Optimization-Based Domain Decomposition Methods for H(div)-Elliptic Problem

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Abstract: In this paper, we propose some optimization-based domain decomposition methods for H(div)-elliptic problem. Convergent properties are examined by choosing proper parameters. Some numerical testes are presented to demonstrate the effectiveness of the method.

Key words: optimization domain decomposition method, H(div)-elliptic problem

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求解 H(div)-椭圆问题的最优型区域分解算法

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[摘要] 本文提出了几种求解 H(div)-椭圆问题的最优型区域分解算法,通过适当参数的选取,论证了算法的收敛性.数值实验验证了算法的有效性.

[关键词] 最优型区域分解算法,H(div)-椭圆问题

In this paper, we show how to use optimization-based domain decomposition methods to solve the following boundary value problem of vector fields,

$$\begin{aligned} -\operatorname{grad}(\operatorname{div}\boldsymbol{u}) + \beta\boldsymbol{u} &= \boldsymbol{f} \quad \text{in} \quad \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded polyhedral domain in $\mathbf{R}^d$ $(d=2,3)$, $\boldsymbol{n}$ is its unit outward normal vector, β is a positive parameter, and $\boldsymbol{f}\in(L^2(\Omega))^d$.

Domain decomposition methods (DDMs) have become increasingly important tools for solving PDEs for their parallelism. They have been extensively studied and become a very active area of research in the past few years. DDMs contain both overlapping and nonoverlapping decompositions, which subdivide the computational domain into overlapping or nonoverlapping subdomains. For a comprehensive account of the theory and the algorithm in domain decomposition methods, we refer to monographs [1,2,3], see also the review article [4] on nonoverlapping DDMs. A class of nonoverlapping domain decomposition method which based on optimization strategies has been previously proposed in [5-10]. The basic idea of these methods is that an appropriate cost functional is minimized so that the optimal solutionsatisfies the partial differential equations, which are linear or nonlinear and the constraints force the solutions on the two subdomains to agree on the common interface.

In recent years, there have been many techniques to solve H(div) problem, such as multigrid and multilevel methods [11-13], two-level overlapping Schwarz preconditioners [14], iterative substructuring methods [15] and Neumann-Neumann methods [16]. For the background on H(div)-elliptic problem, we refer to [17], see also mono-

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graph [18], where its application to the stabilized mixed formulations of the Stokes equations was introduced. Some more significant applications of $H(\mathrm{div})$ problem were proposed in [2]. To the best of our knowledge, there exit no work on the optimization-based nonoverlapping method to solve $H(\mathrm{div})$-elliptic problem. In this paper, we extend this method for solving $H(\mathrm{div})$-elliptic problem.

The paper is organized as follows. In section 1, some optimized-based domain decomposition methods are provided for $H(\mathrm{div})$-elliptic problem, and convergent properties are examined by choosing proper parameters. In section 2, some numerical results are presented to illustrate the effectiveness of the methods.

1 Domain Decomposition for $H(\mathrm{div})$-Elliptic Problem

For problem(1), for simply, we consider the case that Ω is partitioned into two connected nonoverlapping sub-domains Ω_1 and Ω_2, so that $\Omega=\overline{\Omega}_1\cup\overline{\Omega}_2$. $\Gamma=\overline{\Omega}_1\cap\overline{\Omega}_2$ is denoted the interface between the two sub-domains. Let $\Gamma_1=\overline{\Omega}_1\cap\partial\Omega$ and $\Gamma_2=\overline{\Omega}_2\cap\partial\Omega$. (A two dimensional example is denoted by Figure 1).

Let a pair of functions $\boldsymbol{u}_1, \boldsymbol{u}_2$ satisfy the given equations in the subdomains:

$$-\mathrm{grad}(\mathrm{div}\boldsymbol{u}_i)+\boldsymbol{u}_i=\boldsymbol{f} \quad \text{in} \quad \Omega_i, \qquad (2)$$
$$\boldsymbol{u}_i\cdot\boldsymbol{n}=0 \quad \text{on} \quad \Gamma_i, \quad \text{for } i=1,2.$$

We can easily prove that, if the following interface conditions hold [19]:

$$\boldsymbol{u}_1\cdot\boldsymbol{n}_1=\boldsymbol{u}_2\cdot\boldsymbol{n}_1=\lambda \quad \text{on} \quad \Gamma, \qquad (3)$$
$$\mathrm{div}\boldsymbol{u}_1=\mathrm{div}\boldsymbol{u}_2=g \quad \text{on} \quad \Gamma, \qquad (4)$$

then $\{\boldsymbol{u}: \boldsymbol{u}|_{\Omega_i}=\boldsymbol{u}_i, i=1,2\}$ is the solution of(1), here $\boldsymbol{n}_1$ is the unit outward normal of Ω_1 on Γ.

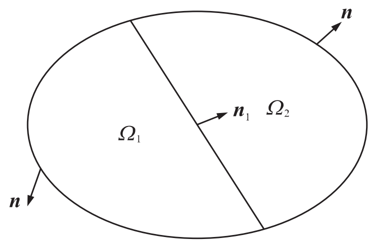

Fig. 1 Nonoverlapping domain decomposition of Ω into two sub-domains

The main idea of optimization-based domain decomposition method is to construct an energy functional $J(\boldsymbol{u}_1,\boldsymbol{u}_2,\lambda,g)$ over $\boldsymbol{u}_1,\boldsymbol{u}_2,\lambda$ and g in suitable case, subject to(2), and make that $\{\boldsymbol{u}|_{\Omega_1},\boldsymbol{u}|_{\Omega_2},\boldsymbol{u}\cdot\boldsymbol{n}_1|_\Gamma,\mathrm{div}\boldsymbol{u}|_\Gamma\}$ is its minimizer, here $\boldsymbol{u}$ is the unique solution of problem(1). For an elaborate description of optimization-based domain decomposition method, please refer to [5,6].

There are many choices of functional $J(\boldsymbol{u}_1,\boldsymbol{u}_2,\lambda,g)$. In this paper, we consider the following functional

$$J(\boldsymbol{u}_1,\boldsymbol{u}_2,\lambda,g)=\frac{1}{2}\sum_{i=1}^{2}\{\|\boldsymbol{u}_i\cdot\boldsymbol{n}_1-\lambda\|_{X_i}^2+\|\mathrm{div}\boldsymbol{u}_i-g\|_{Y_i}^2\}, \qquad (5)$$

with approximate Hilbert spaces X_i and Y_i equipped with norms $\|\cdot\|_{X_i}$ and $\|\cdot\|_{Y_i}(i=1,2)$, respectively.

Before proceeding to more detailed discussion, we define some technical terms. First, define the Sobolev space

$$H(\mathrm{div};\Omega)=\{\boldsymbol{v}\in(L^2(\Omega)^d):\mathrm{div}\boldsymbol{v}\in L^2(\Omega)\}.$$

Then, the subspace of vectors in $H(\mathrm{div};\Omega)$ with vanishing normal component on $\partial\Omega$ is denoted by $H_0(\mathrm{div};\Omega)$:

$$H_0(\mathrm{div};\Omega)=\{\boldsymbol{v}\in H(\mathrm{div};\Omega):\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega}=0\}.$$

Besides, let the harmonic extensions $\boldsymbol{w}_i=\{R_i(\theta), i=1,2\}$ be defined by

$$\begin{aligned} \mathrm{grad}(\mathrm{div}\,\boldsymbol{w}_i)+\beta\boldsymbol{w}_i&=0 \quad \text{in} \quad \Omega_i,\\ \boldsymbol{w}_i\cdot\boldsymbol{n}&=0 \quad \text{on} \quad \Gamma_i,\\ \boldsymbol{w}_i\cdot\boldsymbol{n}_1&=\theta \quad \text{on} \quad \Gamma. \end{aligned} \qquad (6)$$

Then we can define the Steklov-Poincaré operator S_i by

$$S_i(\theta)=(-1)^{i+1}\mathrm{div}\boldsymbol{w}_i, \quad \text{on} \quad \Gamma, \qquad (7)$$

where $\boldsymbol{w}_i$ are the solutions of(6) for $i=1,2$, respectively. We may note that the Steklov-Poincaré oprators $S_i(i=1,2)$ and their inverses are bounded, and they are self-adjiont definite. Thus, the interface conditions(3) and(4) are equivalent to

$$S_1(\lambda)+S_2(\lambda)=\mathrm{div}\hat{\boldsymbol{w}}_2-\mathrm{div}\hat{\boldsymbol{w}}_1 \quad \text{on} \quad \Gamma, \qquad (8)$$

where $\hat{\boldsymbol{w}}_i \in H_0(\mathrm{div}, \Omega_i)$ are the solutions of the following equations

$$\begin{aligned} -\mathrm{grad}(\mathrm{div}\hat{\boldsymbol{w}}_i) + \beta\hat{\boldsymbol{w}}_i = f \quad &\text{in} \quad \Omega_i, \\ \hat{\boldsymbol{w}}_i \cdot \boldsymbol{n} = 0 \quad &\text{on} \quad \Gamma_i, \\ \hat{\boldsymbol{w}}_i \cdot \boldsymbol{n}_1 = 0 \quad &\text{on} \quad \Gamma, \end{aligned} \tag{9}$$

for $i = 1, 2$, respectively.

From [6] we know that we may construct optimization-based domain decomposition method via Neumann control or Dirichlet control. In this paper, we restrict our interest in Neumann control, and try to find a g that minimizes the suitable norm of $\boldsymbol{u}_1 \cdot \boldsymbol{n}_1 - \boldsymbol{u}_2 \cdot \boldsymbol{n}_1$ on the common interface Γ. Hence, letting X, Y be two Hilbert spaces, we have

$$\begin{gathered} \mathrm{Min} \qquad J(g) = \| \boldsymbol{u}_1 \cdot \boldsymbol{n}_1 - \boldsymbol{u}_2 \cdot \boldsymbol{n}_1 \|_X^2, \\ \text{over } g \in Y \text{ and } \boldsymbol{u}_1, \boldsymbol{u}_2 \text{ are subject to (2) and (4)}. \end{gathered} \tag{10}$$

To solve the minimization problem (10), we apply the following gradient type iterative method. Let J' be the Fréchet derivative, i. e., $(J'(u), v)H = (\mathrm{d}J(u + tv) = \mathrm{d}t)|_{t=0}$. Then the gradient iteration is given by choosing some approximate sequence $\alpha_k > 0$ and setting

$$g_{k+1} = g_k - \alpha_k J'(g_k). \tag{11}$$

For convenience, we define operators P and Q by

$$(u, Pv)_\Gamma = (u, v)_X, \quad \forall u, v \in X, \tag{12}$$

$$(u, Qv)_\Gamma = (u, v)_Y, \quad \forall u, v \in Y, \tag{13}$$

where $(\cdot, \cdot)_\Gamma$ denotes the standard inner product in $L^2(\Gamma)$. Through direct calculation, we can obtain the Fréchet derivative by

$$(J'(g), Qv)_\Gamma = (J'(g), v)_Y = (\boldsymbol{u}_1 \cdot \boldsymbol{n}_1 - \boldsymbol{u}_2 \cdot \boldsymbol{n}_1, P(S_1^{-1} + S_2^{-1})v)_\Gamma, \quad \forall g, v \in Y.$$

Thus, viewing $\boldsymbol{u}_1 \cdot \boldsymbol{n}_1$ and $\boldsymbol{u}_2 \cdot \boldsymbol{n}_1$ as functions of g, we have on Γ,

$$J'(g) = Q^{-1}(S_1^{-1} + S_2^{-1})P(\boldsymbol{u}_1 \cdot \boldsymbol{n}_1 - \boldsymbol{u}_2 \cdot \boldsymbol{n}_1).$$

Then, we can apply gradient-type iteration (11) to solve minimization problem (10).

Theorem 1 Given $\alpha_k > \theta > 0$, if the iteration (11) is convergent and

$$\boldsymbol{u}|_{\Omega_i} = \lim_{k \to \infty} \boldsymbol{u}_i^k \quad \text{for} \quad i = 1, 2,$$

then $\boldsymbol{u}$ is the solution of (1) on the whole domain Ω.

Let $e_k = g_k - \mathrm{div}\boldsymbol{u}$, then we have

$$e_{k+1} = [I - \alpha_k Q^{-1}(S_1^{-1} + S_2^{-1})P(S_1^{-1} + S_2^{-1})]e_k = E(\alpha_k)e_k. \tag{14}$$

Therefore, we obtain the following convergent result.

Theorem 2 The iterative procedure (11) is convergent if and only if, for any e_0,

$$\left\| \prod_{i=1}^{k} E(\alpha_k)e_0 \right\| \to 0 \quad \text{as} \quad k \to \infty \tag{15}$$

in some suitable norm and with some suitably chosen sequence $\{\alpha_k\}$.

In the next subsections, we are going to provide elaborate algorithms based on different examples of P and Q.

1.1 Algorithm 1

In this subsection we consider the case $X = Y = L^2(\Gamma)$. In this case, $P = Q = I$, the iteration is given by

$$g_{k+1} = g_k - \alpha_k(S_1^{-1} + S_2^{-1})(\boldsymbol{u}_1^k \cdot \boldsymbol{n}_1 - \boldsymbol{u}_2^k \cdot \boldsymbol{n}_1) \quad \text{on} \quad \Gamma. \tag{16}$$

Let $H_{\Gamma_i}(\mathrm{div}; \Omega_i)$ represent a subspace of $H(\mathrm{div}; \Omega_i)$ whose normal component vanish on Γ_i. Then the corresponding algorithm can be implemented in the following three steps.

(i) Find $\boldsymbol{u}_i^k \in H_{\Gamma_i}(\mathrm{div}; \Omega_i)$ $(i = 1, 2)$ from

$$\begin{aligned} -\mathrm{grad}(\mathrm{div}\boldsymbol{u}_i^k) + \beta\boldsymbol{u}_i^k = \boldsymbol{f} \quad &\text{in} \quad \Omega_i, \\ \boldsymbol{u}_i^k \cdot \boldsymbol{n} = 0 \quad &\text{on} \quad \Gamma_i, \\ \mathrm{div}\boldsymbol{u}_i^k = g_k \quad &\text{on} \quad \Gamma; \end{aligned} \tag{17}$$

(ii) Find $\boldsymbol{w}_i^k \in H_{\Gamma_i}(\mathrm{div};\Omega_i)$ $(i=1,2)$ from

$$\begin{aligned} -\mathrm{grad}(\mathrm{div}\boldsymbol{w}_i^k)+\beta\boldsymbol{w}_i^k=0 &\quad \text{in} \quad \Omega_i, \\ \boldsymbol{w}_i^k\cdot\boldsymbol{n}=0 &\quad \text{on} \quad \Gamma_i, \\ \mathrm{div}\boldsymbol{w}_i^k=(-1)^{i+1}(\boldsymbol{u}_1^k\cdot\boldsymbol{n}_1-\boldsymbol{u}_2^k\cdot\boldsymbol{n}_1) &\quad \text{on} \quad \Gamma; \end{aligned} \tag{18}$$

(iii) Update the iteration data

$$g_{k+1}=g_k-\alpha_k(\boldsymbol{w}_1^k\cdot\boldsymbol{n}_1+\boldsymbol{w}_2^k\cdot\boldsymbol{n}_1) \quad \text{on} \quad \Gamma. \tag{19}$$

To consider the convergence of iterative procedure(16), we know that the error equation is

$$e_{k+1}=[I-\alpha_k(S_1^{-1}+S_2^{-1})^2]e_k.$$

Thus, if we choose α_k satisfy the equation

$$-1<\sup_{v\in L^2(\Gamma)}\frac{[I-\alpha_k(S_1^{-1}+S_2^{-1})^2),v,v]_\Gamma}{(v,v)_\Gamma}<1,$$

then the iterative sequence(16) is convergent. In particular, if we take

$$\tilde{\beta}>\sup_{v\in L^2(\Gamma)}\frac{\|(S_1^{-1}+S_2^{-1})v\|_\Gamma}{\|v\|_\Gamma},$$

and choose

$$0<\theta\leqslant\alpha_k\leqslant\frac{2}{\tilde{\beta}^2}, \tag{20}$$

then the iterative procedure(16) is convergent. Therefore, we have the following result.

Theorem 3 The iterative procedure(16) is convergent for any e_0 and for any sequence α_k satisfying(20).

1.2 Algorithm 2

Here we consider the case $X=H^{1/2}(\Gamma)$ and $Y=L^2(\Gamma)$. We may have some choices for the norms, the details can refer to [5,6]. For example, we can take that $P=(S_1^{-1}+S_2^{-1})^{-1}$ and $Q=I$, then the iteration is given by

$$g_{k+1}=g_k-\alpha_k(\boldsymbol{u}_1^k\cdot\boldsymbol{n}_1-\boldsymbol{u}_2^k\cdot\boldsymbol{n}_1) \quad \text{on} \quad \Gamma, \tag{21}$$

where $\boldsymbol{u}_1^k$ and $\boldsymbol{u}_2^k$ are the solutions of equations(17).

For convergence, we have the error equation

$$e_{k+1}=[I-\alpha_k(S_1^{-1}+S_2^{-1})]e_k.$$

Using the similar analysis used in the above subsection, if we take

$$\tilde{\beta}>\sup_{v\in L^2(\Gamma)}\frac{((S_1^{-1}+S_2^{-1})v,v)_\Gamma}{\|v\|_\Gamma^2}$$

and choose

$$0<\theta\leqslant\alpha_k\leqslant\frac{2}{\hat{\beta}}, \tag{22}$$

then we have the following result.

Theorem 4 The iterative procedure(21) is convergent for any e_0 and for any sequence α_k satisfying(22).

1.3 Algorithm 3

Here we consider the case $X=H^{1/2}(\Gamma)$ and $Y\in H^{-1/2}(\Gamma)$. We may have some choices for the norms, the details we can refer to [5,6]. For example, we can take that $P=S_2$ and $Q=S_1^{-1}$, then the iteration is given by

$$g_{k+1}=g_k-\alpha_k(S_1+S_2)(\boldsymbol{u}_1^k\cdot\boldsymbol{n}_1-\boldsymbol{u}_2^k\cdot\boldsymbol{n}_1) \quad \text{on} \quad \Gamma. \tag{23}$$

The corresponding algorithm can be implemented in the following three steps.

(i) Find $\boldsymbol{u}_i^k \in H_{\Gamma_i}(\mathrm{div};\Omega_i)$ $(i=1,2)$ from

$$\begin{aligned} -\mathrm{grad}(\mathrm{div}\boldsymbol{u}_i^k)+\beta\boldsymbol{u}_i^k=\boldsymbol{f} &\quad \text{in} \quad \Omega_i, \\ \boldsymbol{u}_i^k\cdot\boldsymbol{n}=0 &\quad \text{on} \quad \Gamma_i, \\ \mathrm{div}\boldsymbol{u}_i^k=g_k &\quad \text{on} \quad \Gamma; \end{aligned} \tag{24}$$

(ii) Find $w_i^k \in H_{\Gamma_i}(\mathrm{div};\Omega_i)$ ($i=1,2$) from

$$\begin{aligned} -\mathrm{grad}(\mathrm{div}\boldsymbol{w}_i^k) + \beta \boldsymbol{w}_i^k &= 0 \quad \text{in} \quad \Omega_i, \\ \boldsymbol{w}_i^k \cdot \boldsymbol{n} &= 0 \quad \text{on} \quad \Gamma_i, \\ \boldsymbol{w}_i^k \cdot \boldsymbol{n}_1 &= (\boldsymbol{u}_1^k \cdot \boldsymbol{n}_1 - \boldsymbol{u}_2^k \cdot \boldsymbol{n}_1) \quad \text{on} \quad \Gamma; \end{aligned} \tag{25}$$

(iii) Update the iteration data

$$g_{k+1} = g_k - \alpha_k(\mathrm{div}\boldsymbol{w}_1^k - \mathrm{div}\boldsymbol{w}_2^k) \quad \text{on} \quad \Gamma. \tag{26}$$

For convergence, we have the error equation

$$e_{k+1} = [I - \alpha_k(S_1^{-1} + S_2^{-1})(S_1 + S_2)] e_k.$$

From the result of well-known Dirichlet-Neumann alternating methods, we have the following geometric convergence result.

Theorem 5 There exists two constants c and C such that for any e_0 and for any sequence $\{\alpha_k\}$ satisfying

$$0 < c \leqslant \alpha_k \leqslant C,$$

such that the iterative sequence(23) is geometrically convergent.

2 Numerical Experiments

In this section, we report some numerical experiments. The test problem is a two-dimensional equations of (1) in $\Omega = \{(x,y):(0<x<2, 0<y<1)\}$, where the coefficient β is set to be 1. We set the right-hand side function so that the exact solution is given by

$$\boldsymbol{u}(x,y) = \begin{pmatrix} \boldsymbol{u}_1(x,y) \\ \boldsymbol{u}_2(x,y) \end{pmatrix} = \begin{pmatrix} x(2-x) \\ y(1-y) \end{pmatrix}.$$

Our numerical experiments are performed using MATLAB. The machine is a PC-Intlel(R), Pentium(R) Dual-Core CPU E5300 2.60 GHz, 1.96 G of RAM. Ω is divided into two parts $\Omega_1 = \{(x,y): 0<x<1; 0<y<1\}$ and $\Omega_1 = \{(x,y): 1<x<2; 0<y<1\}$ with the interface $\Gamma = \{(x,y): x=1, 0<y<1\}$. The finite element tested here are the lowest rectangular Raviart-Thomas element. We set the initial value $g_0 = 0$ in each iterative procedure. The iteration is stopped when $\text{Error} = \sum_{i=1}^{2} (\| \boldsymbol{u}_i^k - \boldsymbol{u}_i \|_{0;\Omega_i}^2)^{1/2} \leqslant 5 \times 10^{-2}$.

Numerical results are provided in Table 1 - 3, which shows iteration numbers for different parameter $\{\alpha_k\}$ for algorithms 1 - 3. The iteration number is denoted by Iter.

Table 1 Numerical results for algorithm 1

h	α	Iter	Error	h	α	Iter	Error
$\frac{1}{8}$	$7\cdot10^{-5}$	1 100	0.049 989	$\frac{1}{16}$	$7\cdot10^{-5}$	1 107	0.049 988
	$7\cdot10^{-4}$	759	0.049 050		$5\cdot10^{-5}$	1 549	0.049 952
	$3\cdot10^{-4}$	257	0.049 430		$1\cdot10^{-4}$	774	0.049 987
	$1\cdot10^{-3}$	77	0.048 592		$2\cdot10^{-4}$	387	0.049 840
	$1.1\cdot10^{-3}$	70	0.048 438		$3\cdot10^{-4}$	258	0.049 691
	$1.3\cdot10^{-3}$	59	0.048 788		$4\cdot10^{-4}$	diverges	
	$1.4\cdot10^{-3}$	diverges					

Table 2 Numerical results for algorithm 2

h	α	Iter	Error	h	α	Iter	Error
$\frac{1}{8}$	$1\cdot10^{-3}$	506	0.049 802	$\frac{1}{16}$	$1\cdot10^{3}$	509	0.049 856
	$5\cdot10^{-3}$	101	0.049 227		$5\cdot10^{-3}$	102	0.048 614
	$7\cdot10^{-3}$	72	0.049 097		$7\cdot10^{-3}$	73	0.047 822
	$1\cdot10^{-2}$	50	0.049 746		$1\cdot10^{-2}$	51	0.047 434
	$3\cdot10^{-2}$	17	0.041 269		$2\cdot10^{-4}$	25	0.048 248
	$5\cdot10^{-2}$	10	0.039 967		$3\cdot10^{-4}$	diverges	
	$5.5\cdot10^{-2}$	diverges					

Table 3 Numerical results for algorithm 3

h	α	Iter	Error	h	α	Iter	Error
	$1\cdot10^{-3}$	253	0.049 577		$1\cdot10^{3}$	254	0.049 962
	$3\cdot10^{-3}$	84	0.049 332		$3\cdot10^{-3}$	85	0.048 382
	$5\cdot10^{-3}$	50	0.049 746		$5\cdot10^{-3}$	51	0.047 434
$\frac{1}{8}$	$1\cdot10^{-2}$	25	0.047 217	$\frac{1}{16}$	$1\cdot10^{-2}$	25	0.048 248
	$1.5\cdot10^{-2}$	17	0.041 269		$1.2\cdot10^{-2}$	21	0.045 830
	$2\cdot10^{-2}$	12	0.048 547		$1.3\cdot10^{-2}$	19	0.049 322
	$2.7\cdot10^{-2}$	diverges			$1.4\cdot10^{-2}$	diverges	

3 Conclusions

In this paper, three optimization-based domain decomposition methods are presented to solve $H(\mathrm{div})$-elliptic problem, and their convergent properties are obtained by choosing proper parameters. From the theory analysis, we conclude that when choosing the bigger parameter, the algorithms will convergence more fast. However, choosing too big parameter may result in the divergence of algorithms. These observations are verified by the numerical experiments, which are provided in Table 1-3. From the Table 1-3, we observe that the computational complexity of algorithm 2 is as much as that of algorithm 3, but they are both much lower than that of algorithm 1.

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