

Fuzzy Bayesian Estimation for Parameters Based on Normal Distribution

Tang Shengdao, Yin Shimao

(School of Mathematics and Physics, Anhui University of Technology, Ma'anshan 243002, China)

Abstract: Lifetime data is important to parameter estimation. Conventional estimation needs a lot of precise data. In real world, however, the collected data is often imprecise or vague due to all kinds of factors. Thus, the fuzzy Bayesian estimation is just essential. In this paper, the fuzzy bayesian estimators of two parameters based on normal distribution are given by combining fuzzy sets theory with Bayesian estimation approach. In the final section, a numerical example is used to demonstrate the proposed approach.

Key words: Bayesian estimation, fuzzy lifetime, membership functions, normal distribution

CLC number: O213.2 **Document code:** A **Article ID:** 1001-4616(2015)01-0013-08

正态分布下参数的模糊贝叶斯估计

汤胜道, 殷世茂

(安徽工业大学数理学院, 安徽 马鞍山 243002)

[摘要] 在参数估计中, 寿命数据是非常重要的. 传统的估计是基于完全精确的寿命数据. 然而, 在实际中由于种种原因, 有时收集的数据往往是不精确的. 这样, 参数的模糊估计方法就十分必要. 本文将贝叶斯估计方法与模糊集理论相结合, 给出了正态总体中两参数的模糊贝叶斯估计. 最后, 用一个数值例子演示了本文的方法.

[关键词] 贝叶斯估计, 模糊寿命, 隶属函数, 正态分布

Classical (or conventional) parameter estimations are based on a large amount of precise lifetime data. However, in real world, sometimes, the available lifetime data is not only limited, but also imprecise due to all kinds of factors, in this case, it is impossible to estimating lifetime distribution parameters by conventional statistical analysis methods. Fuzzy Bayesian approach has been developed to handle such difficulties. Wu^[1,2] discuss fuzzy Bayesian estimation to parameters based on binomial distribution and pascal distribution, and give the algorithm for α -level sets of these fuzzy Bayes estimators. Recently, fuzzy Bayesian approach has had some successful applications^[3,4]. Up to now, there have been many literatures related to fuzzy Bayesian estimation^[1-7], however, the most of them are limited to single parameter distribution, such as exponential, binomial, and poisson distributions. Although Huang^[8,9] discuss the fuzzy Bayesian estimation to parameters based on normal distribution by nervous net technology, they cannot give the analytical formulae of fuzzy Bayes estimators with respect to two parameters μ and σ^2 . The main reason is that the prior and posterior distributions of the two parameters are difficult to give. In this paper, the analytical formulae of fuzzy Bayes estimators with respect to the two parameters are given by combining fuzzy set theory with Bayesian estimation.

1 Bayesian Approach

Let X is a continuous random variable, $X \sim f(x, \theta) \theta \in \Theta$ with continuous parameter space Θ . From Bayesian

Received data: 2014-01-29.

Foundation item: Supported by the National Natural Science Foundation of China (31300125).

Corresponding author: Tang Shengdao, Ph. D, professor, majored in reliability theory. E-mail: shdtang@ahut.edu.cn

point of view, θ is also a random variable, the distribution followed by θ is called as prior distribution of θ , denoted by $\pi(\theta)$. For n precise sample observations x_1, x_2, \dots, x_n . According to the Bayesian theory, the conditional distribution

$$\pi(\theta|x_1, x_2, \dots, x_n) = \frac{L(x_1, x_2, \dots, x_n, \theta) \pi(\theta)}{\int_{\Theta} L(x_1, x_2, \dots, x_n, \theta) \pi(\theta) d\theta}, \quad \forall \theta \in \Theta$$

is called as posterior distribution of parameter θ .

There $L(x_1, x_2, \dots, x_n, \theta) = \prod_{i=1}^n f(x_i|\theta)$, $\forall \theta \in \Theta$ is the likelihood function of the observations with given parameter θ . The Bayes point estimator of parameter θ is the mean of the posterior distribution under a square error loss function, that is, $\hat{\theta} = E(\theta|x_1, x_2, \dots, x_n)$.

Mainly because of its use of subjective prior beliefs, the approach to statistical inference based on the Bayes' theorem has been controversial within the field of statistics for many years. It is important to note that the prior distribution cannot be specified arbitrarily and it should based on prior information. Several effective methods have been reported to determine prior distribution, such as non-informative priors, conjugate priors, Jeffreys' priors, bootstrap priors, and random weight priors^[10-14].

2 Fuzzy Set Theory

2.1 Fuzzy real numbers and fuzzy random variables

Definition 1^[15] Let U is universe of discourse, \tilde{x} is a fuzzy subset on the U , variable x belongs to \tilde{x} in some degree $\zeta(\zeta \in (0, 1))$, then ζ is called as membership function of the fuzzy subset \tilde{x} , denoted as $\zeta_{\tilde{x}}(\cdot)$.

Let \tilde{x} is a fuzzy subset of R with membership function $\zeta_{\tilde{x}}(\cdot)$, then \tilde{x} is called a fuzzy real number iff satisfying following two conditions^[15]

(1) \tilde{x} is a convex fuzzy set, α -level set of \tilde{x} is given by $\tilde{x}_{\alpha} = \{x: \zeta_{\tilde{x}}(x) \geq \alpha\}$ for any $\alpha \in [0, 1]$, and \tilde{x}_{α} is a bounded closed interval.

(2) \tilde{x} is a normal fuzzy set, ie. there exists unique $x \in R$, such that $\zeta_{\tilde{x}}(x) = 1$.

Definition 2^[1] Assume that Ω is a sample space, Ψ_R is the set of all fuzzy real numbers, let $\tilde{X}: \Omega \rightarrow \Psi_R$ is fuzzy valued function, \tilde{X} is called a fuzzy random variable, iff \tilde{X} is measurable.

According to the definition of the fuzzy real numbers, \tilde{x}_{α} is a bounded closed interval, for $\forall \alpha \in [0, 1]$, it is denoted by $[\tilde{x}_{\alpha}^L, \tilde{x}_{\alpha}^U]$.

Proposition 1^[16] Let $\tilde{X}: \Omega \rightarrow \Psi_R$ is a fuzzy valued function, then \tilde{X} is a fuzzy random variable, if and only if both \tilde{X}_{α}^L and \tilde{X}_{α}^U are all (conventional) random variables, for $\alpha \in [0, 1]$.

Proposition 1 is obviously holded.

2.2 Fuzzy Bayes point estimators

For the sake of convinence, it is without loss of generality to assume that θ is a single parameter. Let \tilde{X} is a fuzzy random variable with fuzzy parameter $\tilde{\theta}$, from Proposition 1, we can say that $\tilde{\theta}_{\alpha}^L$ and $\tilde{\theta}_{\alpha}^U$ are the parameters of the random variables \tilde{X}_{α}^L and \tilde{X}_{α}^U , respectively, for $\alpha \in [0, 1]$. Our purpose is to estimate fuzzy parameter $\tilde{\theta}$ under fuzzy enviroments. In order to apply the Bayesian approach, the fuzzy parameter $\tilde{\theta}$ is assumed as a fuzzy random variable, according to Proposition 1, both $\tilde{\theta}_{\alpha}^L$ and $\tilde{\theta}_{\alpha}^U$ are all (conventional) random variables. Since $\tilde{\theta}$ is a fuzzy real number, we see that $\tilde{\theta}_{\alpha}^L$ and $\tilde{\theta}_{\alpha}^U$ are continuous with respect to α by definition, therefore the closed intervals $[\tilde{\theta}_{\alpha}^L, \tilde{\theta}_{\alpha}^U]$, for $\alpha \in [0, 1]$, are continuously shrinking with respect to α . Then for any parameter $\theta \in [\tilde{\theta}_{\alpha}^L, \tilde{\theta}_{\alpha}^U]$, we have, $\theta = \tilde{\theta}_{\beta}^L$ or $\theta = \tilde{\theta}_{\beta}^U$ for some $\beta \geq \alpha$, since $\tilde{\theta}_1^L = \tilde{\theta}_1^U$, thus for any parameter $\theta \in [\tilde{\theta}_{\alpha}^L, \tilde{\theta}_{\alpha}^U]$ we can find a Bayes point estimator of θ , denoted by $\hat{\theta}$.

Let

$$A_\alpha = [\min\{\inf_{\alpha \leq \beta \leq 1} \hat{\theta}_\beta^L, \inf_{\alpha \leq \beta \leq 1} \hat{\theta}_\beta^U\}, \max\{\sup_{\alpha \leq \beta \leq 1} \hat{\theta}_\beta^L, \sup_{\alpha \leq \beta \leq 1} \hat{\theta}_\beta^U\}].$$

Then this interval will contain all of the Bayes point estimators for each $\theta \in [\hat{\theta}_\alpha^L, \hat{\theta}_\alpha^U]$.

The fuzzy Bayes point estimator of Fuzzy parameter $\tilde{\theta}$ is denoted by $\hat{\tilde{\theta}}, \hat{\tilde{\theta}}$ with membership function^[15]

$$\xi_{\tilde{\theta}}(r) = \sup_{0 \leq \alpha \leq 1} \alpha \cdot \chi_{A_\alpha}(r).$$

There $\chi_{A_\alpha}(r)$ is characteristic function of A_α , that is,

$$\chi_{A_\alpha}(r) = \begin{cases} 1, & \text{if } r \in A_\alpha, \\ 0, & \text{if } r \notin A_\alpha. \end{cases}$$

3 The Bayes Point Estimators of the Parameters Based on Normal Distribution

In this section, we will discuss the point estimators of the two parameters based on normal distribution.

Definition 3^[17] Assume that random variable X follows Gamma distribution $Ga(w, \lambda)$, then say $Y = X^{-1}$ follows inverse Gamma distribution, denoted by $Y \sim IG(w, \lambda)$ its probability density function (p.d.f.) is given by

$$f(y, w, \lambda) = \frac{\lambda^w}{\Gamma(w)} \left(\frac{1}{y}\right)^{w-1} e^{-\frac{\lambda}{y}} \frac{1}{y^2} = \frac{\lambda^w}{\Gamma(w)} y^{-(w+1)} e^{-\frac{\lambda}{y}}, y > 0.$$

Definition 4^[17] Assume that the p.d.f. of random variable Y is given by

$$f(y, \nu, u, \tau) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{\nu\pi}\tau} \left[1 + \frac{1}{\nu} \left(\frac{y-u}{\tau}\right)^2\right]^{-\frac{\nu+1}{2}}, y \in \mathbf{R},$$

then, call Y follows general t distribution, there ν is called degrees of freedom, $u \in \mathbf{R}$ is referred to as location parameter, $\tau \in \mathbf{R}^+$ is called as scale parameter.

Proposition 2 Assume that random variable Y follows general t distribution, the p.d.f. is given by

$$f(y, \nu, u, \tau) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{\nu\pi}\tau} \left[1 + \frac{1}{\nu} \left(\frac{y-u}{\tau}\right)^2\right]^{-\frac{\nu+1}{2}}, y \in \mathbf{R},$$

then

$$EY = u.$$

Proof

$$EY = \int_{-\infty}^{+\infty} y f(y, \nu, u, \tau) dy.$$

Let

$$\frac{y-u}{\tau} = t,$$

we have

$$EY = \int_{-\infty}^{+\infty} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{\nu\pi}} (\tau t + u) \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} dt = u \int_{-\infty}^{+\infty} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{\nu\pi}\tau} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} dt = u \int_{-\infty}^{+\infty} f(t, \nu) dt = u.$$

Because

$$f(t, \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{\nu\pi}} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

is the p.d.f. of the t distribution with degrees of freedom ν .

Definition 5^[17] Distribution family $\{f_{\theta}(x), \theta \in \Theta\}$ is called Cramér-Rao normal family, if satisfying

- (1) Θ is an open rectangle on R^k ;
- (2) $\frac{\partial \ln f_{\theta}(x)}{\partial \theta_i}, i=1, 2, \dots, k$, for $\theta \in \Theta$ are all existed;
- (3) support $A = \{x: f_{\theta}(x) > 0\}$ is independent on θ ;
- (4) the integration and the differentiation to $f_{\theta}(x)$ are exchangable;
- (5) $E \left| \frac{\partial \ln f_{\theta}(x)}{\partial \theta_i} \cdot \frac{\partial \ln f_{\theta}(x)}{\partial \theta_j} \right| < \infty$, for $1 \leq i, j \leq k, \forall \theta \in \Theta$. if $i=j$, then $\left| \frac{\partial \ln f_{\theta}(x)}{\partial \theta_i} \cdot \frac{\partial \ln f_{\theta}(x)}{\partial \theta_j} \right| = \left(\frac{\partial \ln f_{\theta}(x)}{\partial \theta_i} \right)^2$.

Definition 6^[17] Let $I(\theta)$ is Fisher information matrix with respect to parameter θ , then non-information prior density function $\pi(\theta) = [\det I(\theta)]^{1/2}$ is called Jeffreys prior distribution of the parameter θ .

Now we discuss the point estimations to parameters μ and σ^2 based on normal distribution $X \sim N(\mu, \sigma^2)$. In order to apply Bayes' approach, μ and σ^2 are assumed as two random variables. Firstly, we will determine their prior distributions. Because normal distribution family is Cramér-Rao normal family, the Fisher information is existed, thereby, we can apply Jeffreys' priors.

Let x_1, x_2, \dots, x_n are n sample observations come from a normal population $N(\mu, \sigma^2)$, then logarithmic likelihood function is given by

$$\ell(\mu, \sigma^2, x_1, x_2, \dots, x_n) = -\frac{1}{2} \ln(2\pi) - n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2.$$

The Fisher information matrix is given by

$$I(\mu, \sigma^2) = \begin{bmatrix} E\left(-\frac{\partial^2 \ell}{\partial \mu^2}\right) & E\left(-\frac{\partial^2 \ell}{\partial \mu \partial \sigma^2}\right) \\ E\left(-\frac{\partial^2 \ell}{\partial \mu \partial \sigma^2}\right) & E\left(-\frac{\partial^2 \ell}{\partial \sigma^4}\right) \end{bmatrix} = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{bmatrix}.$$

The determinant of the information matrix is given by

$$\det I(\mu, \sigma^2) = \frac{n^2}{2} \sigma^{-6}.$$

Thus, the Jeffreys joint prior distribution of parameters μ and σ^2 is given by

$$\pi(\mu, \sigma^2) \propto \sigma^{-3}.$$

There are a few special cases:

- (i) if σ^2 is given, then $I(\mu) = \frac{n}{\sigma^2}, \pi(\mu) = 1$;
- (ii) if μ is given, then $I(\sigma^2) = E\left(-\frac{\partial^2 \ell}{\partial \sigma^4}\right) = \frac{n}{2\sigma^4}, \pi(\sigma^2) = \frac{1}{\sigma^2}$;
- (iii) if μ and σ^2 are independent each other, then $\pi(\mu, \sigma^2) = \frac{1}{\sigma^2}$.

Jeffreys' priors show that the non-informative prior distribution of μ and σ^2 are dependent, the ultimate shape recommended by Jeffreys is $\pi(\mu, \sigma^2) = \frac{1}{\sigma^2}$. Most of people adopt the ultimate recommendation.

In this paper, the ultimate shape is also adopted, that is,

$$\pi(\mu, \sigma^2) = \frac{1}{\sigma^2}.$$

Easily, the joint posterior distribution of μ and σ^2 can be given by

$$\pi(\mu, \sigma^2 | \bar{x}, s) \propto \sigma^{-(n+2)} \exp \left\{ -\frac{(n-1)s^2 + n(\bar{x} - \mu)^2}{2\sigma^2} \right\},$$

there

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i, s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2,$$

in fact, the joint conditional distribution with respect to \bar{x}, s is given by

$$\begin{aligned} f(x_1, x_2, \dots, x_n | \mu, \sigma^2) &= (2\pi)^{-n/2} \sigma^{-n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \right\} = \\ &= (2\pi)^{-n/2} \sigma^{-n} \exp \left\{ -\frac{1}{2\sigma^2} [n(\bar{x} - \mu)^2 + (n-1)s^2] \right\} \triangleq f(\bar{x}, s | \mu, \sigma^2). \end{aligned}$$

Then, the joint distribution is given by

$$f(\bar{x}, s, \mu, \sigma^2) = f(\bar{x}, s | \mu, \sigma^2) \cdot \pi(\mu, \sigma^2) = (2\pi)^{-n/2} \sigma^{-(n+2)} \exp \left\{ -\frac{1}{2\sigma^2} [n(\bar{x} - \mu)^2 + (n-1)s^2] \right\}.$$

The marginal distribution with respect to \bar{x}, s^2 is given by

$$f(\bar{x}, s) = \int_0^{+\infty} d\sigma^2 \int_{-\infty}^{+\infty} f(\bar{x}, s, \mu, \sigma^2) d\mu.$$

Thereby, the joint posterior distribution of μ and σ^2 is given by

$$\pi(\mu, \sigma^2 | \bar{x}, s) = \frac{f(\bar{x}, s, \mu, \sigma^2)}{f(\bar{x}, s)} = (2\pi)^{-n/2} \sigma^{-(n+2)} \exp \left\{ -\frac{1}{2\sigma^2} [n(\bar{x} - \mu)^2 + (n-1)s^2] \right\} / f(\bar{x}, s).$$

Obviously, the joint posterior distribution of μ and σ^2 can be given by

$$\pi(\mu, \sigma^2 | \bar{x}, s) \propto \sigma^{-(n+2)} \exp \left\{ -\frac{(n-1)s^2 + n(\bar{x} - \mu)^2}{2\sigma^2} \right\},$$

thereby, the marginal distribution with respect to μ is given by

$$\pi(\mu | \bar{x}, s) \propto \int_0^{+\infty} (\sigma^2)^{-\frac{n}{2}-1} \exp \left\{ -\frac{[(n-1)s^2 + n(\bar{x} - \mu)^2]}{2\sigma^2} \right\} d\sigma^2.$$

We see that the integrand just is the kernel of the inverse Gamma distribution $IG(w, \lambda)$, $w = n/2, \lambda = [(n-1)s^2 + n(\bar{x} - \mu)^2]/2$, using the normality, we have

$$\pi(\mu | \bar{x}, s) \propto [(n-1)s^2 + n(\mu - \bar{x})^2]^{-\frac{n}{2}} \propto \left[1 + \frac{1}{n-1} \left(\frac{\mu - \bar{x}}{s/\sqrt{n}} \right)^2 \right]^{-\frac{(n-1)+1}{2}}.$$

Note that $\left[1 + \frac{1}{n-1} \left(\frac{\mu - \bar{x}}{s/\sqrt{n}} \right)^2 \right]^{-\frac{(n-1)+1}{2}}$ is the kernel of general t distribution, according to Proposition 2, the Bayes point estimator of μ under a square error loss function is given by

$$\hat{\mu} = E(\mu | \bar{x}, s) = \bar{x},$$

where $\hat{\mu}$ is same with conventional point estimator of μ .

Likely, the marginal distribution with respect to σ^2 is given by

$$\pi(\sigma^2 | \bar{x}, s) \propto \sigma^{-(n+2)} \exp \left\{ -s^2/2\sigma^2 \right\} \int_{-\infty}^{+\infty} \exp \left\{ -n(\mu - \bar{x})^2/2\sigma^2 \right\} d\mu.$$

We find that it is the kernel of inverse Gamma distribution $IG(w, \lambda)$, $w = \frac{n-1}{2}, \lambda = s^2/2$, consequently, the posterior expectation of σ^2 is given by

$$E(\sigma^2 | \bar{x}, s) = \frac{\lambda}{w-1} = \frac{s^2}{n-3}.$$

Thus, the Bayes point estimator of σ^2 under the square error loss function is given by

$$\hat{\sigma}^2 = \frac{s^2}{n-3}.$$

For n imprecise sample observations x_1, x_2, \dots, x_n , we regard them as n fuzzy real numbers $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$, now we will give the fuzzy Bayes estimators of the fuzzy parameters $\tilde{\mu}$ and $\tilde{\sigma}^2$. From the preceding discussing, we have

$$\tilde{\mu}_\alpha = [\tilde{\mu}_\alpha^L, \tilde{\mu}_\alpha^U], \quad \tilde{\sigma}_\alpha^2 = [\tilde{\sigma}_\alpha^{2L}, \tilde{\sigma}_\alpha^{2U}], \quad \text{for } \alpha \in [0, 1].$$

The Bayes point estimators of these parameters are given by

$$\hat{\mu}_{\alpha}^L = \bar{x}_{\alpha}^L, \quad \hat{\mu}_{\alpha}^U = \bar{x}_{\alpha}^U, \quad \hat{\sigma}_{\alpha}^{2L} = \frac{\bar{s}_{\alpha}^{2L}}{n-3}, \quad \hat{\sigma}_{\alpha}^{2U} = \frac{\bar{s}_{\alpha}^{2U}}{n-3}.$$

As a result, the fuzzy Bayes estimators of the fuzzy parameters $\tilde{\mu}$ and $\tilde{\sigma}^2$ are given by, respectively,

$$\begin{aligned} \hat{\mu}_{\alpha} &= [\bar{x}_{\alpha}^L, \bar{x}_{\alpha}^U], \quad \text{for } \alpha \in [0, 1], \\ \hat{\sigma}_{\alpha}^2 &= \left[\frac{\bar{s}_{\alpha}^{2L}}{n-3}, \frac{\bar{s}_{\alpha}^{2U}}{n-3} \right], \quad \text{for } \alpha \in [0, 1]. \end{aligned}$$

4 Numerical Analysis

In this section, we introduce a special kind of fuzzy number \tilde{x} , we say that \tilde{x} is a triangle fuzzy number, if its membership function is given by

$$\zeta_{\tilde{x}} = \begin{cases} (r-a_1)/(a_2-a_1), & \text{if } a_1 \leq r \leq a_2, \\ (a_3-r)/(a_3-a_2), & \text{if } a_2 < r \leq a_3, \\ 0, & \text{orthwise.} \end{cases}$$

The fuzzy number are often used by [12]. We denote by

$$\tilde{x} = (a_1, a_2, a_3).$$

Then, \tilde{x} is a fuzzy real number, for any $\alpha \in [0, 1]$, its α -level set is given by

$$\tilde{x}_{\alpha} = [(a_2-a_1)\alpha+a_1, (a_2-a_3)\alpha+a_3].$$

We consider a normal population $X \sim N(\mu, \sigma^2)$, the lifetime data cannot be exactly recordered dut to the homan errors, or machine errors and some unexpected situations, then the n sample observations are regarded as n fuzzy real numbers $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$, their values are given by

$$(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) = (\widetilde{20}, \widetilde{30}, \widetilde{40}, \widetilde{50}).$$

Where n equals 4, and

$$\widetilde{20} = (15, 20, 25), \quad \widetilde{30} = (25, 30, 35), \quad \widetilde{40} = (35, 40, 45), \quad \widetilde{50} = (45, 50, 55),$$

the α -level sets with respect to all 4 fuzzy real numbers are given differently by

$$\widetilde{20}_{\alpha} = [15+5\alpha, 25-5\alpha], \quad \widetilde{30}_{\alpha} = [25+5\alpha, 35-5\alpha], \quad \widetilde{40}_{\alpha} = [35+5\alpha, 45-5\alpha], \quad \widetilde{50}_{\alpha} = [45+5\alpha, 55-5\alpha],$$

for $\alpha \in [0, 1]$.

In order to give out the fuzzy Bayes estimators of parameters μ and σ^2 , Lemma 1 is necessary.

Lemma 1^[15] Assume that $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ are two interval numbers, $*$ $\in \{+, -, \times, \div, \vee, \wedge\}$, and I_2 cannot be zero if $*$ implies \div , then $I_1 * I_2$ is also interval number, and

$$\begin{aligned} I_1 + I_2 &= [a_1 + a_2, b_1 + b_2], \quad I_1 - I_2 = [a_1 - b_2, b_1 - a_2], \\ I_1 \vee I_2 &= [a_1 \vee a_2, b_1 \vee b_2], \quad I_1 \wedge I_2 = [a_1 \wedge a_2, b_1 \wedge b_2], \\ I_1 \times I_2 &= [c, d], \text{ where } c = \min\{a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2\}, d = \max\{a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2\}, \\ I_1 \div I_2 &= \left[\frac{1}{b_2}, \frac{1}{a_2} \right]. \end{aligned}$$

According to Lamma 1 and preceding interval numbers, we can get

$$\bar{x}_{\alpha}^L = \frac{1}{4} [(15+5\alpha) + (25+5\alpha) + (35+5\alpha) + (45+5\alpha)] = 30+5\alpha,$$

$$\bar{x}_{\alpha}^U = \frac{1}{4} [(25-5\alpha) + (35-5\alpha) + (45-5\alpha) + (55-5\alpha)] = 40-5\alpha,$$

$$\bar{\tilde{x}}_{\alpha} = [30+5\alpha, 40-5\alpha],$$

$$\tilde{x}_{1\alpha} - \bar{\tilde{x}}_{\alpha} = [10\alpha-25, -5-10\alpha],$$

$$\begin{aligned}
 \tilde{x}_{2\alpha} - \bar{\tilde{x}}_{\alpha} &= [10\alpha - 15, 5 - 10\alpha], \\
 \tilde{x}_{3\alpha} - \bar{\tilde{x}}_{\alpha} &= [10\alpha - 5, 15 - 10\alpha], \\
 \tilde{x}_{4\alpha} - \bar{\tilde{x}}_{\alpha} &= [10\alpha + 5, 25 - 10\alpha], \\
 (\tilde{x}_{1\alpha} - \bar{\tilde{x}}_{\alpha})^2 &= [\min \{25 + 100\alpha + 100\alpha^2, 125 + 200\alpha - 100\alpha^2, 625 - 500\alpha + 100\alpha^2\}; \\
 &\quad \max \{25 + 100\alpha + 100\alpha^2, 125 + 200\alpha - 100\alpha^2, 625 - 500\alpha + 100\alpha^2\}] \triangleq \tilde{y}_{1\alpha}, \\
 (\tilde{x}_{2\alpha} - \bar{\tilde{x}}_{\alpha})^2 &= [\min \{25 - 100\alpha + 100\alpha^2, -75 + 200\alpha - 100\alpha^2, 225 - 300\alpha + 100\alpha^2\}; \\
 &\quad \max \{25 - 100\alpha + 100\alpha^2, -75 + 200\alpha - 100\alpha^2, 225 - 300\alpha + 100\alpha^2\}] \triangleq \tilde{y}_{2\alpha}.
 \end{aligned}$$

Likely,

$$(\tilde{x}_{3\alpha} - \bar{\tilde{x}}_{\alpha})^2 = \tilde{y}_{2\alpha} \triangleq \tilde{y}_{3\alpha}, (\tilde{x}_{4\alpha} - \bar{\tilde{x}}_{\alpha})^2 = \tilde{y}_{1\alpha} \triangleq \tilde{y}_{4\alpha},$$

thereby

$$\tilde{s}_{\alpha}^2 = \frac{\tilde{y}_{1\alpha} + \tilde{y}_{2\alpha} + \tilde{y}_{3\alpha} + \tilde{y}_{4\alpha}}{3} = \frac{2}{3}(\tilde{y}_{1\alpha} + \tilde{y}_{2\alpha}).$$

Thus, the fuzzy Bayes estimator of μ is given by

$$\hat{\mu}_{\alpha} = \bar{\tilde{x}}_{\alpha} = [30 + 5\alpha, 40 - 5\alpha].$$

It imply that the confidence interval with confidence degree α with respect to μ is given by $[30 + 5\alpha, 40 - 5\alpha]$. The fuzzy Bayes estimator of σ^2 is given by

$$\hat{\sigma}_{\alpha}^2 = \frac{\tilde{s}_{\alpha}^2}{n-3} = \tilde{s}_{\alpha}^2 = \frac{2}{3}(\tilde{y}_{1\alpha} + \tilde{y}_{2\alpha}).$$

Especially, if the fuzzy real numbers are all real numbers, then, membership degree α equals 1,

$$\hat{\mu}_{\alpha} = \hat{\mu} = \bar{x} = 35, \quad \hat{\sigma}_{\alpha}^2 = \tilde{s}_{\alpha}^2 = s^2 = \frac{1}{3} \sum_{i=1}^4 (x_i - \bar{x})^2 = \frac{500}{3}.$$

Likely, we can get the fuzzy Bayes estimator of μ with respect to $\alpha = 0.95$,

$$A_{0.95} = [34.75, 35.25].$$

We are just intrrested in considering the $\mu \in A_0 = [30, 40]$, because of A_0 containing all Bayes point estimators of μ . According to Wu Hsienchung's^[1] simplified method, we have

① if $\mu < 35$, then, to solve $\zeta_{\mu}^{\sim}(\mu) = \max \{ \alpha \in [0, 1] : g(\alpha) = \hat{\mu}_{\alpha}^L = 30 + 5\alpha \leq \mu \}$, easily, we get

$$\zeta_{\mu}^{\sim}(\mu) = \max \{ \alpha \in [0, 1] : g(\alpha) = \hat{\mu}_{\alpha}^L = 30 + 5\alpha = \mu \};$$

② if $\mu > 35$, then, to solve $\zeta_{\mu}^{\sim}(\mu) = \max \{ \alpha \in [0, 1] : h(\alpha) = \hat{\mu}_{\alpha}^L = 40 - 5\alpha \geq \mu \}$, easily, we get

$$\zeta_{\mu}^{\sim}(\mu) = \max \{ \alpha \in [0, 1] : h(\alpha) = \hat{\mu}_{\alpha}^L = 40 - 5\alpha = \mu \}.$$

For preceding numerical example, we can also figure out the membership degree of the Bayes point estimator $\mu_0 = 36.5$.

Firstly, we assume that the allowable error $\varepsilon = 0.05$, let $\alpha_0 = 0.5$.

Since $\mu_0 > 35$, so to solve problem ②, using the dichotomy iteration program to find the membership degree for μ_0 , by following operating steps:

Step 1 Let $\varepsilon = 0.05, \alpha_0 = 0.5, a = 0, b = 1$; Find out $h(\alpha_0) = 37.5 > 36.5$, and $37.5 - 36.5 = 1 > \varepsilon$, go to step 2;

Step 2 Let $a = (a+b)/2 = 0.5, \alpha = (a+b)/2 = 0.75$; Find out $h(\alpha) = 36.25 < 36.5$, and $36.5 - 36.25 = 0.25 > \varepsilon$, go to step 3;

Step 3 Let $b = 0.75, \alpha = (a+b)/2 = 0.625$; Find out $h(\alpha) = 36.875 > 36.5$, and $36.875 - 36.5 = 0.375 > \varepsilon$, go to step 4;

Step 4 Let $a = 0.625, \alpha = (a+b)/2 = 0.6875$; Find out $h(\alpha) = 36.5625 > 36.5$, and $36.5625 - 36.5 = 0.0625 > \varepsilon$, go to step 5;

Step 5 Let $a = 0.6875$, $\alpha = (a+b)/2 = 0.71875$; Find out $h(\alpha) = 36.40625 < 36.5$, and $36.5 - 36.40625 = 0.09375 > \varepsilon$, go to step 6;

Step 6 Let $b = 0.71875$, $\alpha = (a+b)/2 = 0.703125$; Find out $h(\alpha) = 36.484375 < 36.5$, and $36.5 - 36.484375 = 0.015625 < \varepsilon$, satisfying the accuracy criteria, stop.

We get

$$\zeta_{\mu}^{-}(36.5) = 0.703125.$$

Likely, we can also find the membership degree for given σ^2 (or point estimator) such as

$$\zeta_{\sigma^2}^{-}\left(\frac{500}{3}\right) = 1, \zeta_{\sigma^2}^{-}(106.34) = 0.74.$$

5 Conclusion

In this paper, we have given the Bayesian estimators to two parameters μ and σ^2 using Jefferys prior distribution, further, the fuzzy Bayes point estimators are obtained by combining the fuzzy set theory with parameter estimation. There, the membership degree α can be regarded as degree of belief. In large sample situation, the approach developed by this paper can be applied to estimate parameters based on other distributions.

Acknowledgement We would like to thank editors and referees for their constructive comments that lead to several improvements on an earlier version of the paper.

[参考文献]

- [1] Wu Hsienchung. Fuzzy reliability estimation using Bayesian approach[J]. Computer and Industrial Engineering, 2004, 46: 467–493.
- [2] Wu Hsienchung. Fuzzy Bayesian system reliability assessment based on exponential distribution[J]. Applied Mathematical Modeling, 2006, 30: 509–530.
- [3] Taheri S M, Zarei R. Bayesian system reliability assessment under the vague environment[J]. Applied Soft Computing, 2011, 11: 614–622.
- [4] Latife Gökemli, Selda Kapan Ulusoy. Fuzzy Bayesian reliability and availability analysis of production systems[J]. Computers and Industrial Engineering, 2010, 59: 690–696.
- [5] Chou K C, Yuan J. Fuzzy Bayesian approach to reliability of existing structures[J]. Struct Eng, 1993, 119: 3276–3290.
- [6] Viertl R. On statistical inference for non-precise data[J]. Environmentrics, 1997, 8: 541–568.
- [7] Viertl R, Gurker W. Reliability estimation based on fuzzy lifetime data[C]//Onisawa T, Kacprzyk J. Reliability and Safety Analyses Under Fuzziness. Heidelberg: Physica Verlag, 1995: 163–168.
- [8] Huang Hongzhong, Zuo Ming J, Sun Zhanquan, et al. Bayesian reliability analysis for fuzzy lifetime data[J]. Fuzzy Sets and Systems, 2006, 157: 674–686.
- [9] Liu Yu, Huang Hongzhong, Li Yanfeng, et al. Optimal preventive maintenance policy under fuzzy Bayesian reliability assessment environments[J]. IIE Transactions, 2010, 42: 734–745.
- [10] Berger J O. Statistical Decision Theory and Bayesian Analysis[M]. New York: Springer, 1985: 32–38.
- [11] Kotz S, Wu X Z. Modern Bayes Statistics[M]. Beijing: Chinese Statistic Press, 2000.
- [12] Press S J. Bayesian Statistics: Principles, Models, and Applications[M]. New York: Wiley, 1989.
- [13] Smith J Q. Decision Analysis: A Bayesian Approach[M]. London: Chapman and Hall, 1988.
- [14] Zhang R T, Chen H F. Bayes Statistics[M]. Beijing: Science Press, 1991.
- [15] Peng Z Z, Sun Y Y. Fuzzy Mathematics and its Application[M]. Wuhan: Wuhan University Press, 2007.
- [16] Pur M L, Ralescu D A. Fuzzy random variables[J]. J Math Anal, 1986, 114: 409–422.
- [17] Mao S S, Wang J L, Pu X L, et al. Advanced Mathematics Statistics[M]. Beijing: Advanced Education Press, 2004.

[责任编辑: 丁 蓉]