Soliton-Antisoliton Solutions of a Supersymmetric and Noncommutative Modified Chiral Model

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Abstract: We study a supersymmetric and noncommutative extension of the modified $\boldsymbol{U}(n)$ chiral model in 2+1 dimensions. A large family of multi-soliton solutions of this model were constructed by a super generalized dressing approach. In this paper, we apply the same method to construct a class of soliton-antisoliton solutions of the super extended model, and present two explicit non-abelian $\boldsymbol{U}(2)$ soliton-antisoliton configurations.

Key words: supersymmetric, noncommutative, Grassmann variable, soliton-antisoliton

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超对称非交换修正手征模型的孤子-反孤子解

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[**摘要**] 研究了 2+1 维修正 $\boldsymbol{U}(\boldsymbol{n})$ 手征模型的超对称非交换扩张模型. Lechtenfeld 等人利用超对称推广的穿衣方法构造了该模型的一大类多孤子解. 本文运用同样的方法构造了该模型的一类孤子-反孤子解,并具体给出两个 $\boldsymbol{U}(\boldsymbol{2})$ 孤子-反孤子解构型.

[**关键词**] 超对称,非交换,格拉斯曼变量,孤子-反孤子

An $N \leqslant 8$ supersymmetric generalization of the noncommutative modified $\boldsymbol{U}(n)$ 2+1 chiral model was introduced by Lechtenfeld and Popov[1]. Since this model can be formulated as the compatibility conditions of a linear system of differential equations involving a spectral parameter, a powerful solution-generating technique called "dressing method" can be applied to construct multi-soliton solutions of the extended model as in the non-supersymmetric case[2,3] and also in the commutative setup[4,5].

Taking a dressing ansatz for the solution $\boldsymbol{\psi}$ of the associated linear system with only first-order poles (which are all complex numbers with imaginary part < 0) in the spectral parameter yields no-scattering soliton configurations[1]. By allowing for second-order poles in the dressing ansatz, two-soliton configurations with nontrivial scattering were constructed[6]. These constructions can be viewed as just the supersymmetric generalization of the noncommutative bosonic cases[2,3]. A class of soliton-antisoliton solutions of the noncommutative modified chiral model were constructed by making the dressing ansatz with a pair of conjugate complex numbers poles[7]. Inspired by all of these constructions, we set out in this paper to construct soliton-antisoliton solutions of the supersymmetric and noncommutative modified chiral model, and illustrate these solutions are configurations with soliton and antisoliton interacting by discussing the scattering property of the bosonic subsector of two explicit examples.

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The plan of the paper is as follows: We present the $N\leqslant 8$ supersymmetric and noncommutative modified $\boldsymbol{U}(n)$ chiral model in 2+1 dimensions and the linear system associated to it in Section 1. In Section 2, we first briefly review the dressing constructions[1,6,7], then generalize the ones[7] to the supersymmetric case and derive a class of soliton-antisoliton solutions of the super extended model. In Section 3, we construct explicit $\boldsymbol{U}(2)$ soliton-antisoliton solutions and exhibit the scattering property of their bosonic subsector to show the solutions constructed by us are configurations with genuine soliton-antisoliton interaction.

1 The Supersymmetric and Noncommutative Modified Chiral Model

An $N\leqslant 8$ supersymmetric and noncommutative modified $\boldsymbol{U}(n)$ chiral model in 2+1 dimensions describes the dynamics of a $\boldsymbol{U}(n)$-valued superfield $\boldsymbol{\Phi}(t,x,y,\eta_i^\alpha)$, living on the antichiral superspace $\mathbf{R}^{2,1|N}$ with coordinates (t,x,y,η_i^α) for $\alpha=1,2$ and $i=1,\cdots,\frac{1}{2}N\leqslant 4$, here η_i^α are called fermionic coordinates, which are Grassmann variables[8], so they obey the anticommutation rules

$$\eta_i^\alpha\eta_j^\alpha+\eta_j^\alpha\eta_i^\alpha=0$$

for $1\leqslant i,j\leqslant\frac{1}{2}N$. In particular, when $i=j$, we have $(\eta_i^\alpha)^2=0$. Hence the degree of η_i^α in any function of the Grassmann variables η_i^α is no more than 1 for $\alpha=1,2$ and $i=1,\cdots,\frac{1}{2}N\leqslant 4$.

Definition 1[1,6] The $U(n)$-valued superfield $\boldsymbol{\Phi}(t,x,y,\eta_i^\alpha)$ of the supersymmetric and noncommutative modified chiral model satisfies the classical field equations

$$\begin{aligned}
&\partial_x(\boldsymbol{\Phi}^\dagger*\partial_x\boldsymbol{\Phi})+\partial_y(\boldsymbol{\Phi}^\dagger*\partial_y\boldsymbol{\Phi})-\partial_t(\boldsymbol{\Phi}^\dagger*\partial_t\boldsymbol{\Phi})+\partial_y(\boldsymbol{\Phi}^\dagger*\partial_t\boldsymbol{\Phi})-\partial_t(\boldsymbol{\Phi}^\dagger*\partial_y\boldsymbol{\Phi})=0,\\
&\partial_1^i(\boldsymbol{\Phi}^\dagger*\partial_x\boldsymbol{\Phi})-\partial_t(\boldsymbol{\Phi}^\dagger*\partial_2^i\boldsymbol{\Phi})+\partial_y(\boldsymbol{\Phi}^\dagger*\partial_2^i\boldsymbol{\Phi})=0,\\
&\partial_1^i(\boldsymbol{\Phi}^\dagger*\partial_t\boldsymbol{\Phi})+\partial_1^i(\boldsymbol{\Phi}^\dagger*\partial_y\boldsymbol{\Phi})-\partial_x(\boldsymbol{\Phi}^\dagger*\partial_2^i\boldsymbol{\Phi})=0,\\
&\partial_1^i(\boldsymbol{\Phi}^\dagger*\partial_2^j\boldsymbol{\Phi})+\partial_1^j(\boldsymbol{\Phi}^\dagger*\partial_2^i\boldsymbol{\Phi})=0,
\end{aligned}\tag{1}$$

and the unitarity condition

$$\boldsymbol{\Phi}^\dagger*\boldsymbol{\Phi}=\boldsymbol{\Phi}*\boldsymbol{\Phi}^\dagger=\boldsymbol{I}_n,\tag{2}$$

where $\partial_\alpha^i:=\partial/\partial\eta_i^\alpha$, '$\dagger$' denotes hermitian conjugation.

The noncommutative star product on the antichiral superspace is defined by

$$(f*g)(t,x,y,\eta_i^\alpha)=f(t,x,y,\eta_i^\alpha)\exp\left\{\frac{i}{2}\theta(\overleftarrow{\partial_x}\overrightarrow{\partial_y}-\overleftarrow{\partial_y}\overrightarrow{\partial_x}\right\}g(t,x,y,\eta_i^\alpha).\tag{3}$$

Note that the time coordinate remains commutative and no derivatives with respect to the Grassmann variables η_i^α. The nonlocality of the star product renders explicit calculations cumbersome. It is therefore helpful to pass over to the operator formalism, which trades the star product for operator-valued spatial coordinates $(\hat{x},\hat{y})$ or their complex combinations $(\hat{z},\hat{\bar{z}})$, subject to

$$[t,\hat{x}]=[t,\hat{y}]=0,\quad[\hat{x},\hat{y}]=i\theta\Rightarrow[\hat{z},\hat{\bar{z}}]=2\theta.\tag{4}$$

The later equation suggests the introduction of creation and annihilation operators,

$$a=\frac{1}{\sqrt{2\theta}}\hat{z}\quad\text{and}\quad a^\dagger=\frac{1}{\sqrt{2\theta}}\hat{\bar{z}}\quad\text{with}\quad[a,a^\dagger]=1,\tag{5}$$

which act on a harmonic-oscillator Fock space $\mathscr{D}$ with an orthonormal basis $\{|\ell\rangle,\ell=0,1,2,\cdots\}$ such that

$$a|\ell\rangle=\sqrt{\ell}\,|\ell-1\rangle\quad\text{and}\quad a^\dagger|\ell\rangle=\sqrt{\ell+1}\,|\ell+1\rangle.\tag{6}$$

Any superfields $f(t,z,\bar{z},\eta_i^\alpha)$ on $\mathbf{R}^{2,1|N}$ can be related to an operator-valued superfield $\hat{f}(t,\eta_i^\alpha)\equiv F(t,a,a^\dagger,\eta_i^\alpha)$ on $\mathbf{R}^{1|N}$ acting in $\mathscr{D}$, with the help of the Moyal-Weyl map

$$f(t,z,\bar{z},\eta_i^\alpha)\to\hat{f}(t,\eta_i^\alpha)=\text{Weyl-ordered}\,f(t,\sqrt{2\theta}a,\sqrt{2\theta}a^\dagger,\eta_i^\alpha).\tag{7}$$

The inverse transformation recovers the ordinary superfield,

$$\hat{f}(t,\eta_i^\alpha)\equiv F(t,a,a^\dagger,\eta_i^\alpha)\to f(t,z,\bar{z},\eta_i^\alpha)=F_*(t,\frac{z}{\sqrt{2\theta}},\frac{\bar{z}}{\sqrt{2\theta}},\eta_i^\alpha),\tag{8}$$

where F_* is obtained from F by replacing ordinary products with star products. Under the Moyal-Weyl map, we have

$$f*g\to\hat{f}\hat{g},\tag{9}$$

and the spatial derivatives are mapped into commutators,

$$\partial_z f\to\hat{\partial}_z\hat{f}=-\frac{1}{\sqrt{2\theta}}[a^\dagger,\hat{f}]\quad\text{and}\quad\partial_{\bar{z}}f\to\hat{\partial}_{\bar{z}}\hat{f}=\frac{1}{\sqrt{2\theta}}[a,\hat{f}].\tag{10}$$

Note that the product between $\hat{f}$ and $\hat{g}$ is the ordinary product. For notational simplicity we will from now on omit the hats over the operators except when confusion may arise.

The model (1) is the compatibility condition for the following linear system of differential equations involving a spectral parameter $\zeta\in\mathbf{C}\cup\{\infty\}$[1]

$$\begin{cases}(\zeta\partial_x-\partial_u)\boldsymbol{\psi}=\boldsymbol{A}\boldsymbol{\psi},\\(\zeta\partial_v-\partial_x)\boldsymbol{\psi}=\boldsymbol{B}\boldsymbol{\psi},\\(\zeta\partial_1^i-\partial_2^i)\boldsymbol{\psi}=\boldsymbol{C}^i\boldsymbol{\psi},\quad i=1,\cdots,\frac{1}{2}N,\end{cases}\tag{11}$$

where $u=\frac{1}{2}(t+y)$, $v=\frac{1}{2}(t-y)$, $\boldsymbol{\psi}$ depends on $(t,x,y,\eta_i^\alpha,\zeta)$ or equivalently on $(x,u,v,\eta_i^\alpha,\zeta)$, and is an $n\times n$ matrix whose elements act as operators in the Fock space $\mathscr{D}$. The $n\times n$ matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}^i$ are superfields on $(t,x,y,\eta_i^\alpha)\in\mathbf{R}^{2,1|N}$ independent of the spectral parameter ζ. Moreover, $\boldsymbol{\psi}$ is subject to the following reality condition

$$\boldsymbol{\psi}(t,x,y,\eta_i^\alpha,\zeta)[\boldsymbol{\psi}(t,x,y,\eta_i^\alpha,\bar{\zeta})]^\dagger=\boldsymbol{I}_n.\tag{12}$$

Then the superfield $\boldsymbol{\Phi}(t,x,y,\eta_i^\alpha)=\boldsymbol{\psi}(t,x,y,\eta_i^\alpha,0)^{-1}:=\boldsymbol{\psi}^{-1}(\zeta=0)$ clearly satisfies the model (1) and the unitarity condition (2).

2 Dressing Approach and Constructions for Soliton-Antisoliton Solutions

The dressing method is a recursive procedure for generating a new solution from an old one. We briefly review the dressing construction. By the reality condition (12), the linear system (11) can be rewritten as

$$\begin{cases}\boldsymbol{\psi}(\partial_u-\zeta\partial_x)\boldsymbol{\psi}^\dagger=\boldsymbol{A},\\\boldsymbol{\psi}(\partial_x-\zeta\partial_v)\boldsymbol{\psi}^\dagger=\boldsymbol{B},\\\boldsymbol{\psi}(\partial_2^i-\zeta\partial_1^i)\boldsymbol{\psi}^\dagger=\boldsymbol{C}^i,\quad i=1,\cdots,\frac{1}{2}N.\end{cases}\tag{13}$$

Given a seed solution $\boldsymbol{\psi}_0$ of (13), we can look for a new solution $\boldsymbol{\psi}$ in the form

$$\boldsymbol{\psi}(t,x,y,\eta_i^\alpha,\zeta)=\boldsymbol{\chi}(t,x,y,\eta_i^\alpha,\zeta)\boldsymbol{\psi}_0(t,x,y,\eta_i^\alpha,\zeta)\tag{14}$$

with the dressing factor

$$\boldsymbol{\chi}=\boldsymbol{I}_n+\sum_{\alpha=1}^{s}\sum_{k=1}^{m}\frac{\boldsymbol{R}_{\alpha k}}{(\zeta-\mu_k)^\alpha},\tag{15}$$

where the $\mu_k(t,x,y,\eta_i^\alpha)$ are complex functions and the $n\times n$ operator-valued matrices $\boldsymbol{R}_{\alpha k}$ are independent of ζ. If we take the seed solution $\boldsymbol{\psi}_0=\boldsymbol{I}_n$, the dressing factor $\boldsymbol{\chi}$ being restricted to containing only first-order poles in ζ and all the poles μ_k are complex constants with $\mathrm{Im}\mu_k<0$[1], then the dressing ansatz yields a multi-soliton solution

$$\boldsymbol{\psi}=\boldsymbol{I}_n+\sum_{k=1}^{m}\frac{\boldsymbol{R}_k}{\zeta-\mu_k},\tag{16}$$

where $\boldsymbol{R}_k$ are given via $n\times r$ operator-valued matrices $\boldsymbol{T}_k$.

The simplest case occurs when $\boldsymbol{\psi}$ has only one pole at $\zeta=-i$, so we drop the index k and obtain

$$\boldsymbol{\psi}=\boldsymbol{I}_n+\frac{\boldsymbol{R}}{\zeta+i}=:\boldsymbol{I}_n-\frac{2i}{\zeta+i}\boldsymbol{P}\quad\text{so that}\quad\boldsymbol{\Phi}=\boldsymbol{I}_n-2\boldsymbol{P}.\tag{17}$$

In this case all configurations are static and parametrized by a hermitian project $\boldsymbol{P}=\boldsymbol{T}(\boldsymbol{T}^{\dagger}\boldsymbol{T})^{-1}\boldsymbol{T}^{\dagger}$ which obeys

$$\begin{aligned}&(\boldsymbol{I}_n-\boldsymbol{P})\partial_z\boldsymbol{P}=0 \quad\Rightarrow\quad (\boldsymbol{I}_n-\boldsymbol{P})a\boldsymbol{T}=0,\\&(\boldsymbol{I}_n-\boldsymbol{P})\bar{\partial}_i\boldsymbol{P}=0 \quad\Rightarrow\quad (\boldsymbol{I}_n-\boldsymbol{P})\bar{\partial}_i\boldsymbol{T}=0,\end{aligned} \tag{18}$$

here we have used the abbreviations $\bar{\partial}_i:=\frac{1}{2}(\partial_1^i+i\partial_2^i)$. Clearly, this can be solved by

$$\boldsymbol{T}=\boldsymbol{T}(z,\eta^i) \quad \text{with} \quad \eta^i=\eta_i^1+i\eta_i^2, \quad i=1,\cdots,\frac{1}{2}N, \tag{19}$$

and the configurations are solitons when $\boldsymbol{T}$ depends on z rationally. The $n\times n$ matrix superfields $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}^i$ are expressed in terms of $\boldsymbol{P}$ as

$$\boldsymbol{A}=-2i\partial_x\boldsymbol{P}, \quad \boldsymbol{B}=2i\partial_y\boldsymbol{P}, \quad \text{and} \quad \boldsymbol{C}^i=-2i\partial_1^i\boldsymbol{P}. \tag{20}$$

By taking the static configuration ψ of (17) as a seed solution and considering a dressing factor $\boldsymbol{\chi}$ of the same form as $\boldsymbol{\psi}$, we have

$$\boldsymbol{\psi}\rightarrow\tilde{\boldsymbol{\psi}}=(\boldsymbol{I}_n-\frac{2i}{\zeta+i}\tilde{\boldsymbol{P}})(\boldsymbol{I}_n-\frac{2i}{\zeta+i}\boldsymbol{P})=\boldsymbol{I}_n-\frac{2i}{\zeta+i}(\boldsymbol{P}+\tilde{\boldsymbol{P}})-\frac{4}{(\zeta+i)^2}\tilde{\boldsymbol{P}}\boldsymbol{P}, \tag{21}$$

where $\tilde{\boldsymbol{P}}$ is some matrix to be determined, which is also a hermitian projector by the reality condition (12), so we can set $\tilde{\boldsymbol{P}}=\tilde{\boldsymbol{T}}(\tilde{\boldsymbol{T}}^{\dagger}\tilde{\boldsymbol{T}})^{-1}\tilde{\boldsymbol{T}}^{\dagger}$ with some $n\times\tilde{r}$ matrix $\tilde{\boldsymbol{T}}$. Obviously, the ansatz for $\tilde{\boldsymbol{\psi}}$ contains a second-order pole. Demanding that $\tilde{\boldsymbol{\psi}}$ is again a solution of the linear system (13) with some new superfields $\tilde{\boldsymbol{A}}$, $\tilde{\boldsymbol{B}}$ and $\tilde{\boldsymbol{C}}^i$, which are independent of ζ, we obtain the following equations

$$\begin{aligned}&(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})(\partial_z\tilde{\boldsymbol{T}}+(\partial_z\boldsymbol{P})\tilde{\boldsymbol{T}})=0,\\&(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})(\partial_t\tilde{\boldsymbol{T}}-2i(\partial_z\boldsymbol{P})\tilde{\boldsymbol{T}})=0,\\&(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})(\frac{1}{2}(\partial_1^i+i\partial_2^i)\tilde{\boldsymbol{T}}+(\partial_1^i\boldsymbol{P})\tilde{\boldsymbol{T}})=0.\end{aligned} \tag{22}$$

After constructing a projector $\tilde{\boldsymbol{P}}$ via a solution $\tilde{\boldsymbol{T}}$ of (22), we obtain a solution $\tilde{\boldsymbol{\psi}}$ of the linear system (13) with a double pole at $\zeta=-i$, and hence a new superfield satisfying the model (1) with nontrivial scattering.

Now we set out to construct soliton-antisoliton solutions of the model (1). As in the non-supersymmetric case[7], we take the static configuration $\boldsymbol{\psi}$ of (17) as a seed solution and choose the dressing factor $\boldsymbol{\chi}=\boldsymbol{I}_n+\frac{2i}{\zeta-i}\tilde{\boldsymbol{P}}$, then

$$\boldsymbol{\psi}\rightarrow\tilde{\boldsymbol{\psi}}=(\boldsymbol{I}_n+\frac{2i}{\zeta-i}\tilde{\boldsymbol{P}})(\boldsymbol{I}_n-\frac{2i}{\zeta+i}\boldsymbol{P}), \tag{23}$$

where $\tilde{\boldsymbol{P}}$ is also a hermitian projector by the reality condition (12), so we can set $\tilde{\boldsymbol{P}}=\tilde{\boldsymbol{T}}(\tilde{\boldsymbol{T}}^{\dagger}\tilde{\boldsymbol{T}})^{-1}\tilde{\boldsymbol{T}}^{\dagger}$ with some $n\times\tilde{r}$ matrix $\tilde{\boldsymbol{T}}$. Since $\tilde{\boldsymbol{\psi}}$ is again a solution of the linear system (13) with some new superfields $\tilde{\boldsymbol{A}}$, $\tilde{\boldsymbol{B}}$ and $\tilde{\boldsymbol{C}}^i$, we derive

$$\begin{aligned}&\tilde{\boldsymbol{A}}(t,x,y,\eta_i^\alpha)=\tilde{\boldsymbol{\psi}}(t,x,y,\eta_i^\alpha,\zeta)(\partial_u-\zeta\partial_x)[\tilde{\boldsymbol{\psi}}(t,x,y,\eta_i^\alpha,\bar{\zeta})]^{\dagger}=(\boldsymbol{I}_n+\frac{2i}{\zeta-i}\tilde{\boldsymbol{P}})(\boldsymbol{A}+(\partial_u-\zeta\partial_x))(\boldsymbol{I}_n-\frac{2i}{\zeta+i}\tilde{\boldsymbol{P}}),\\&\tilde{\boldsymbol{B}}(t,x,y,\eta_i^\alpha)=\tilde{\boldsymbol{\psi}}(t,x,y,\eta_i^\alpha,\zeta)(\partial_x-\zeta\partial_v)[\tilde{\boldsymbol{\psi}}(t,x,y,\eta_i^\alpha,\bar{\zeta})]^{\dagger}=(\boldsymbol{I}_n+\frac{2i}{\zeta-i}\tilde{\boldsymbol{P}})(\boldsymbol{B}+(\partial_x-\zeta\partial_v))(\boldsymbol{I}_n-\frac{2i}{\zeta+i}\tilde{\boldsymbol{P}}),\\&\tilde{\boldsymbol{C}}^i(t,x,y,\eta_i^\alpha)=\tilde{\boldsymbol{\psi}}(t,x,y,\eta_i^\alpha,\zeta)(\partial_2^i-\zeta\partial_1^i)[\tilde{\boldsymbol{\psi}}(t,x,y,\eta_i^\alpha,\bar{\zeta})]^{\dagger}=(\boldsymbol{I}_n+\frac{2i}{\zeta-i}\tilde{\boldsymbol{P}})(\boldsymbol{C}^i+(\partial_2^i-\zeta\partial_1^i))(\boldsymbol{I}_n-\frac{2i}{\zeta+i}\tilde{\boldsymbol{P}}).\end{aligned} \tag{24}$$

The poles at $\zeta=\pm i$ on the right-hand side have to be removable since $\tilde{\boldsymbol{A}}$, $\tilde{\boldsymbol{B}}$ and $\tilde{\boldsymbol{C}}^i$ are independent of ζ. Putting to zero the corresponding residues, we obtain

$$\begin{aligned}&(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})(\partial_z\tilde{\boldsymbol{P}}-(\partial_z\boldsymbol{P})\tilde{\boldsymbol{P}})=0,\\&(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})(\partial_t\tilde{\boldsymbol{P}}-2i(\partial_z\boldsymbol{P})\tilde{\boldsymbol{P}})=0,\\&(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})(\frac{1}{2}(\partial_1^i-i\partial_2^i)\tilde{\boldsymbol{P}}-(\partial_1^i\boldsymbol{P})\tilde{\boldsymbol{P}})=0.\end{aligned} \tag{25}$$

With the help of the identities

$$(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})\tilde{\boldsymbol{P}}\equiv 0 \quad \text{and} \quad (\boldsymbol{I}_n-\tilde{\boldsymbol{P}})\tilde{\boldsymbol{T}}\equiv 0, \tag{26}$$

the equations (25) can be reduced to

$$\begin{aligned}
&(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})(\partial_z\tilde{\boldsymbol{T}}-(\partial_z\boldsymbol{P})\tilde{\boldsymbol{T}})=0,\\
&(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})(\partial_t\tilde{\boldsymbol{T}}-2i(\partial_z\boldsymbol{P})\tilde{\boldsymbol{T}})=0,\\
&(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})(\frac{1}{2}(\partial_1^i-i\partial_2^i)\tilde{\boldsymbol{T}}-(\partial_1^i\boldsymbol{P})\tilde{\boldsymbol{T}})=0.
\end{aligned} \tag{27}$$

Using the operator formalism notations, (27) can be written in the form

$$\begin{aligned}
&(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})(a^\dagger\tilde{\boldsymbol{T}}-[a^\dagger,\boldsymbol{P}]\tilde{\boldsymbol{T}})=0,\\
&(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})(\partial_t\tilde{\boldsymbol{T}}-i\sqrt{\frac{2}{\theta}}[a,\boldsymbol{P}]\tilde{\boldsymbol{T}})=0,\\
&(\boldsymbol{I}_n-\tilde{\boldsymbol{P}})(\frac{1}{2}(\partial_1^i-i\partial_2^i)\tilde{\boldsymbol{T}}-(\partial_1^i\boldsymbol{P})\tilde{\boldsymbol{T}})=0.
\end{aligned} \tag{28}$$

Obviously, a sufficient condition for a solution is

$$\begin{aligned}
&a^\dagger\tilde{\boldsymbol{T}}-[a^\dagger,\boldsymbol{P}]\tilde{\boldsymbol{T}}=\tilde{\boldsymbol{T}}\boldsymbol{Z}_1,\\
&\partial_t\tilde{\boldsymbol{T}}-i\sqrt{\frac{2}{\theta}}[a,\boldsymbol{P}]\tilde{\boldsymbol{T}}=\tilde{\boldsymbol{T}}\boldsymbol{Z}_2,\\
&\frac{1}{2}(\partial_1^i-i\partial_2^i)\tilde{\boldsymbol{T}}-(\partial_1^i\boldsymbol{P})\tilde{\boldsymbol{T}}=\tilde{\boldsymbol{T}}\boldsymbol{Z}_3,
\end{aligned} \tag{29}$$

with some operator-valued superfields $\boldsymbol{Z}_k(t,a,a^\dagger,\eta^i)$ for $k=1,2,3$. After constructing a projector $\tilde{\boldsymbol{P}}$ via a solution $\tilde{\boldsymbol{T}}$ of the equations (29), we can obtain a solution $\tilde{\boldsymbol{\psi}}$ of the linear system (13) and hence a new superfield satisfying the model (1), which is a soliton-antisoliton solution by our ansatz.

3 Explicit Soliton-Antisoliton Solutions

In order to generate some explicit examples of nonabelian soliton-antisoliton solutions, we specialize to the group $\boldsymbol{U}(2)$ and choose $\boldsymbol{Z}_1=a^\dagger, \boldsymbol{Z}_2=\boldsymbol{Z}_3=0$, thus (29) is reduced to

$$\begin{aligned}
&[a^\dagger,\tilde{\boldsymbol{T}}]-[a^\dagger,\boldsymbol{P}]\tilde{\boldsymbol{T}}=0,\\
&\partial_t\tilde{\boldsymbol{T}}-i\sqrt{\frac{2}{\theta}}[a,\boldsymbol{P}]\tilde{\boldsymbol{T}}=0,\\
&\frac{1}{2}(\partial_1^i-i\partial_2^i)\tilde{\boldsymbol{T}}-(\partial_1^i\boldsymbol{P})\tilde{\boldsymbol{T}}=0.
\end{aligned} \tag{30}$$

Moreover, we take as a seed configuration the simplest nontrivial solution of (18), that is,

$$\boldsymbol{P}=\boldsymbol{T}(\boldsymbol{T}^\dagger\boldsymbol{T})^{-1}\boldsymbol{T}^\dagger \quad \text{with} \quad \boldsymbol{T}=\begin{pmatrix}1\\ f(z,\eta^i)\end{pmatrix}, \tag{31}$$

where f depends on z rationally. Inspired by the known form of $\tilde{\boldsymbol{T}}$ in the non-supersymmetric case[7], we make the ansatz

$$\tilde{\boldsymbol{T}}=\boldsymbol{T}_\perp+\boldsymbol{T}(\boldsymbol{T}^\dagger\boldsymbol{T})^{-1}g \quad \text{with} \quad \boldsymbol{T}_\perp=\begin{pmatrix}\overline{f(z,\eta^i)}\\ -1\end{pmatrix} \tag{32}$$

being orthogonal to $\boldsymbol{T}$, i. e.,

$$\boldsymbol{T}^\dagger\boldsymbol{T}_\perp=0 \quad \Rightarrow \quad \boldsymbol{P}\boldsymbol{T}_\perp=0 \quad \text{and} \quad \boldsymbol{I}_2-\boldsymbol{P}=\boldsymbol{T}_\perp(\boldsymbol{T}_\perp^\dagger\boldsymbol{T}_\perp)^{-1}\boldsymbol{T}_\perp^\dagger, \tag{33}$$

where $g=g(t,z,\bar{z},\eta^i,\bar{\eta}^i)$ is an operator-valued superfield to be determined.

Substituting our ansatz into the first equation of (30), we get $[a^\dagger,g]=0$, which means that

$$g=g(t,\bar{z},\eta^i,\bar{\eta}^i)$$

does not depend on z. From the second equation of (30) we obtain $\partial_t g=-i\sqrt{\frac{2}{\theta}}[a,\bar{f}]$, which implies

$$g=-i\sqrt{\frac{2}{\theta}}\left(t[a,\bar{f}]+h(\bar{z},\eta^i,\bar{\eta}^i)\right).$$

From the third one we obtain $\partial_i g=-\partial_i f$, here we have used the abbreviations $\partial_i:=\frac{1}{2}(\partial_1^i-i\partial_2^i)$, thus g is determined by

$$g=-i\sqrt{\frac{2}{\theta}}\left(t[a,\bar{f}]+\tilde{h}(\bar{z},\bar{\eta}^i)\right)-\eta^i\partial_i f \tag{34}$$

with an arbitrary function $\tilde{h}(\bar{z},\bar{\eta}^i)$ of $\bar{z}$ and $\bar{\eta}^i$. Therefore, we obtain an explicit solution

$$\tilde{\boldsymbol{T}}=(\bar{f}-1)+\begin{pmatrix}1\\ f\end{pmatrix}(1+\bar{f}f)^{-1}\left(-i\sqrt{\frac{2}{\theta}}\left(t[a,\bar{f}]+\tilde{h}(\bar{z},\bar{\eta}^i)\right)-\eta^i\partial_i f\right) \tag{35}$$

of the equations (30).

Depending on the explicit forms of $f(z,\eta^i)$ and $\tilde{h}(\bar{z},\bar{\eta}^i)$, the solutions (35) describe different kinds of soliton-antisoliton configurations. The scattering analysis of these supersymmetric configurations seems much more complicated even when $N=2$ and we postpone it to future work. Here, we restrict ourselves only to a bosonic subsector[7]. For this purpose, we expand $f(z,\eta^i)$ in η^i and $\tilde{h}(\bar{z},\bar{\eta}^i)$ in $\bar{\eta}^i$ respectively,

$$f(z,\eta^i)=f_0(z)+\eta^i f_i(z)+\cdots \quad \text{and} \quad \tilde{h}(\bar{z},\bar{\eta}^i)=h_0(\bar{z})+\bar{\eta}^i h_i(\bar{z})+\cdots \tag{36}$$

One choice is $f_0(z)=z$, $h_0(\bar{z})=\frac{1}{2}\bar{z}^2$. For large $r^2\equiv z\bar{z}$, the corresponding bosonic subsector energy density configuration has two lumps accelerating symmetrically towards each other along the x-axis, interacting at the origin(near $t=0$) and decelerating to infinity along the y-axis. Thus a head-on collision of one soliton and one antisoliton results a 90° angle scattering.

Another choice $f_0(z)=z$, $h_0(\bar{z})=\frac{1}{2}\bar{z}^3$ yields, for large $r^2\equiv z\bar{z}$, a configuration with one soliton and two antisolitons, which come together(near $t=0$) forming a bell-like structure and then emerge at an angle of 60° with respect to the original direction.

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