Blow-up Solutions to the Schrödinger-Hartree Equation

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Abstract: In this paper, we study the blow-up solutions for the nonlinear Schrödinger-Hartree equation, we give another characterization of the blow-up solutions.

Key words: Schrödinger equation, blow-up, variational characterization

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Schrödinger-Hartree 方程爆破解的存在性

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[摘要] 本文研究了一类 Schrödinger-Hartree 方程,给出了爆破解的另外一种刻画.

[关键词] 薛定谔方程,爆破,变分刻画

This paper is devoted to the study of the Schrödinger-Poisson system

$$\begin{cases} i\partial_t u+\Delta u=P|u|^{p-2}u, & (t,x)\in \mathbf{R}_+\times\mathbf{R}^d, \\ \Delta P=|u|^p, \\ u(0,x)=u_0(x), \end{cases} \tag{1}$$

where $u:\mathbf{R}_+\times\mathbf{R}^d\mapsto\mathbf{C}$, $n\geqslant 3$ and $1+\dfrac{4}{d}<p<1+\dfrac{4}{d-2}$.

One can verify that (1) is essentially equivalent to the following Schrödinger-Hartree equation

$$\begin{cases} i\partial_t u+\Delta u=-(|\cdot|^{2-d}\times|u|^p)|u|^{p-2}u, & (t,x)\in \mathbf{R}_+\times\mathbf{R}^d, \\ u(0,x)=u_0(x). \end{cases} \tag{2}$$

Miao, Xu and Zhao[1] established global existence for solutions with finite energy in the case of $d=6$ and $p=2$. Genev and Venkov[2] established the existence of solitary wave solutions and local existence with initial data.

In this paper, following Ibrahim, Masmoudi and Nakanishi's ideas[3], we characterize the ground state of in terms of some constrained minimization problem, what's more, we obtain some blow-up criterion for the Schrödinger-Hartree equation (2).

Notation Throughout this paper, we denote the Lebesgue L^q-space on $\mathbf{R}^d$ by $L^q(\mathbf{R}^d)$ with norm $\|\cdot\|_q$, $1\leqslant q<\infty$. We employ inhomogeneous Sobolev space $H^1(\mathbf{R}^d)$, which is defined as $H^1(\mathbf{R}^d)=\{u\in L^2(\mathbf{R}^d): \|u\|_{H^1}^2=\|u\|_2^2+\|\nabla u\|_2^2<\infty\}$.

We write $X\prec Y$ to indicate $X\leqslant CY$ for some constant $C>0$. We use the notation $X\sim Y$ whenever $X\prec Y\prec X$.

For $\phi\in H^1(\mathbf{R}^d)$, we denote the scaling function $\phi_{d,-2}^{\lambda}$ by $\phi_{d,-2}^{\lambda}(x)=e^{d\lambda}\phi(e^{2\lambda}x)$, and the differential operator L acting on any functional $J: H^1(\mathbf{R}^d)\mapsto\mathbf{R}$, by $LJ(\phi)=\dfrac{d}{d\lambda}J(\phi_{d,-2}^{\lambda})|_{\lambda=0}$. Then the scaling derivative of $S(\phi)$ is

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defined by

$$K(\phi)=LS(\phi)=\frac{\mathrm{d}}{\mathrm{d}\lambda}S(\phi_{d,-2}^{\lambda})|_{\lambda=0}=\int_{\mathbf{R}^d}(2|\nabla\phi|^2-\frac{dp-(d+2)}{p}(|\cdot|^{2-d}\times|\phi|^p)|\phi|^p). \quad (3)$$

Define

$$m=\inf\{S(\phi):\phi\in H^1(\mathbf{R}^d)\setminus\{0\},K(\phi)=0\}, \quad (4)$$

where

$$S(\phi)=\frac{1}{2}\int_{\mathbf{R}^d}|\nabla\phi|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbf{R}^d}|\phi|^2\mathrm{d}x-\frac{1}{2p}\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\phi|^p)|\phi|^p \quad (5)$$

and $K(\phi)$ is defined by (3). Now, we are able to state the main result of this paper.

Theorem 1　For $1+\frac{4}{d}<p<1+\frac{4}{d-2}$, and any initial data

$$u_0\in\Sigma:=\{f:f\in H^1(\mathbf{R}^d),xf\in L^2(\mathbf{R}^d)\},$$

that satisfies (i) $K(u_0)<0$, where K is defined by (3); (ii) $S(u_0)<m$, where S is defined by (5) and m is defined by (4).

Then, there exists a finite time $T\in(0,+\infty)$ such that $\lim_{t\to T}\|\nabla u(t)\|_2=+\infty$, where $u\in C([0,T),H^1(\mathbf{R}^d))$ is the corresponding solution to the Cauchy problem (2).

1　Preliminaries

1.1　Some known results

For the Cauchy problem (2), Genev and Venkov[2] established the local existence of weak solution, see also [4].

Proposition 1　For $2p<1+\frac{4}{d-2}$ and an initial data $u_0\in H^1(\mathbf{R}^d)$, there exists $T\in(0,+\infty]$ and a solution $u\in C([0,T),H^1(\mathbf{R}^d))$ of the Cauchy problem (2). Furthermore, u is unique in $C([0,T),H^1(\mathbf{R}^d))$,

(i) either $T=+\infty$, or else $T<+\infty$, and $\lim_{t\to T}\|\nabla u(t)\|_2=+\infty$;

(ii) $u(t)$ satisfies the conservation of mass and energy, that is, for all $t\in[0,T)$.

$$M(u(t))=\int_{\mathbf{R}^d}|u(t)|^2\mathrm{d}x=\int_{\mathbf{R}^d}|u_0|^2\mathrm{d}x,$$

$$E(u(t))=\frac{1}{2}\int_{\mathbf{R}^d}|\nabla u(t)|^2\mathrm{d}x-\frac{1}{2p}\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|u|^p)|u|^p\mathrm{d}x.$$

Looking for standing wave solutions $e^{it}\phi(x)$ for the equation of (2) leads us to consider the stationary equation

$$-\Delta\phi+\phi=(|\cdot|^{2-d}\times|u|^p)|u|^{p-2}u,\quad x\in\mathbf{R}^d. \quad (6)$$

It is easily seen that ϕ is a critical point of the functional (5) in $H^1(\mathbf{R}^n)$. Genev and Venkov[5] established the existence of the ground state solution of (5) in terms of the minimizing problem

$$C_{d,p}=\inf_{\phi\in H^1(\mathbf{R}^d)\setminus\{0\}}J^{d,p}(\phi), \quad (7)$$

with $J^{d,p}(\phi)=\dfrac{\|\phi\|_2^{d+2-(d-2)p}\|\nabla\phi\|_2^{dp-(d+2)}}{\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|u|^p)|u|^p}$.

For minimizing problem (7), it holds that

$$C_{d,p}=\inf_{\phi\in H^1(\mathbf{R}^d)\setminus\{0\}}J^{d,p}(\phi)=\frac{2p}{d+2-(d-2)p}\left(\frac{d+2-(d-2)p}{dp-(d+2)}\right)^{\frac{dp-(d+2)}{2}}\|Q\|_2^{2-2p},$$

where Q the unique positive radial ground state solution of (6) in $H^1(\mathbf{R}^d)$. The following lemma is useful in the subsequent sections.

Lemma 1 (Wely-Heisenberg inequality)[4,6]　For any $u\in H^1(\mathbf{R}^d)$, we have

$$\|u\|_2^2\leqslant\frac{2}{d}\||x|u\|_2\|\nabla u\|_2. \quad (8)$$

1.2 The variational characterization of the ground state

Inspired and motived by [3], we give a new variational characterization of the ground state of (6). Let us decompose K into the quadratic and the nonlinear parts $K(\phi)=K^Q(\phi)+K^N(\phi)$, where

$$K^Q(\phi)=2\int_{\mathbf{R}^d}|\nabla\phi|^2,\quad K^N(\phi)=-\frac{dp-(d+2)}{p}\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\phi|^p)|\phi|^p.$$

For any $\phi\in H^1(\mathbf{R}^d)\setminus\{0\}$, a direct calculation shows that

$$\lim_{\lambda\to-\infty}K^Q(\phi^\lambda_{d,-2})=\lim_{\lambda\to-\infty}2\int_{\mathbf{R}^d}|\nabla\phi^\lambda_{d,-2}|^2=\lim_{\lambda\to-\infty}2\mathrm{e}^{4\lambda}K^Q(\phi)=0. \tag{9}$$

The following lemma shows that K is positive near 0 in the energy space.

Lemma 2 For any bounded sequence $\phi_n\in H^1(\mathbf{R}^d)\setminus\{0\}$ with $\lim_{n\to+\infty}K^Q(\phi_n)=0$, we have $K(\phi_n)>0$, for large n.

Proof Since $K^Q(\phi_n)$ tends to 0 as n tends to 0, we know that $\lim_{n\to+\infty}\|\nabla\phi\|_2^2=0$. Then by Hardy-Littlewood-Sobolev and Gagliardo-Nerenberg inequalities[7], we have for large n,

$$\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}|x-y|^{2-d}|\phi_n(y)|^p|\phi_n(x)|^p\mathrm{d}x\mathrm{d}y\prec\|\phi_n\|^{2p}_{\frac{2dp}{d+2}}\|\nabla\phi_n\|_2^{dp-(d+2)}\|\phi_n\|_2^{(d+2)-(d-2)p}=o(\|\nabla\phi_n\|_2^2),$$

where we use the boundedness of $\|\phi_n\|_2$ and $1+\frac{4}{d}<p<1+\frac{4}{d-2}$. Therefore, based the analysis above, we get

$$K(\phi_n)=2\|\nabla\phi_n\|_2^2-\frac{dp-(d+2)}{p}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}|x-y|^{2-d}|\phi_n(y)|^p|\phi_n(x)|^p\mathrm{d}x\mathrm{d}y\sim\|\nabla\phi_n\|_2^2>0$$

for large n. This ends the proof.

Let us mention here two nonnegative numbers import for the following discussion:

$$\bar{\mu}=\min\{2dp-2(d+2),\max\{4,0\}\}=4,\ \underline{\mu}=\min\{2dp-2(d+2),\min\{4,0\}\}=0.$$

The following lemma plays an important role in the succeeding argument.

Lemma 3

$$(\bar{\mu}-L)S(\phi)=2\|\phi\|_2^2-\frac{(4+d)-dp}{p}\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|u|^p)|u|^p\mathrm{d}x\mathrm{d}x, \tag{10}$$

$$L(\bar{\mu}-L)S(\phi)=(2d+4-2dp)\frac{(d+4)-dp}{p}\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\phi|^p)|\phi|^p\mathrm{d}x. \tag{11}$$

Proof By the definition of L, we have

$$L\|\nabla\phi\|_2^2=4\|\nabla\phi\|_2^2,\quad L\|\phi\|_2^2=0,$$

$$L\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\phi|^p)|\phi|^p\mathrm{d}x=(2dp-2d-4)\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\phi|^p)|\phi|^p\mathrm{d}x,$$

direct calculation implies that

$$(\bar{\mu}-L)S(\phi)=2\|\phi\|_2^2-\frac{(4+d)-dp}{p}\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\phi|^p)|\phi^p\mathrm{d}x,$$

$$L(\bar{\mu}-L)S(\phi)=(2d+4-2dp)\frac{(d+4)-dp}{p}\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\phi|^p)|\phi|^p\mathrm{d}x.$$

This ends the proof.

Using Lemma 3, we are allowed to replace the functional S in (5) with a positive functional H, while extending the minimizing region from "$K(\phi)=0$" to "$K(\phi)\leqslant 0$". Let

$$H(\phi)=\left(1-\frac{L}{\bar{\mu}}\right)S(\phi), \tag{12}$$

then for any $\phi\in H^1(\mathbf{R}^d)\setminus\{0\}$, it follows that $H(\phi)>0, LH(\phi)\geqslant 0$. Now, we can rewrite the minimization problem (4) by using H.

Lemma 4 For the minimization m in (4), we have

$$m=\inf\{H(\phi):\phi\in H^1(\mathbf{R}^d)\setminus\{0\},K(\phi)\leqslant 0\},$$

where H is defined by (12) and K is defined by (3).

Proof For the convenience, we denote $\overline{m}=\inf\{H(\phi):\phi\in H^1(\mathbf{R}^d)\setminus\{0\},K(\phi)\leqslant 0\}$.

Firstly, for any $\phi\in H^1(\mathbf{R}^d)\setminus\{0\}$ with $K(\phi)=0$, we have $H(\phi)=S(\phi)$, we deduce that

$$\overline{m}=\inf\{H(\phi):\phi\in H^1(\mathbf{R}^d)\setminus\{0\},K(\phi)=0\}\geqslant\inf\{H(\phi):\phi\in H^1(\mathbf{R}^d)\setminus\{0\},K(\phi)\leqslant 0\}.$$

Hence, $\overline{m}\geqslant m$.

Finally, for any $\phi\in H^1(\mathbf{R}^d)\setminus\{0\}$ with $K(\phi)<0$, by (9), Lemma 2 and the continuity of K in λ, we deduce that there exists a $\lambda_0<0$ such that

$$K(\phi_{d,-2}^{\lambda_0})=0,\text{and } S(\phi_{d,-2}^{\lambda_0})=H(\phi_{d,-2}^{\lambda_0})\leqslant H(\phi_{d,-2}^{0})=H(\phi),$$

where we used $LH(\phi)\geqslant 0$ in the above inequality. Hence $\overline{m}\leqslant m$.

After these preparations, we can now characterize the ground state of (6) through the minimizer of (4).

Lemma 5 For the minimization m in (4), we have

$$m=S(Q),$$

where S is defined by (5) and Q is some radial ground state of (6).

Proof Let $\phi_n\in H^1(\mathbf{R}^d)$ be a minimizing sequence for (2), i. e.

$$K(\phi_n)\leqslant 0,\phi_n\neq 0,H(\phi_n)\geqslant m,\quad \lim_{n\to+\infty}H(\phi_n)=m. \tag{13}$$

Let ϕ_n^* be the symmetric decreasing rearrangement of ϕ_n. By the Riesz's rearrangement and Polya-Szegö inequalities[7], we have

$$K(\phi_n^*)\leqslant 0,\phi_n\neq 0,H(\phi_n)\geqslant H(\phi_n^*)\geqslant m,\quad \lim_{n\to+\infty}H(\phi_n^*)=m.$$

Therefore, replacing ϕ_n by its symmetric decreasing rearrangement ψ_n, then using (13), we obtain that

$$K(\psi_n)\leqslant 0,\psi_n\neq 0,S(\psi_n)=H(\psi_n),\lim_{n\to+\infty}S(\psi_n)=m. \tag{14}$$

Combing (13) with (5), we have

$$\mu m+(2pd-2d-8)m\leftarrow\mu H(\psi_n)+(2dp-2d-8)S(\psi_n)=2\|\psi_n\|_2^2-\frac{d+4-dp}{p}\int_{\mathbf{R}^5}(|\cdot|^{2-d}\times|\psi_n|^p)|\psi_n|^p+$$

$$(2dp-2d-8)\left[\frac{1}{2}\|\nabla\psi_n\|_2^2+\frac{1}{2}\|\psi_n\|_2^2-\frac{1}{2p}\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\psi_n|^p)|\psi_n|^p\right]=$$

$$2\|\psi_n\|_2^2+(dp-d-4)\|\psi_n\|_{H^1(\mathbf{R}^d)}^2,$$

which implies that ψ_n is bounded in $H^1(\mathbf{R}^d)$. Hence after extracting a subsequence (still denote by ψ_n), we have

$$\psi_n\rightharpoonup\psi \text{ weakly in } H^1(\mathbf{R}^d),\text{for some } \psi \text{ in } H^1(\mathbf{R}^d).$$

By the radial symmetry, we have also

$$\psi_n\to\psi \text{ strongly in } L^q(\mathbf{R}^d),\text{for all } 2<q<\frac{2d}{d-2},$$

$$|\psi_n|^p\to|\psi|^p \text{ strongly in } L^{\frac{2d}{d+2}}(\mathbf{R}^d),$$

therefore, the well-known Hardy-Littlewood-Sobolev inequality[7] implies that

$$\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\psi_n|^p)|\psi_n|^p\to\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\psi|^p)|\psi|^p,$$

it follows that $K(\psi)\leqslant 0$, and $H(\psi)\geqslant m$.

However, we still need to verify $\psi\neq 0$. In fact, if $\psi=0$, then $K(\psi_n)=0$ implies that $\lim_{n\to+\infty}K^Q(\psi_n)=-\lim_{n\to+\infty}K^N(\psi_n)=0$, and by Lemma 2, we have $K(\psi_n)>0$ for large n, which is a contradiction.

Using $LH(\phi)\geqslant 0$, we are allowed to replace ψ by its rescaling (still denote by ψ), such that

$$K(\psi)=0,\ S(\psi)=H(\psi)=m \text{ and } \psi\neq 0,$$

which implies that ψ is a minimizer of (4) and $m=H(\psi)=S(\psi)>0$. Hence, there exists a Lagrange multiplier $\eta\in\mathbf{R}$ such that $S'(\psi)=\eta K'(\psi)$.

Then using the chain rule, denoting we arrive at

$$0=K(\psi)=LS(\psi)=\langle S'(\psi),L\psi\rangle=\eta\langle K'(\psi),L\psi\rangle=\eta L^2S(\psi).$$

Combing (11) with $LS(\psi)=0$, it implies that

$$L^2S(\psi)=-(d+4-dp)(2d+4-2dp)\frac{1}{p}\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\psi|^p)|\psi|^p<0,$$

therefore $\eta=0$ and ψ is a solution of (6), what's more, ψ is nonnegative and radial. Since every solution φ in $H^1(\mathbf{R}^d)$ of (6) satisfies

$$K(\varphi)=\langle S'(\varphi),L\varphi\rangle=0.$$

This implies ψ is the ground state of (6).

We conclude this section with the following lemma, which gives the uniform bounds on the scaling derivative functional K with the functional S below the threshold m and plays an important role for the blow-up analysis.

Lemma 6 Let S be defined by (5), for any $\phi\in H^1(\mathbf{R}^d)$ with $S(\phi)<m$.

(i) If $K(\phi)<0$, then

$$K(\phi)\leqslant-4(m-S(\phi));$$

(ii) If $K(\phi)\geqslant0$, then

$$K(\phi)\geqslant\min\left\{4(m-S(\phi)),\frac{2(dp-d-4)}{dp-d}\|\nabla\phi\|_2^2\right\},$$

where K is defined by (3).

Proof Let $s(\lambda)=S(\phi^\lambda)$, $n(\lambda)=\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\phi_\lambda|^p)|\phi_\lambda|^p\mathrm{d}x$, where $\phi_\lambda=\phi^\lambda_{d,-2}$. By (11), we have

$$s''(\lambda)=\mu s'(\lambda)-\frac{1}{p}(d+4-dp)n'(\lambda). \tag{15}$$

Case 1 If $K(\phi)<0$, then by Lemma 2 together with (9), there exists a real number $\lambda_0<0$ such that $K(\phi^{\lambda_0})=0$ and $K(\phi^\lambda)<0$ for $\lambda_0<\lambda\leqslant0$. By (15), we get

$$s''(\lambda)=4s'(\lambda)-\frac{1}{p}(d+4-dp)n'(\lambda)=4s'(\lambda)-\frac{1}{p}(d+4-dp)(2d+4-2dp)n(\lambda)\leqslant4s'(\lambda).$$

Integrating from λ_0 to 0, we get

$$K(\phi)=s'(0)\leqslant4[s(0)-s(\lambda_0)]=4[S(\phi)-S(\phi^{\lambda_0})]\leqslant-4[m-S(\phi)].$$

Case 2 $K(\phi)\geqslant0$. We divide it into two sub cases:

When

$$2\mu K(\phi)<\frac{1}{p}(d+4-dp)(2d+4-2dp)\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\phi|^p)|\phi|^p,$$

applying (15), we have

$$s''(\lambda)<-\mu s'(\lambda) \tag{16}$$

for $\lambda=0$. On the other hand, $n''(\lambda)=\frac{1}{p}(d+4-dp)(2d+4-2dp)n(\lambda)>0$, implies that the right hand side of (16) is negative and decreasing as long as $n'>0$. Hence, we have

$$s'(\lambda)\leqslant s'(0)+\int_0^\lambda s''(\tau)\mathrm{d}\tau\leqslant s'(0)+\int_0^\lambda\left[\mu s'(\tau)-\frac{1}{p}(n+4-np)n'(\tau)\right]\mathrm{d}\tau\leqslant$$

$$s'(0)+\int_0^\lambda\left[\mu s'(\tau)-\frac{1}{p}(n+4-np)n'(\tau)\right]\mathrm{d}\tau\leqslant s'(0)+\left[\mu s'(0)-\frac{1}{p}(n+4-np)n'(0)\right]\lambda$$

as long as $n'>0$ holds. Hence (16) is preserved until λ reaches at some finite $\lambda_0>0$, i. e. there exists a finite $\lambda_0>0$ such that $s'(\lambda_0)=0$. Now integrating (16) from 0 to λ_0, we have

$$0=K(\phi^{\lambda_0})\leqslant K(\phi)-4\int_0^{\lambda_0}s'(\tau)\mathrm{d}\tau=K(\phi)-4[S(\phi^{\lambda_0})-S(\phi)],$$

this means

$$K(\phi)\geqslant4(m-S(\phi)).$$

When

$$2\mu K(\phi)\geqslant\frac{1}{p}(n+4-np)(2n+4-2np)\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\phi|^p)|\phi|^p,$$

since

$$\frac{1}{p}(d+4-dp)(2d+4-2dp)\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\phi|^p)|\phi|^p\mathrm{d}x=-4\|\nabla\phi\|_2^2+2(4+d-dp)K(\phi),$$

then we have

$$2\mu K(\phi)\geqslant-4\|\nabla\phi\|_2^2+2(4+d-dp)K(\phi),$$

which implies that

$$K(\phi)\geqslant\frac{2(dp-d-4)}{dp-d}\|\nabla\phi\|_2^2.$$

This finishes the proof.

2 Blow-up Threshold

In this section, we proof Theorem 1. To investigate the blow-up phenomena, we introduce the so-called "virial identity".

Lemma 7[4] (Virial identity) For $1+\frac{4}{d}<p<1+\frac{4}{d-2}$ and any initial $u_0\in\Sigma=\{f;f\in H^1(\mathbf{R}^d),|x|f\in L^2(\mathbf{R}^d)\}$, there exists a unique maximal solution $u\in C([0,T),\Sigma)$ of (2). Moreover, the first variance $V(t)=\int_{\mathbf{R}^d}|x|^2|u(t,x)|^2\mathrm{d}x$, belongs to $C^2([0,T),\Sigma)$, and satisfies the virial identity

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}V(t)=16E(u)-4\left(d-\frac{d+4}{p}\right)\int_{\mathbf{R}^d}(|\cdot|^{2-d}\times|\phi|^p)|\phi|^p=4K(u). \tag{17}$$

Proof of Theorem 1 Let u_0 satisfy $K(u_0)<0, S(u_0)<m$ and $u(t)$ be the solution of (2) with initial data u_0. Since $u(t)$ satisfies the conservation of mass and energy, we have u_0 satisfy $K(u_0)<0, S(u_0)<m$ and $u(t)$ be the solution of (2) with initial data u_0. Since $u(t)$ satisfies the conservation of mass and energy, we have

$$E(u(t))=E(u_0),\quad M(u(t))=M(u_0),$$

$$S(u(t))=\frac{1}{2}E(u(t))+\frac{1}{2}M(u(t))=\frac{1}{2}E(u_0)+\frac{1}{2}M(u_0)<m.$$

In addition, there exists $\delta>0$ such that

$$S(u(t))\leqslant(1-\delta)m.$$

Thus, by (17) and Lemma 7, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}V(t)=4K(u(t))\leqslant-4(m-S(u(t)))\leqslant-4\delta,$$

which implies that there exists a positive finite time T such that $\lim_{t\to T}V(t)=0$. By Weyl-Heisenberg inequality (8) and the conservation of mass, we have $\lim_{t\to T}\|\nabla u(t)\|_2^2=+\infty$ and the proof is finished.

[参考文献]

[1] Miao C, Xu G, Zhao L. Global well-posedness and scattering for the energy-critical, defocusing Hartree Equation in $R^\wedge(1+n)$[J]. Comm Partial Differential Equations, 2010, 36(5): 729-776.

[2] Genev H, Venkov G. Soliton and singular solutions to the Schrodinger-Hartree equation[J]. AIP Conference Proceedings, 2010, 1293: 107.

[3] Ibrahim S, Masmoudi N, Nakanishi K. Scattering threshold for the focusing nonlinear Klein-Gordon equation[J]. Analysis and PDE, 2011, 4(3): 405-460.

[4] Cazenave T. Semilinear Schrodinger Equations[M]. Providence: American Mathematical Society, 2003.

[5] Genev H, Venkov G. Symmetry and uniqueness of minimizers of Hartree type equations with external Coulomb potential[J]. Journal of Differential Equations, 2011, 251(2): 420-438.

[6] Genev H, Venkov G. Soliton and blow-up solutions to the time-dependent Schrodinger-Hartree equation[J]. Discrete and Continuous Dynamical Systems S, 2012, 5(5): 903-923.

[7] Lieb E H, Loss M. Analysis[M]. Providence: American Mathematical Society, 2001.

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