi,g}(q)\geqslant h(\mu)+\int g\mathrm{d}\mu+\int q\varphi\mathrm{d}\mu\geqslant\underline{\lim_{n\to\infty}}\frac{1}{n}\log|P_n|+q\alpha-q\varepsilon=$$

$$\underline{\lim_{n\to\infty}}\frac{1}{n}\log\sup_{\substack{F_n\text{ is an }(n,\delta)\\\text{separated set for }F_\alpha^{\varepsilon,N}}}\sum_{y\in F_n}\exp(S_ng(y))+q\alpha-q\varepsilon.$$

Let $\delta\to0$, we have $T_{\mathscr{B},\varphi,g}(q)\geqslant\underline{CP}_{F_\alpha^{\varepsilon,N}}(g)+q\alpha-q\varepsilon$. Taking the supremum over all $N$ yields

$$T_{\mathscr{B},\varphi,g}(q)\geqslant P_g(\alpha)+q\alpha-q\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, this implies

$$T_{\mathscr{B},\varphi,g}(q)\geqslant P_g(\alpha)+q\alpha,$$

which completes the proof.

**Theorem 2** For any $g \in \mathscr{C}(X)$, the domain of $P_g(\alpha)$ is bounded by the following:

$$\alpha_{\min} = \inf\{\alpha \in \mathbf{R} \mid T_{\mathscr{B},\varphi,g}(q) \geqslant q\alpha - \max|g| \text{ for all } q\},$$
$$\alpha_{\max} = \sup\{\alpha \in \mathbf{R} \mid T_{\mathscr{B},\varphi,g}(q) \geqslant q\alpha - \max|g| \text{ for all } q\}.$$

That is, $K_\alpha^{\mathscr{B}} = \varnothing$ for every $\alpha < \alpha_{\min}$ and every $\alpha > \alpha_{\max}$.

**Proof** Suppose that $K_\alpha^{\mathscr{B}}$ is non-empty; that is, there exists $x \in X$ such that $\varphi^+(x) = \lim_{n\to\infty}\frac{1}{n}S_n\varphi(x) = \alpha$. Consider the empirical measures

$$\mu_{n,x} = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}.$$

Choose any subsequence $n_k$ such that $\mu_{n_k,x}$ converges in the weak* topology to $\mu \in M^f(X)$. Then by Lemma 2, we have $\int\varphi \mathrm{d}\mu = \alpha$, and in particular,

$$T_{\mathscr{B},\varphi,g}(q) \geqslant h(\mu) + \int q\varphi \mathrm{d}\mu + \int g \mathrm{d}\mu \geqslant q\alpha - \max|g|$$

for every $q \in \mathbf{R}$.

**Theorem 3** Suppose that $T_{\mathscr{B},\varphi,g}$ is $\mathscr{C}^r$ on $(q_1, q_2)$ for some $r \geqslant 1$, and that for each $q \in (q_1, q_2)$, there exists a (not necessarily unique) equilibrium state $\nu_{q,g}$ for the potential function $q\varphi + g$. Then

$$P_g(\alpha) = T^{L_2}_{\mathscr{B},\varphi,g}(\alpha) = \inf_{q\in\mathbf{R}}(T_{\mathscr{B},\varphi,g}(q) - q\alpha)$$

for all $\alpha \in (\alpha_1, \alpha_2)$, where $\alpha_i = T'_{\mathscr{B},\varphi,g}(q_i)$. In particular, $P_g(\alpha)$ is strictly concave on $(\alpha_1, \alpha_2)$, and $\mathscr{C}^r$ except at points corresponding to intervals on which $T_{\mathscr{B},\varphi,g}$ is affine.

Theorem 3 is an easy consequence of the following Lemmas.

**Lemma 4** If $\nu_{q,g}$ is an ergodic equilibrium state for $q\varphi + g$, then the Brikhoff spectrum $P_g$ is concave at

$$\alpha = \alpha(\nu_{q,g}) = \int_X \varphi \mathrm{d}\nu_{q,g} \tag{2}$$

in the following sense; there exists a line $l \subset \mathbf{R}^2$ through $(\alpha, P_g(\alpha))$ such that the graph of $P_g$ lies on or below $l$.

**Proof** Observe that since $\nu_{q,g}$ is ergodic, we have $\nu_{q,g}(K_\alpha^{\mathscr{B}}) = 1$, and hence $h(\nu_{q,g}) + \int g\mathrm{d}\nu_{q,g} \leqslant P_{K_\alpha^{\mathscr{B}}}(g)$. Now given $\alpha' \in \mathbf{R}$, we have

$$P_g(\alpha') \leqslant (P_g^{L_1})^{L_2}(\alpha') = (T_{\mathscr{B},\varphi,g})^{L_2}(\alpha') = \inf_{q'\in\mathbf{R}}(T_{\mathscr{B},\varphi,g}(q') - q'\alpha') \leqslant$$
$$T_{\mathscr{B},\varphi,g}(q) - q\alpha' = P_g^{L_1}(q) - q\alpha' = h(\nu_{q,g}) + \int q\varphi\mathrm{d}\nu_{q,g} + \int g\mathrm{d}\nu_{q,g} - q\alpha' \leqslant$$
$$P_{K_\alpha^{\mathscr{B}}}(g) + q(\alpha - \alpha') = P_g(\alpha) + q(\alpha - \alpha').$$

Thus we may take $l$ to be the line through $(\alpha, P_g(\alpha))$ with slope $-q$.

**Corollary 1** For any $\alpha$ as in (2), we have

$$P_g(\alpha) = T^{L_2}_{\mathscr{B},\varphi,g}(\alpha).$$

**Proof** Follows from Lemma 4 and $\alpha = \alpha'$.

**Lemma 5** (Ruelle's formula for the derivative of pressure) Let $\psi$ and $\phi$ be Borel measurable functions. If the function

$$q \qquad P_Z(\psi + q\phi)$$

is differentiable at $q$, and if in addition $\nu_q$ is an equilibrium state for $\psi + q\phi$, then

$$\frac{\mathrm{d}}{\mathrm{d}q}P_Z(\psi + q\phi) = \int_X \phi\mathrm{d}\nu_q.$$

**Proof** See Proposition 9.3 in [7].

**Lemma 6** If $T_{\mathscr{B},\varphi,g}$ is continuously differentiable on $(q_1, q_2)$ and $q\varphi + g$ has an equilibrium state $\nu_{q,g}$ for each $q \in (q_1, q_2)$, then every $\alpha \in (\alpha_1, \alpha_2) = (T'_{\mathscr{B},\varphi,g}(q_1), T'_{\mathscr{B},\varphi,g}(q_2))$ is of the form (2) for some ergodic

$\nu_{q,g}$.

**Proof** Since $T'_{\mathcal{B},\varphi,g}$ is continuous, the Intermediated Value Theorem implies that for every such $\alpha$ there exist $q$ such that $T'_{\mathcal{B},\varphi,g}(q) = \alpha$. Thus applying Lemma 5 with $\psi = g$ and $\phi = \varphi$, we see that any equilibrium state $\nu_{q,g}$ for $q\varphi + g$ have $\nu_{q,g}(\varphi) = \alpha$. If $\nu$ is not ergodic, then some element $\nu_{q,g}$ in its ergodic decomposition is also an equilibrium state, and we are done.

## [References]


[1] Halsey T, Jensen M, Kadanoff L, et al. Fractal measures and their singularities: the characterization of strange sets [J]. Phys Rev A, 1986, 33(2): 1 141-1 151.

[2] Barreira L, Pesion Y, Schmeling J. Dimension and product structure of hyperbolic measures [J]. Ann of Math, 1999, 149(3): 755-783.

[3] Brin M, Katok A. On Local Entropy [M] // Geometric Dynamics. Berlin: Springer-Verlag, 1983: 30-38.

[4] Bowen R. Topological entropy for non-compact sets [J]. Trans Amer Math Soc, 1973, 49: 125-136.

[5] Barreira L, Pesin Y, Schmeling J. On a general concept of multifractality: Multifractal spectra for dimensions, entropies, and lyapunov exponents. Multifractal rigidity [J]. Chaos, 1997, 7(1): 27-38.

[6] Takens F, Verbitski E. Multifractal analysis of local entropies for expansive homeomorphisms with specification [J]. Comm Math Phys, 1999, 203: 593-612.

[7] Climenhaga V. Multifractal formalism devived from thermodynamics for general dynamical systems [J]. Electron Res Announc Math Sci, 2010, 17: 1-11.

[8] Pesin Y. Dimension Theory in Dynamical Systems, Contemporary Views and Applications [M]. Chicago: University of Chicago Press, 1998.

[9] Climenhaga V. Bowen's equation in the non-uniform setting [J]. Ergoclic Theory Dynam Systems, 2011, 31(4): 1 163-1 182.


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