具有一对共轭复不变直线的三次系统的中心判定问题

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[摘要] 对于一类具有一对共轭复不变直线和中心-焦点型奇点的三次系统,证明它以原点为中心的充要条件是其前五阶焦点量全为零. 此中心条件是通过不变代数曲线构造积分因子或对称原理得以证明.

[关键词] 三次微分系统,中心条件,积分因子,对称原理

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Center Determination Problem for a Class of Cubic System with a Pair of Invariant Conjugate Imaginary Straight Lines

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Abstract: A class of cubic systems with a pair of invariant conjugate imaginary straight lines and a center-focus type singular point, is proved to have a center at the origin if and only if the first five focal values vanish. The presence of a center at the origin is proved by constructing integrating factor formed from invariant algebraic curves or by symmetry principle.

Key words: cubic differential systems, center conditions, integrating factor, symmetry principle

1 研究背景

当线性孤立奇点是中心时,其非线性项的影响可使相图是非退化中心或是稳定焦点或不稳定焦点,这类判定问题称为中心焦点判定. 中心焦点问题是常微分方程定性理论中的经典难题,它对 Arnold 问题、Hilbert 第十六问题后半部分的解决具有重要意义. Bautin N N 完整解决了二次系统的中心焦点判定问题;Sibirskii K S 解决了缺少二次项的三次系统的中心判定问题;Sadovskii A P[1] 利用 Cherkas 方法解决了一类可约化为 Liénard 系统的三次系统的中心判定问题. 但对于一般三次系统,目前还没有彻底的结论.

对多项式微分系统

$$\begin{cases} \dfrac{dx}{dt}=-y+P_n(x,y), \\ \dfrac{dy}{dt}=x+Q_n(x,y) \end{cases} \tag{1}$$

而言,焦点量是推导中心条件的重要判定量,这里多项式 P_n、Q_n 的次数都不超过 n 且不含常数项和线性项. 一方面,当多项式系统在原点处的各阶焦点量都为零时,系统以该点为中心;另一方面由 Hilbert 有限基定理,所有焦点量生成的有理数域上的多项式理想是有限生成的,因此中心焦点问题可在有限步内解决.

一般来讲,中心焦点问题的最终解决依赖于焦点量的计算,但当计算量过大时,可以通过增加条件的

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方法加以解决. 例如:刘一戎等[2]定义基本李不变量,给出了广义对称原理;Lloyd N 等[3]以 Gröbner 基为工具,寻找双线性变换将一类多项式系统化为时间可逆系统,从而确定中心条件;对于一般三次系统,Cozma D 等[4,5]讨论了 3 条不变直线与中心的共存问题;而文[6]讨论了两条相交直线、1 条二次曲线与中心的共存问题,其中这 3 条不变代数曲线共点.

考虑右端函数为 n 次多项式的二维微分自治系统:

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=X_n(x,y),\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=Y_n(x,y).\end{cases}\tag{2}$$

定义 1[7]　设 $f(x,y)$ 是一个 $m>0$ 次多项式,如果存在多项式 $h(x,y)$,使得

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{(2)}=h(x,y)f(x,y),\tag{3}$$

则称 $f=0$ 是系统(2)的 m 次不变代数曲线,并称 f 是系统(2)的代数积分,h 称为 f 的余因子.

注 1　由上面的定义,多项式 f 是系统(2)的代数积分的充要条件是在纯字典序 plex(x,y)下,$\left.\dfrac{\mathrm{d}f}{\mathrm{d}t}\right|_{(2)}$ 关于 f 的余式 $R\equiv 0$.

引理 1[7]　设 $f_1,f_2,\cdots,f_m$ 是系统(2)的 m 个独立的代数积分,满足

$$\left.\frac{\mathrm{d}f_k}{\mathrm{d}t}\right|_{(2)}=h_k(x,y)f_k(x,y),k=1,2,\cdots,m,\tag{4}$$

如果存在一组不全为零的复常数 $\alpha_1,\alpha_2,\cdots,\alpha_m$,使得

$$\alpha_1h_1+\alpha_2h_2+\cdots+\alpha_mh_m=-\left(\frac{\partial X_n}{\partial x}+\frac{\partial Y_n}{\partial y}\right),\tag{5}$$

则 $f=f_1^{\alpha_1}f_2^{\alpha_2}\cdots f_m^{\alpha_m}$ 是系统(2)的一个 Darboux 积分因子.

引理 2[7]　系统(1)以原点为中心的充要条件是该系统在原点附近存在解析的积分因子 $\mu(x,y)$,且 $\mu(0,0)\neq 0$.

2　主要结果

考虑三次系统

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=y+A_{2,0}x^2+A_{2,1}xy+A_{2,2}y^2+A_{3,0}x^3+A_{3,1}x^2y+A_{3,2}xy^2+A_{3,3}y^3,\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=-x+B_{2,0}x^2+B_{2,1}xy+B_{2,2}y^2+B_{3,0}x^3+B_{3,1}x^2y+B_{3,2}xy^2+B_{2,2}^2y^3,\end{cases}\tag{6}$$

其中 $B_{2,2}\neq 0$. 设系统(6)具有不变代数曲线:

$$p(x,y)=B_{2,2}^2y^2+(B_{2,2}x+1)^2=0,\tag{7}$$

即该系统具有两条复不变代数直线:

$$B_{2,2}y+\mathrm{i}(B_{2,2}x+1)=0,$$
$$B_{2,2}y-\mathrm{i}(B_{2,2}x+1)=0.$$

由注 1,系统(6)具有二次不变代数曲线 $p(x,y)=0$ 的充要条件是 $\left.\dfrac{\mathrm{d}p}{\mathrm{d}t}\right|_{(6)}$ 关于 $p(x,y)$ 的余式 $R(y)\equiv 0$,而这又等价于系统(6)可表为

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=y-\dfrac{(B_{2,2}B_{2,1}-B_{3,1})x^2}{B_{2,2}}+\dfrac{(B_{3,0}+2B_{2,2}^2+A_{3,1})xy}{B_{2,2}}+B_{2,2}y^2+(-B_{2,2}B_{2,1}+B_{3,1})x^3+A_{3,1}x^2y+(B_{2,2}^2-B_{2,2}B_{2,1})xy^2,\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=-x-\dfrac{(-B_{3,0}+B_{2,2}^2)x^2}{B_{2,2}}+B_{2,1}xy+B_{2,2}y^2+B_{3,0}x^3+B_{3,1}x^2y+(A_{3,1}+B_{3,0})xy^2+B_{2,2}^2y^3.\end{cases}\tag{8}$$

为了简化上述系统中的某些系数,对系统(8)作伸缩变换

$$x\rightarrow\frac{x}{B_{2,2}},y\rightarrow\frac{y}{B_{2,2}},\tag{9}$$

则系统(8)变为

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=y-\dfrac{(B_{2,2}B_{2,1}-B_{3,1})x^2}{B_{2,2}^2}+\dfrac{(B_{3,0}+2B_{2,2}^2+A_{3,1})xy}{B_{2,2}^2}+y^2+\dfrac{(-B_{2,2}B_{2,1}+B_{3,1})x^3}{B_{2,2}^2}+\dfrac{A_{3,1}x^2y}{B_{2,2}^2}+\dfrac{(B_{2,2}^2-B_{2,2}B_{2,1})xy^2}{B_{2,2}^2},\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=-x+y^2+\dfrac{B_{2,1}xy}{B_{2,2}}-\dfrac{(-B_{3,0}+B_{2,2}^2)x^2}{B_{2,2}^2}+y^3+\dfrac{(A_{3,1}+B_{3,0})xy^2}{B_{2,2}^2}+\dfrac{B_{3,1}x^2y}{B_{2,2}^2}+\dfrac{B_{3,0}x^3}{B_{2,2}^2}.\end{cases} \tag{10}$$

系统(10)虽具有5个参数 $A_{3,1},B_{2,1},B_{2,2},B_{3,0},B_{3,1}$,但可以通过下面的系数替换

$$\frac{B_{2,1}}{B_{2,2}}=b_{2,1},\frac{B_{3,1}}{B_{2,2}^2}=b_{3,1},\frac{A_{3,1}}{B_{2,2}^2}=a_{3,1},\frac{B_{3,0}}{B_{2,2}^2}=b_{3,0}, \tag{11}$$

转化为仅含4个独立参数的系统

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=y-(b_{2,1}-b_{3,1})x^2+(2+b_{3,0}+a_{3,1})yx+y^2+(-b_{2,1}+b_{3,1})x^3+a_{3,1}yx^2+(1-b_{2,1})y^2x,\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=-x-(1-b_{3,0})x^2+b_{2,1}yx+y^2+b_{3,0}x^3+b_{3,1}yx^2+(a_{3,1}+b_{3,0})y^2x+y^3.\end{cases} \tag{12}$$

定理1　系统(8)以原点为中心的充要条件是下列六组条件之一成立:

(1) $B_{3,1}=B_{2,1}B_{2,2},A_{3,1}=-B_{2,1}B_{2,2}+B_{2,2}^2,B_{3,0}=-B_{2,2}^2$;

(2) $B_{2,1}=-B_{2,2},B_{3,1}=-B_{2,2}^2,A_{3,1}=-2B_{3,0}$;

(3) $B_{2,1}=0,B_{3,1}=-B_{2,2}^2,B_{3,0}=0$;

(4) $B_{2,1}=\dfrac{1}{2}\ \dfrac{(-1+\mu)B_{2,2}B_{3,0}}{B_{3,0}+B_{2,2}^2},B_{3,1}=-\dfrac{1}{4}\ \dfrac{(B_{2,2}^2\mu+B_{2,2}^2-B_{3,0}\mu+3B_{3,0})B_{2,2}^2}{B_{3,0}+B_{2,2}^2},A_{3,1}=-\dfrac{1}{2}\ \dfrac{(B_{2,2}^2\mu+B_{2,2}^2+2B_{3,0})B_{3,0}}{B_{3,0}+B_{2,2}^2}$;

(5) $B_{2,1}=-\dfrac{1}{2}\ \dfrac{(1+\mu)B_{2,2}B_{3,0}}{B_{3,0}+B_{2,2}^2},B_{3,1}=\dfrac{1}{4}\ \dfrac{(B_{2,2}^2\mu-B_{2,2}^2-B_{3,0}\mu-3B_{3,0})B_{2,2}^2}{B_{3,0}+B_{2,2}^2},A_{3,1}=\dfrac{1}{2}\ \dfrac{(B_{2,2}^2\mu-B_{2,2}^2-2B_{3,0})B_{3,0}}{B_{3,0}+B_{2,2}^2}$;

(6) $A_{3,1}=0,B_{2,1}=0,B_{3,0}=0.$

其中 $\mu=\sqrt{-\dfrac{7B_{2,2}^2+8B_{3,0}}{B_{2,2}^2}}$.

3　主要结果的证明

利用文献[8]的约化焦点量算法,可得系统(12)的前五阶焦点量. 其中前三阶焦点量依次为:

$$W_1=b_{3,0}(-\frac{1}{3}+\frac{1}{3}b_{3,1})-\frac{1}{3}b_{3,1}a_{3,1}-\frac{1}{3}a_{3,1}+\frac{1}{3}b_{2,1}a_{3,1},$$

$$\begin{aligned}W_2=&\frac{1}{15}b_{2,1}b_{3,0}^2a_{3,1}+\frac{4}{15}b_{3,0}b_{2,1}a_{3,1}+\frac{1}{15}b_{2,1}b_{3,1}b_{3,0}+\frac{2}{15}b_{2,1}a_{3,1}+\frac{4}{5}b_{3,1}b_{3,0}^3+\frac{1}{15}b_{2,1}^2b_{3,0}-\frac{4}{15}b_{2,1}b_{3,0}^3-\\&\frac{4}{15}b_{3,0}^2b_{2,1}+\frac{1}{15}b_{3,0}^2a_{3,1}+\frac{1}{3}b_{2,1}b_{3,0}+\frac{6}{5}b_{3,0}^2b_{3,1}-\frac{4}{15}b_{3,1}b_{3,0}-\frac{2}{3}b_{2,1}b_{3,1}-\frac{1}{15}a_{3,1}^2b_{2,1}b_{3,0}+\frac{11}{15}b_{2,1}^2b_{3,1}b_{3,0}-\\&\frac{2}{5}b_{2,1}b_{3,0}b_{3,1}^2+\frac{2}{5}b_{2,1}^2-\frac{4}{15}b_{2,1}^3+\frac{2}{15}b_{3,0}^2-\frac{1}{3}b_{2,1}+\frac{2}{15}b_{3,0}-\frac{4}{15}b_{2,1}^3b_{3,0}+\frac{14}{15}b_{2,1}^2b_{3,1}-\frac{2}{5}b_{3,0}b_{3,1}^2-\frac{3}{5}b_{2,1}b_{3,1}^2-\\&\frac{1}{15}a_{3,1}^2b_{2,1}+\frac{1}{15}a_{3,1}^2b_{3,0},\end{aligned}$$

$$\begin{aligned}W_3=&\frac{82}{105}b_{3,0}^3b_{2,1}a_{3,1}+\frac{193}{315}b_{2,1}b_{3,0}^2a_{3,1}-\frac{16}{315}b_{3,0}b_{2,1}a_{3,1}+\frac{8}{21}b_{2,1}b_{3,0}^4a_{3,1}+\frac{8}{21}b_{2,1}b_{3,1}b_{3,0}^3+\frac{32}{15}b_{3,0}^2b_{2,1}b_{3,1}+\frac{39}{35}b_{2,1}b_{3,1}b_{3,0}-\\&\frac{10}{63}b_{2,1}a_{3,1}+\frac{22}{21}b_{3,1}b_{3,0}^3-\frac{9}{35}b_{2,1}^2b_{3,0}-\frac{134}{105}b_{3,0}^3a_{3,1}-\frac{16}{21}b_{3,0}^4a_{3,1}-\frac{92}{315}b_{2,1}b_{3,0}^3+\frac{56}{45}b_{3,0}^2b_{2,1}-\frac{53}{63}b_{3,0}^2a_{3,1}-\frac{8}{105}a_{3,1}b_{3,0}+\\&\frac{137}{315}b_{2,1}b_{3,0}+\frac{16}{105}b_{3,0}^4b_{2,1}+\frac{4}{21}b_{2,1}b_{3,0}^5+\frac{4}{7}b_{3,0}^4b_{3,1}-\frac{52}{35}b_{3,0}^2b_{3,1}-\frac{212}{315}b_{3,1}b_{3,0}+\frac{76}{315}b_{2,1}b_{3,1}-\frac{4}{63}b_{2,1}^3b_{3,1}b_{3,0}+\\&\frac{2}{63}a_{3,1}^2b_{2,1}^2b_{3,0}+\frac{4}{21}a_{3,1}^2b_{2,1}b_{3,0}^3+\frac{12}{35}a_{3,1}^2b_{3,0}^2b_{2,1}-\frac{4}{315}a_{3,1}b_{2,1}^2b_{3,0}+\frac{4}{63}a_{3,1}b_{2,1}^2b_{3,0}^2-\frac{1}{35}a_{3,1}b_{2,1}^3b_{3,0}+\frac{5}{21}a_{3,1}^2b_{2,1}b_{3,0}-\\&\frac{43}{105}b_{2,1}^2b_{3,0}^2b_{3,1}-\frac{8}{21}b_{2,1}^2b_{3,1}b_{3,0}^3-\frac{23}{105}b_{2,1}^2b_{3,1}b_{3,0}-\frac{22}{63}b_{2,1}^4-\frac{22}{315}b_{2,1}^2+\frac{59}{63}b_{2,1}^3-\frac{146}{315}b_{3,0}^2-\frac{8}{7}b_{3,0}^3-\frac{4}{7}b_{3,0}^5-\frac{148}{105}b_{3,0}^4+\end{aligned}$$

$\frac{2}{63}b_{2,1}^4b_{3,0}-\frac{6}{35}b_{2,1}^4b_{3,1}+\frac{11}{35}b_{2,1}^3b_{3,1}^2+\frac{62}{63}b_{2,1}^3b_{3,1}+\frac{16}{105}b_{2,1}^3b_{3,0}^2+\frac{4}{21}b_{2,1}^3b_{3,0}^3-\frac{6}{7}b_{3,0}^2b_{3,1}^2+\frac{13}{315}a_{3,1}^2b_{2,1}^2-\frac{4}{21}a_{3,1}^2b_{3,0}^3-$
$\frac{16}{105}a_{3,1}^2b_{3,0}^2-\frac{8}{315}a_{3,1}b_{2,1}^2-\frac{2}{105}a_{3,1}b_{2,1}^3-\frac{299}{315}b_{2,1}^2b_{3,0}^2-\frac{10}{63}b_{2,1}^2b_{3,0}^3-\frac{1}{7}b_{2,1}^2b_{3,1}^3-\frac{20}{21}b_{2,1}^2b_{3,1}^2+\frac{2}{7}b_{2,1}b_{3,1}^3+\frac{41}{315}b_{2,1}-$
$\frac{16}{63}b_{3,0}+\frac{16}{105}b_{2,1}^3b_{3,0}-\frac{541}{315}b_{2,1}^2b_{3,1}-\frac{4}{7}b_{3,0}b_{3,1}^2+\frac{83}{105}b_{2,1}b_{3,1}^2+\frac{43}{315}a_{3,1}^2b_{2,1}-\frac{37}{315}a_{3,1}^2b_{3,0}.$

而第四阶、第五阶焦点量 W_4、W_5 分别多达 103 项、135 项,在此从略.

定理 1 的证明

必要性　首先计算系统(12)的前五阶焦点量的约化 Gröbner 基 G;然后在变量序 $a_{3,1}>b_{3,1}>b_{2,1}>b_{3,0}$ 下,利用吴特征集法[9]对多项式组 G 进行零点分解,共得到六组独立的系数条件;最后根据这些条件和系数转化关系(11),便得到定理 1 中的六组独立条件.

充分性　当条件(1)成立时,系统(8)变为

$$\begin{cases}\frac{\mathrm{d}x}{\mathrm{d}t}=y+\frac{(2B_{2,2}^2-B_{2,2}B_{2,1})xy}{B_{2,2}}+B_{2,2}y^2+(B_{2,2}^2-B_{2,2}B_{2,1})x^2y+(B_{2,2}^2-B_{2,2}B_{2,1})xy^2,\\ \frac{\mathrm{d}y}{\mathrm{d}t}=-x-2B_{2,2}x^2+B_{2,1}xy+B_{2,2}y^2-B_{2,2}^2x^3+B_{2,2}B_{2,1}x^2y-B_{2,2}B_{2,1}xy^2+B_{2,2}^2y^3,\end{cases}\tag{13}$$

系统(13)以

$$R_1(x,y)=\frac{1}{(1+B_{2,2}x-B_{2,1}y)(B_{2,2}^2x^2+2B_{2,2}x+B_{2,2}^2y^2+1)(1+B_{2,2}x-B_{2,1}x)}$$

为积分因子,因此由引理 2,它以原点为中心.

当条件(2)成立时,系统(8)变为

$$\begin{cases}\frac{\mathrm{d}x}{\mathrm{d}t}=y+\frac{(2B_{2,2}^2-B_{3,0})xy}{B_{2,2}}+B_{2,2}y^2-2B_{3,0}x^2y+2B_{2,2}^2xy^2,\\ \frac{\mathrm{d}y}{\mathrm{d}t}=-x-\frac{(B_{2,2}^2-B_{3,0})x^2}{B_{2,2}}-B_{2,2}xy+B_{2,2}y^2+B_{3,0}x^3-B_{2,2}^2x^2y-B_{3,0}xy^2+B_{2,2}^2y^3,\end{cases}\tag{14}$$

系统(14)以

$$R_2(x,y)=\frac{1}{(1+2B_{2,2}x)(B_{2,2}-B_{3,0}x+B_{2,2}^2y)(1+2B_{2,2}x-4B_{2,2}^2x^2-4B_{2,2}^2y^2)}$$

为积分因子,因此由引理 2,它以原点为中心.

当条件(3)成立时,系统(8)变为

$$\begin{cases}\frac{\mathrm{d}x}{\mathrm{d}t}=y-B_{2,2}x^2+\frac{(2B_{2,2}^2+A_{3,1})xy}{B_{2,2}}+B_{2,2}y^2-B_{2,2}^2x^3+A_{3,1}x^2y+B_{2,2}^2xy^2,\\ \frac{\mathrm{d}y}{\mathrm{d}t}=-x-B_{2,2}x^2+B_{2,2}y^2-x^2yB_{2,2}^2+A_{3,1}xy^2+B_{2,2}^2y^3,\end{cases}\tag{15}$$

系统(15)以

$$R_3(x,y)=(1+2B_{2,2}x+B_{2,2}^2x^2+B_{2,2}^2y^2)^{-1}R_{3,1}(x,y)^{-1}$$

为积分因子,其中

$$R_{3,1}(x,y)=B_{2,2}+(A_{3,1}+2B_{2,2}^2)x+2B_{2,2}^2y+(-B_{2,2}^3+A_{3,1}B_{2,2})x^2+(2B_{2,2}^3+A_{3,1}B_{2,2})yx+B_{2,2}^3y^2,$$

因此由引理 2,它以原点为中心.

当条件(4)成立时,系统(8)变为

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y-\left(\frac{1}{2}\ \frac{B_{2,2}(-1+\mu)B_{3,0}}{B_{3,0}+B_{2,2}^2}+\frac{1}{4}\ \frac{(B_{2,2}^2\mu+B_{2,2}^2-B_{3,0}\mu+3B_{3,0})B_{2,2}}{B_{3,0}+B_{2,2}^2}\right)x^2+$$
$$\left(2B_{2,2}^2+B_{3,0}-\frac{1}{2}\ \frac{(B_{2,2}^2\mu+B_{2,2}^2+2B_{3,0})B_{3,0}}{B_{3,0}+B_{2,2}^2}\right)xyB_{2,2}^{-1}+B_{2,2}y^2+$$
$$\left(-\frac{1}{2}\ \frac{B_{2,2}^2(-1+\mu)B_{3,0}}{B_{3,0}+B_{2,2}^2}-\frac{1}{4}\ \frac{(B_{2,2}^2\mu+B_{2,2}^2-B_{3,0}\mu+3B_{3,0})B_{2,2}^2}{B_{3,0}+B_{2,2}^2}\right)x^3-$$

$$\frac{1}{2}\frac{(B_{2,2}^2\mu+B_{2,2}^2+2B_{3,0})B_{3,0}x^2y}{B_{3,0}+B_{2,2}^2}+\left(B_{2,2}^2-\frac{1}{2}\frac{B_{2,2}^2(-1+\mu)B_{3,0}}{B_{3,0}+B_{2,2}^2}\right)y^2x,$$

$$\frac{dy}{dt}=-x-\frac{(B_{2,2}^2-B_{3,0})x^2}{B_{2,2}}+\frac{1}{2}\frac{(-1+\mu)B_{2,2}B_{3,0}xy}{B_{3,0}+B_{2,2}^2}+B_{2,2}y^2+B_{3,0}x^3-$$

$$\frac{1}{4}\frac{(B_{2,2}^2\mu+B_{2,2}^2-B_{3,0}\mu+3B_{3,0})B_{2,2}^2x^2y}{B_{3,0}+B_{2,2}^2}+$$

$$\left(-\frac{1}{2}\frac{(B_{2,2}^2\mu+B_{2,2}^2+2B_{3,0})B_{3,0}}{B_{3,0}+B_{2,2}^2}+B_{3,0}\right)y^2x+B_{2,2}^2y^3.$$

此系统以

$$R_4(x,y)=(1+2B_{2,2}x+B_{2,2}^2x^2+B_{2,2}^2y^2)^{-1}R_{4,1}(x,y)^{-1}$$

为积分因子,其中

$$R_{4,1}(x,y)=-4(B_{3,0}+B_{2,2}^2)^2+2(B_{3,0}+B_{2,2}^2)B_{2,2}(-5B_{3,0}-4B_{2,2}^2+B_{3,0}\mu)x-2(B_{3,0}+B_{2,2}^2)B_{2,2}(B_{2,2}^2\mu+B_{2,2}^2+2B_{3,0})y+(B_{3,0}+B_{2,2}^2)(-3B_{2,2}^4+B_{2,2}^4\mu-B_{2,2}^2B_{3,0}+3B_{3,0}B_{2,2}^2\mu+4B_{3,0}^2)x^2-2B_{2,2}^2(B_{2,2}^4\mu+B_{2,2}^4+B_{3,0}B_{2,2}^2\mu+7B_{2,2}^2B_{3,0}-\mu B_{3,0}^2+7B_{3,0}^2)yx,$$

因此由引理2,它以原点为中心.

当条件(5)成立时,系统(8)变为

$$\frac{dx}{dt}=y-\left(-\frac{1}{2}\frac{B_{2,2}(\mu+1)B_{3,0}}{B_{3,0}+B_{2,2}^2}-\frac{1}{4}\frac{(B_{2,2}^2\mu-B_{2,2}^2-B_{3,0}\mu-3B_{3,0})B_{2,2}}{B_{3,0}+B_{2,2}^2}\right)x^2+$$

$$\left(2B_{2,2}^2+B_{3,0}+\frac{1}{2}\frac{(B_{2,2}^2\mu-B_{2,2}^2-2B_{3,0})B_{3,0}}{B_{3,0}+B_{2,2}^2}\right)B_{2,2}^{-1}xy+B_{2,2}y^2+$$

$$\left(\frac{1}{2}\frac{B_{2,2}^2(\mu+1)B_{3,0}}{B_{3,0}+B_{2,2}^2}+\frac{1}{4}\frac{(B_{2,2}^2\mu-B_{2,2}^2-B_{3,0}\mu-3B_{3,0})B_{2,2}^2}{B_{3,0}+B_{2,2}^2}\right)x^3+$$

$$\frac{1}{2}\frac{(B_{2,2}^2\mu-B_{2,2}^2-2B_{3,0})B_{3,0}x^2y}{B_{3,0}+B_{2,2}^2}+\left(B_{2,2}^2+\frac{1}{2}\frac{B_{2,2}^2(\mu+1)B_{3,0}}{B_{3,0}+B_{2,2}^2}\right)xy^2,$$

$$\frac{dy}{dt}=-x-\frac{(B_{2,2}^2-B_{3,0})x^2}{B_{2,2}}-\frac{1}{2}\frac{(\mu+1)B_{2,2}B_{3,0}xy}{B_{3,0}+B_{2,2}^2}+B_{2,2}y^2+B_{3,0}x^3+$$

$$\frac{1}{4}\frac{(B_{2,2}^2\mu-B_{2,2}^2-B_{3,0}\mu-3B_{3,0})B_{2,2}^2x^2y}{B_{3,0}+B_{2,2}^2}+$$

$$\left(\frac{1}{2}\frac{(B_{2,2}^2\mu-B_{2,2}^2-2B_{3,0})B_{3,0}}{B_{3,0}+B_{2,2}^2}+B_{3,0}\right)xy^2+B_{2,2}^2y^3.$$

此系统以

$$R_5(x,y)=(1+2B_{2,2}x+B_{2,2}^2x^2+B_{2,2}^2y^2)^{-1}R_{5,1}(x,y)^{-1}$$

为积分因子,其中

$$R_{5,1}(x,y)=4(B_{3,0}+B_{2,2}^2)^2+2(B_{3,0}+B_{2,2}^2)B_{2,2}(4B_{2,2}^2+5B_{3,0}+B_{3,0}\mu)x-2(B_{3,0}+B_{2,2}^2)B_{2,2}(B_{2,2}^2\mu-B_{2,2}^2-2B_{3,0})y+(B_{3,0}+B_{2,2}^2)(3B_{2,2}^4+B_{2,2}^4\mu+B_{2,2}^2B_{3,0}+3B_{3,0}B_{2,2}^2\mu-4B_{3,0}^2)x^2-2B_{2,2}^2(B_{2,2}^4\mu-B_{2,2}^4+B_{3,0}B_{2,2}^2\mu-7B_{2,2}^2B_{3,0}-\mu B_{3,0}^2-7B_{3,0}^2)yx,$$

因此由引理2,它以原点为中心.

当条件(6)成立时,系统(8)变为

$$\begin{cases}\dfrac{dx}{dt}=y+\dfrac{B_{3,1}x^2}{B_{2,2}}+2B_{2,2}xy+B_{2,2}y^2+B_{3,1}x^3+B_{2,2}^2xy^2,\\ \dfrac{dy}{dt}=-x-B_{2,2}x^2+B_{2,2}y^2+B_{3,1}x^2y+B_{2,2}^2y^3,\end{cases}\qquad(16)$$

通过分式变换

$$x\to\frac{y}{1-B_{2,2}y},y\to\frac{x}{1-B_{2,2}y},$$

系统(16)变为关于 y 轴对称的系统,故而由 Poincaré 对称原理,它以原点为中心.

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