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# Bogdanov-Takens Bifurcation Analysis of an Epidemic Model

Wan Hui[1], Li Yongfeng[2]

(1. Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China)
(2. Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou 450000, China)

**Abstract:** In this paper, a SIR epidemic model is proposed to understand the impact of limited medical resource on infectious disease transmission, and Bogdanov-Takens bifurcation is analyzed. Our results suggest that the model may exhibit vital dynamics when the basic reproduction number $\mathscr{R}_0$ is equal to a subthreshold value and the unique equilibrium is a cusp of codimension 2.

**Key words:** epidemic model, medical resource, Bogdanov-Takens bifurcation, codimension 2

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## 一个传染病模型的 Bogdanov-Takens 分支分析

万　辉，李永凤

(1. 江苏省大规模复杂系统数值模拟重点实验室，南京师范大学数学科学学院，江苏 南京 210023)
(2. 郑州轻工业大学数学与信息科学系，河南 郑州 450000)

[摘要]　为了研究有限的医疗资源对传染病传播影响，本文考虑了一个 SIR 传染病模型，并着重分析了模型的 Bogdanov-Takens 分支问题. 结果表明，当基本再生数等于一个子阈值时，模型将出现非常复杂的分支现象. 此时，模型惟一的平衡点为余维 2 尖点.

[关键词]　传染病模型，医疗资源，Bogdanov-Takens 分支，余维 2

In recent years, a lot of realistic mathematical models have been proposed [1-7]. The development of such models aims at understanding the epidemiological transmission patterns and predicting the consequences of the introduction of public health interventions to control the possible outbreak and spread of the disease. Besides, some interesting dynamical behaviors, like Hopf bifurcation, bi-stability, backward bifurcation, etc., have been observed in these models.

In this paper, we will focus on the Bogdanov-Takens bifurcation of an epidemic model considering the impact of limited medical resource or treatment capacity on infectious disease transmission.

The organization of this paper is as follows. In section 1, we introduce the SIR epidemic model briefly. In Section 2, Bogdanov-Takens bifurcation is presented. Section 3 is the discussion.

## 1　The Model

Let $S(t)$, $I(t)$ and $R(t)$ denote the number of susceptible, infective, recovered individuals at time $t$, respectively. Providing that the infected individuals can not recover unless they were given timely treatment in hospitals, based on standard SIR model with mass action incidence, we can construct a model

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**Corresponding author:** Wan Hui, lecture, majored in mathematical biology. E-mail: wanh2046@163.com

$$\begin{cases}\dfrac{\mathrm{d}S}{\mathrm{d}t}=A-\mathrm{d}S-\beta SI,\\ \dfrac{\mathrm{d}I}{\mathrm{d}t}=\beta SI-(d+v)I-\dfrac{cI}{b+I},\\ \dfrac{\mathrm{d}r}{\mathrm{d}t}=\dfrac{cI}{b+I}-\mathrm{d}R,\end{cases} \tag{1}$$

where all the parameters are positive, and $A$ is the recruitment rate of susceptible population; $d$ is natural death rate; $v$ is the disease-induced death rate; $c$ is the maximum of treatment per unit of time, and $b$, the infected size at which is 50% saturation($h(b)=c/2$), measures how soon saturation occurs; $\beta$ is the transmission rate.

Note that the first two equations are independent of the third one, we need only to study the following reduced model:

$$\begin{cases}\dfrac{\mathrm{d}S}{\mathrm{d}t}=A-\mathrm{d}S-\beta SI,\\ \dfrac{\mathrm{d}I}{\mathrm{d}t}=\beta SI-(d+v)I-\dfrac{cI}{b+I},\end{cases} \tag{2}$$

Model(2) has one disease free equilibrium at $E_0=\left(\dfrac{A}{d},0\right)$. Using the formulae in [8], a straightforward calculation gives the reproduction number:

$$\mathscr{R}_0=\frac{Ab\beta}{d(bd+bv+c)}. \tag{3}$$

The disease-free equilibrium $E_0$ has two eigenvalues $-d$ and $R_0-1$. Therefore we have the following proposition:

**Proposition 1** For the model(2), the disease free equilibrium $E_0$ is locally asymptotically stable if $\mathscr{R}_0<1$, and unstable if $\mathscr{R}_0>1$.

Let

$$\begin{aligned}&a_1=\frac{(b\beta+d)(d+v)+c\beta-A\beta}{\beta(d+v)},\\ &a_2=\frac{bd(d+v)+cd-\beta Ab}{\beta(d+v)}=\frac{bd(d+v)+cd}{\beta(d+v)}(1-\mathscr{R}_0),\\ &\Delta=a_1^2-4a_2.\end{aligned} \tag{4}$$

For the existence of endemic equilibrium, we have the following theorem:

**Theorem 1** For model(2), we have

1. If $\mathscr{R}_0>1$, there exists a unique positive equilibrium $E^*(S^*,I^*)$.

2. If $\mathscr{R}_0=1$, there is a positive equilibrium $E^*(S^*,I^*)$ when $a_1<0$, otherwise there is no positive equilibrium.

3. If $\mathscr{R}_0<1$ and $a_1\geqslant 0$, there is no positive equilibrium.

4. If $\hat{\mathscr{R}}_0<\mathscr{R}_0<1$ and $a_1<0$, there are two positive equilibria $E^*$ and $E_*$.

5. If $\mathscr{R}_0=\hat{\mathscr{R}}_0$ and $a_1<0$, $E^*$ and $E_*$ coalesce together as a unique equilibrium of multiplicity two.

6. If $\mathscr{R}_0<\hat{\mathscr{R}}_0$ and $a_1<0$, there is no positive equilibrium.

Where when exist, $E^*(S^*;I^*)$ and $E_*(S_*,I_*)$ are the corresponding equilibria, and

$$S^*=h(I^*),S_*=h(I_*),I^*=\frac{-a_1+\sqrt{\Delta}}{2},I_*=\frac{-a_1-\sqrt{\Delta}}{2} \tag{5}$$

and

$$\hat{\mathscr{R}}_0(c)=\frac{Ab(d+v)[b(d+v)+(\sqrt{A}+\sqrt{c})^2]}{(c+b(d+v))[b(d+v)(b(d+v)+2(A+c))+(A-c)^2]}. \tag{6}$$

**Proof** Let the right hand side of(2) be zero and eliminate $S$. Then, we have the following equation:

$$I^2 + a_1 I + a_2 = 0. \tag{7}$$

If $\mathscr{R}_0 > 1$, then $a_2 < 0$ and there is a unique positive root for (7) which implies a unique endemic equilibrium $E^*(S^*, I^*)$ exists.

If $\mathscr{R}_0 = 1$, then $a_2 = 0$ and there is a unique nonzero solution of (7) $I = -a_1$ which is positive if and only if $a_1 < 0$. Then, there is a unique endemic equilibrium $E^*(S^*, I^*)$ when $a_1 < 0$, and there is not endemic equilibria when $a_1 \geqslant 0$.

If $\mathscr{R}_0 < 1$, then $a_2 > 0$. For (7) to have at least one positive root we must have

$$a_1 < 0 \text{ and } \Delta \geqslant 0.$$

Solving $\Delta = 0$ in terms of $\mathscr{R}_0$, one get $\mathscr{R}_0 = \hat{\mathscr{R}}_0$ where

$$\hat{\mathscr{R}}_0(c) = \frac{Ab(d+v)\left[b(d+v) + (\sqrt{A} + \sqrt{c})^2\right]}{(c + b(d+v))\left[b(d+v)(b(d+v) + 2(A+c)) + (A-c)^2\right]}. \tag{8}$$

One can verify that, providing $a_1 < 0$, model (2) has exactly 0, 1 and 2 endemic equilibria for $\mathscr{R}_0 < \hat{\mathscr{R}}_0$, $\mathscr{R}_0 = \hat{\mathscr{R}}_0$, $\mathscr{R}_0 > \hat{\mathscr{R}}_0$, respectively.

## 2 Bogdanov-Takens Bifurcation Analysis

In this section, we focus on the Bogdanov-Takens bifurcation when $\mathscr{R}_0 = \hat{\mathscr{R}}_0(c)$. Evaluating the Jacobian of the model (2) at $E^*(S^*; I^*)$ gives

$$\boldsymbol{J} = \begin{pmatrix} -d - \beta I^* & -\beta S^* \\ \beta I^* & \dfrac{cI^*}{(b+I^*)^2} \end{pmatrix} \tag{9}$$

Then the characteristic equation about $E^*$ is given by

$$\lambda^2 + H(I^*, c)\lambda + I^* G(I^*, c) = 0, \tag{10}$$

where

$$H(I^*, c) = d + \beta I^* - \frac{cI^*}{(b+I^*)^2}, G(I, c) = \frac{A\beta^2}{d + \beta I^*} - \frac{c(d + \beta I^*)}{(b+I^*)^2}. \tag{11}$$

It follows from Theorem 1 that when $\mathscr{R}_0 = \hat{\mathscr{R}}_0(c)$ and $a_1 < 0$, two equilibria $E_*$ and $E^*$ coalesce at the equilibrium $E^*(S^*, I^*)$, where

$$I^* = \frac{A\beta - c\beta - (b\beta + d)(d+v)}{2\beta(d+v)}, S^* = \frac{A}{d + \beta I^*}. \tag{12}$$

Substituting (12) into which in (11), note that $H(I^*; c) = G(I^*; c) = 0$ is equivalent to

$$\begin{cases} A = A^* = \dfrac{d^2(d+v)^3}{\beta(\beta b + v)^2(d - \beta b)}, \\ c = C^* = \dfrac{(d+v)(\beta^2 b^2 - d\beta b + v\beta b + d^2)^2}{\beta(\beta b + v)^2(d - \beta b)} \end{cases} \tag{13}$$

with $d > \beta b$. One can easily get that $C^* > c^*$ where

$$c^* = \frac{b^2\beta(d+v)}{d - b\beta}. \tag{14}$$

Thus (9) has a zero eigenvalue with multiplicity 2 under the assumption (13). This suggest that (2) may admit a Bogdanov-Takens bifurcation. We confirm this by giving the following theorem.

**Theorem 2** Suppose $A = A^*$, $c = C^*$ and $d \geqslant 2\beta b + v$. Then the equilibrium $E^*$ of system (2) is a cusp of codimension 2 (a Bogdanov-Takens bifurcation point).

**Proof** Suppose $A = A^*$, $c = C^*$. (12) becomes

$$I^* = \frac{d(d - \beta b)}{\beta(v + \beta b)}, S^* = \frac{d(d+v)^2}{\beta(v + \beta b)(d - \beta b)}. \tag{15}$$

The translation

$$x = S - S^*, y = I - I^* \tag{16}$$

bring $E^*$ to the origin. Expanding the right hand side of the resulting system in a Taylor series about the origin, we obtain

$$\begin{cases} \dfrac{dx}{dt} = -(d + \beta I^*)x - \beta S^* y - \beta xy, \\ \dfrac{dx}{dt} = \beta I^* x + \dfrac{cI^*}{(b + I^*)^2} y + \beta xy + \dfrac{bc}{(b + I^*)^3} y^2 + O(y^3). \end{cases} \tag{17}$$

Since the Jacobian $\boldsymbol{J} \neq 0$, there exist real linearly independent vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ such that $\boldsymbol{J}\boldsymbol{x}_1 = 0$ and $\boldsymbol{J}\boldsymbol{x}_2 = \boldsymbol{x}_1$. These vectors are given by

$$\boldsymbol{x}_1 = \left( -\frac{cI^*}{(b + I^*)^2}, \beta I^* \right)^{\mathrm{T}}, \boldsymbol{x}_2 = (1, 0)^{\mathrm{T}}. \tag{18}$$

Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\dfrac{cI^*}{(b + I^*)^2} & 1 \\ \beta I^* & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{19}$$

or

$$X = \frac{y}{\beta I^*}, Y = x + \frac{cy}{\beta(b + I^*)^2}, \tag{20}$$

then(17) becomes

$$\begin{cases} \dfrac{dx}{dt} = Y - \dfrac{c\beta I^{*2}}{(b + I^*)^3} X^2 + \beta XY + R_{10}(X, Y), \\ \dfrac{dY}{dt} = \dfrac{c\beta(\beta b - d) I^{*2}}{(b + I^*)^3} X^2 + d\beta XY + R_{20}(X, Y), \end{cases} \tag{21}$$

where $R_{i0}(i = 1, 2)$ is $C^\infty$ in$(X, Y)$ and $R_{i0} = O(|(X, Y)|^3)$. Making the near-identity transformation

$$X = X, Z = Y - \frac{c\beta I^{*2}}{(b + I^*)^3} X^2 + \beta XY + R_{10}(X, Y), \tag{22}$$

we obtain

$$\begin{cases} \dfrac{dX}{dt} = Z, \\ \dfrac{dZ}{dt} = a_{20} X^2 + a_{11} XZ + \beta Z^2 + R_{21}(X, Z), \end{cases} \tag{23}$$

where $R_{21}$ is $C^\infty$ in$(X, Z)$ and $R_{i0} = O(|(X, Z)|^3)$, and

$$\begin{cases} a_{20} = -\dfrac{\beta(d + v) d^2 (d - \beta b)^2}{(v + \beta b)(\beta^2 b^2 - d\beta b + v\beta b + d^2)}, \\ a_{11} = -\dfrac{d\beta\,[\,(d - \beta b)^2 + d^3 - \beta b(2\beta bd + v^2 + 2\beta bv)\,]}{(v + \beta b)(\beta^2 b^2 - d\beta b + v\beta b + d^2)}. \end{cases} \tag{24}$$

Note $d > 2\beta b + v$, we have $d^3 - \beta b(2\beta bd + v^2 + 2\beta bv) > \beta b(d^2 - 2\beta bd - v^2 - 2\beta bv) > 0$. Hence $a_{20} < 0$, $a_{11} < 0$.

Using a time reparameterization $dt = (1 - \beta X)\, d\tau$, (23) becomes

$$\left\{ \frac{dX}{d\tau} = (1 - \beta X) Z, \frac{dZ}{d\tau} = (1 - \beta X)(a_{20} X^2 + a_{11} XZ + \beta Z^2 + R_{21}(X, Z)). \right. \tag{25}$$

Introducing new variables $u = X$ and $v = (1 - \beta X) Z$, then(25) becomes

$$\begin{cases} \dfrac{du}{d\tau} = v, \\ \dfrac{dv}{d\tau} = a_{20} u^2 + a_{11} uv + O(|(u, v)|^3). \end{cases} \tag{26}$$

Thus $E^*$ is a cusp point of codimension 2 [9].

Next we consider(2) for all $A$ and $c$ with $|A - A^*|$ and $|c - C^*|$ small. Thus we let

$$\begin{cases} A = A^* + \varepsilon_1, \\ c = C^* + \varepsilon_2, \end{cases} \tag{27}$$

in(2) and we study the bifurcations of the resulting system

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = A^* + \varepsilon_1 - dS - \beta SI, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \beta SI - (d+v)I - \dfrac{(C^* + \varepsilon_2)I}{b+I}. \end{cases} \tag{28}$$

Translating $E^*$ to the origin, system(28) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \varepsilon_1 - (d + \beta I^*)x - \beta S^* y - \beta xy, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -\dfrac{\varepsilon_2 I^*}{b+I^*} + \beta I^* x + \dfrac{C^* I^* - b\varepsilon_2}{(b+I^*)^2} y + \beta xy + \dfrac{bC^* + b\varepsilon_2}{(b+I^*)^3} y^2 + O(y^3). \end{cases} \tag{29}$$

Making translation

$$X = x, Y = \varepsilon_1 - (d + \beta I^*)x - \beta S^* y - \beta xy, \tag{30}$$

we obtain

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = Y, \\ \dfrac{\mathrm{d}Y}{\mathrm{d}t} = p_{00} + p_{10}X + p_{01}Y + p_{20}X^2 + p_{11}XY + p_{02}Y^2 + R_{21}(X,Y), \end{cases} \tag{31}$$

where

$$p_{00} = \frac{-C^* \beta S^* I^* (b+I^*)\varepsilon_1 + b\beta S^* (b+I^*)\varepsilon_1\varepsilon_2 - b(C^* + \varepsilon_2)\varepsilon_1^2 + \beta^2 S^{*2} I^* (b+I^*)^2 \varepsilon_2}{\beta S^* (b+I^*)^3}$$

$$p_{10} = \frac{1}{\beta S^{*2}(b+I^*)^3}\left[2bdS^* \varepsilon_1\varepsilon_2 - b\beta S^{*2}(b+I^*)(d+\beta I^*)\varepsilon_2 + b(C^* + \varepsilon_2)\varepsilon_1^2 + 2b\beta S^* I^* \varepsilon_1\varepsilon_2 + \beta^2 S^{*2} I^* (b+I^*)^2\varepsilon_2 - \beta^2 S^{*2}(b+I^*)^3\varepsilon_1 + 2bdS^*\varepsilon_1 + 2bC^*\beta S^* I^* \varepsilon_1\right],$$

$$p_{01} = \frac{2b\varepsilon_1\varepsilon_2 - b\beta S^*(b+I^*)\varepsilon_2 + 2bC^*\varepsilon_1 - \beta(b+I^*)^3\varepsilon_1}{\beta S^*(b+I^*)^3},$$

$$p_{20} = \frac{d\beta(d-b\beta)^2}{\beta^2 b^2 - d\beta b + v\beta b + d^2} + p_{20\varepsilon},$$

$$p_{20\varepsilon} = -\frac{1}{\beta S^{*3}(b+I^*)^3}\left[2bS^*(d+\beta I^*)\varepsilon_1\varepsilon_2 + b(C^* + \varepsilon_2)\varepsilon_1^2 + 2bC^* S^*(d+\beta I^*)\varepsilon_1 + bS^{*2}(d+\beta I^*)^2\varepsilon_2\right],$$

$$p_{02} = \frac{\beta(v+\beta b)(d(d-\beta b)^2 - \beta b(\beta b + v)^2)}{d(d+v)^2(\beta^2 b^2 - d\beta b + v\beta b + d^2)} - \frac{b\varepsilon_2}{\beta S^*(b+I^*)^3},$$

$$p_{11} = \frac{\beta\left[(d-\beta b)^3 + dv(d-\beta b) + d^3 - \beta b(2\beta bd + v^2 + 2\beta bv)\right]}{(\beta^2 b^2 - d\beta b + v\beta b + d^2)(d+v)} + p_{11\varepsilon},$$

$$p_{11\varepsilon} = -\frac{1}{\beta S^{*2}(b+I^*)^3}\left[2b(C^* + \varepsilon_1)\varepsilon_2 + 2bS^*(d+\beta I^*)\varepsilon_2 - \beta(b+I^*)^3\varepsilon_1\right],$$

and $R_{21}$ is $C^\infty$ in$(X,Y)$ and $R_{21} = O(|(X,Y)|^3)$.

Introduce a new time variable $\tau$ by $\mathrm{d}t = (1 - p_{02}X)\,\mathrm{d}\tau$. Rewriting $\tau$ as $t$, we have

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = (1 - p_{02}X)Y, \\ \dfrac{\mathrm{d}Y}{\mathrm{d}t} = (1 - p_{02}X)(p_{00} + p_{10}X + p_{01}Y + p_{20}X^2 + p_{11}XY + p_{02}Y^2 + R_{21}(X,Y)). \end{cases} \tag{32}$$

Let $u = X$ and $v = (1 - p_{02}X)Y$, then(32) becomes

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = p_{00} + (p_{10} - 2p_{00}p_{02})u + p_{01}v + (p_{20} + p_{00}p_{02}^2 - 2p_{10}p_{02})u^2 + (p_{11} - p_{01}p_{02})uv + R_{22}(u,v). \end{cases} \tag{33}$$

$R_{22}$ is $C^{\infty}$ in$(u,v)$ and $R_{22}=O(|(u,v)|^3)$.

Because

$$\lim_{\varepsilon_1\to 0,\varepsilon_2\to 0} p_{11}=\frac{\beta[(d-\beta b)^3+dv(d-\beta b)+d^3-\beta b(2\beta bd+v^2+2\beta bv)]}{(\beta^2b^2-d\beta b+v\beta b+d^2)(d+v)},$$

We have $d^3-\beta b(2\beta bd+v^2+2\beta bv)>\beta b(d^2-2\beta bd-v^2-2\beta bv)>0$ when $d>2\beta b+v$.
Hence

$$\lim_{\varepsilon_1\to 0,\varepsilon_2\to 0}(p_{11}-p_{01}p_{02})>0.$$

By setting $u=w-\dfrac{p_{01}}{p_{11}-p_{01}p_{02}}$, rewriting $w$ as $u$, we have

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t}=v,\\ \dfrac{\mathrm{d}v}{\mathrm{d}t}=q_{00}+q_{10}u+q_{20}u^2+q_{11}uv+R_{23}(u,v),\end{cases}\tag{34}$$

where $R_{23}$ is $C^{\infty}$ in$(u,v)$ and $R_{23}=O(|(u,v)|^3)$, and

$$\begin{aligned}&q_{00}=p_{00}-\frac{p_{01}(p_{10}-2p_{00}p_{02})}{p_{11}-p_{01}p_{02}}+\frac{p_{01}^2(p_{20}+p_{00}p_{02}^2-2p_{10}p_{02})}{(p_{11}-p_{01}p_{02})^2},\\ &q_{10}=p_{10}-2p_{00}p_{02}-\frac{2p_{01}(p_{20}+p_{00}p_{02}^2-2p_{10}p_{02})}{p_{11}-p_{01}p_{02}},\\ &q_{20}=p_{20}p_{00}p_{02}^2-2p_{10}p_{02},\\ &q_{11}=p_{11}-p_{01}p_{02}.\end{aligned}\tag{35}$$

By(31) and(35) we have

$$\lim_{\varepsilon_1\to 0,\varepsilon_2\to 0} q_{20}=\lim_{\varepsilon_1\to 0,\varepsilon_2\to 0} p_{20}=\frac{d\beta(d-b\beta)^2}{\beta^2b^2-d\beta b+v\beta b+d^2}>0.$$

and

$$\lim_{\varepsilon_1\to 0,\varepsilon_2\to 0} q_{11}=\lim_{\varepsilon_1\to 0,\varepsilon_2\to 0} p_{11}>0.$$

By making the change of variables $\xi=\dfrac{q_{11}^2u}{q_{20}}$, $\eta=\dfrac{q_{11}^3v}{q_{20}^2}$ and $t=\dfrac{q_{20}}{q_{11}}\tau$ in a small neighborhood of the origin and renaming$(\xi,\eta)$ and $\tau$ as$(u,v)$ and $t$ respectively, we have

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t}=v,\\ \dfrac{\mathrm{d}v}{\mathrm{d}t}=\dfrac{q_{00}q_{11}^4}{q_{20}^3}+\dfrac{q_{10}q_{11}^2}{q_{20}^2}v+u^2+uv+R_{24}(u,v),\end{cases}\tag{36}$$

where $R_{24}$ is $C^{\infty}$ in$(u,v)$ and $R_{24}=O(|(u,v)|^3)$.

Using the theorems in [9] or [10], we obtain the following local representations of the bifurcation curves in a small neighborhood of the origin.

**Theorem 3** Suppose $d\geqslant 2\beta b+v$ at the Bogdanov point. Then the model(2) has the following bifurcation behavior in a small neighborhood of $E^*$:

(1) there is a saddle-node bifurcation curve

$$SN=\{(\varepsilon_1,\varepsilon_2):4q_{20}q_{00}=q_{10}^2+o(|(\varepsilon_1,\varepsilon_2)|^2)\};$$

(2) there is a Hopf bifurcation curve

$$H=\{(\varepsilon_1,\varepsilon_2):q_{00}^2=o(|(\varepsilon_1,\varepsilon_2)|^2),q_{10}<0\};$$

(3) there is a homoclinic bifurcation curve

$$HL=\{(\varepsilon_1,\varepsilon_2):25q_{20}q_{00}+6q_{10}^2=o(|(\varepsilon_1,\varepsilon_2)|^2)\}.$$

## 3 Discussion

In this paper, for a SIR epidemic model with saturation recovery to understand the impact of limited medical resource on infectious disease transmission, existence of equilibria were introduced under different conditions and Bogdanov-Takens bifurcation was analyzed. Our results suggest that the model may exhibit vital dynamics when the basic reproduction number $\mathscr{R}_0$ equal to a subthreshold value $\hat{\mathscr{R}}_0(c)$ and the unique equilibrium is a cusp of codimension 2.

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