Preservation Property of NBUCA Under L-S Transforms

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Abstract: Some characterization results of the NBUCA class of life distribution are obtained. The preservation properties of NBUCA aging properties under Laplace-Stieltjes transforms are investigated.

Key words: preservation property, Laplace-Stieltjes transform, NBUCA

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NBUCA 寿命分布在 L-S 变换下的封闭性质

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[**摘要**] 研究了 NBUCA 寿命分布在单调变换下的封闭性质,证明了 NBUCA 寿命分布在 L-S 变换下都具有封闭性质.

[**关键词**] 封闭性,L-S 变换,NBUCA

1 Introduction and Motivation

Non-parametric distribution classes and stochastic orders have been defined in reliability literature to model positive aging, i. e., the adverse effect of age on the random residual lifetime of a unit or component. Since in the analysis of system reliability one is often concerned with properties of a system life distribution which can be guaranteed from properties of component life distributions, general closure properties of non-parametric aging classes with respect to reliability operations have been extensively studied. Besides the theoretical interest in the problem, closure properties furnish both rules to choose among various types of replacements and maintenance policies. The closure properties of life distribution class and stochastic order under some transforms, such as increasing convex(concave) transform, increasing anti star-shaped transforms(increasing star-shaped transforms), have been investigated[1-11].

Given two non-negative random variables X and Y with distribution functions F, G and survival functions $\bar{F}=1-F$, $\bar{G}=1-G$, respectively. Let $X_t=(X-t\mid X>t)$ be the residual life at age $t>0$ of the random life X, denote its distribution function and survival function by F_t and $\bar{F}_t=1-F_t$.

Definition 1[12] X is said to be smaller than Y in increasing convex average order(denoted by $X\leqslant_{icxa}Y$) if, for all $t>0$,

$$\int_0^{+\infty}\int_t^{+\infty}\bar{F}(x)\,\mathrm{d}x\mathrm{d}t\leqslant\int_0^{+\infty}\int_t^{+\infty}\bar{G}(x)\,\mathrm{d}x\mathrm{d}t.$$

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In the economics theory, the above order is known as weak third-order stochastic dominance.

The classes NBU and NBUC have proved to be very useful in performing analysis of life lengths as well as usable in replacement policies. Hence a lot of results related to these two classes have been obtained in the literature. Along a similar line, Ahmad et al proposed the new better than used in increasing convex average order class of life distributions, expanding the NBUC class to a much bigger and a more practical one, whose definition is also recalled here.

Definition 2[12] X is NBUCA(NWUCA) if $X_t \leqslant_{icxa} (\geqslant_{icxa}) X$ for all $t>0$.

Equivalently, X is NBUCA if and only if, for all $t>0$,

$$\int_0^{+\infty}\int_x^{+\infty} \bar{F}(u+t)\,\mathrm{d}u\mathrm{d}x \leqslant \int_0^{+\infty}\int_x^{+\infty} \bar{F}(u)\,\mathrm{d}u\mathrm{d}x. \tag{1}$$

Its dual class is new worse than used in increasing convex average ordering, denoted by NWUCA, which is defined by reversing above inequality.

In recent decades, many authors have devoted themselves to investigating preservation properties of some positive aging conceptions which can be regarded as extensions of IFR and NBU. For examples, Cai and Wu (1997), Li et al(2000) and Pellerey and Petakos(2002) obtained the preservation property of NBUC under the formation of parallel systems of independent units. The closure properties of NBU(2)(NWU(2)) classes and NBUC(NWUC) classes under anti-star-shaped transformations and Laplace-Stieltjes transform are studied in the literature.

2 Main Results

Let $\mathscr{F}$ denote the family of life distributions which have no non-zero mass at origin and have finite mean. For any $F \in \mathscr{F}$, let $\Phi(s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{d}F(t)$, $(s \geqslant 0)$ be the Laplace-Stieltjes transform of F. Define that

$$\beta_n(s) = (-1)^n \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left\{\frac{1-\Phi(s)}{s}\right\}, \quad n=1,2,\cdots, s\geqslant 0$$

and

$$\alpha_0(s) = 1, \alpha_{n+1}(s) = s^{n+1}\beta_n(s), n=1,2,\cdots, s\geqslant 0.$$

Then we have

$$\alpha_n(s) = s\int_0^\infty \frac{(sx)^{n-1}}{(n-1)!}\mathrm{e}^{-sx}\bar{F}(x)\,\mathrm{d}x, \quad n=1,2,\cdots.$$

It is not difficult to show that $\alpha_n(s)$ is decreasing in n for any fixed $s\geqslant 0$. Then $\{\alpha_n(s), n\geqslant 0\}$ may be regarded as the survival function of a random variable taking nonnegative integer values.

Let $\{p_n, n\geqslant 0\}$ be a discrete distribution with $p_0 = 0$ and let $\{\bar{p}_n, n\geqslant 0\}$ denote its survival function, where $\bar{p}_n = \sum_{k>n} p_k$. It is noteworthy that if $F = \{p_n, n\geqslant 0\}$ with $p_0 = 0$ is a discrete life distribution, we can obtain the definition of discrete NBUCA class and NWUCA class as follows.

Definition 3 If $F = \{p_n, n\geqslant 0\}$ with $p_0 = 0$ is a discrete life distribution, F is NBUCA(NWUCA) class if, for all positive integers j and k,

$$\sum_{j=0}^{\infty}\sum_{i=j+k}^{\infty}\bar{P}_i \leqslant (\geqslant) \bar{P}_k \sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\bar{P}_i.$$

To prove the following theorem, we need some expression as follows.

$$\int_0^u (u-v)^{j-1}v^{i-1}\mathrm{d}v = \frac{(i-1)!\ (j-1)!}{(i+j-1)!}u^{i+j-1}, \quad i,j=1,2,\cdots \tag{2}$$

$$\sum_{i=1}^{k}\frac{(vs)^{i-1}}{(i-1)!}\mathrm{e}^{-vs} = s\int_v^\infty \frac{(ws)^{k-1}}{(k-1)!}\mathrm{e}^{-ws}\mathrm{d}w, \tag{3}$$

$$\sum_{i=k+1}^{\infty}\frac{(vs)^{i-1}}{(i-1)!}\mathrm{e}^{-vs} = s\int_0^v \frac{(ws)^{k-1}}{(k-1)!}\mathrm{e}^{-ws}\mathrm{d}w. \tag{4}$$

In fact, let $v=uy$, then the left-hand side of (2) becomes $u^{i+j-1}B(i,j)$ which is equal to the right-hand side of (2), where $B(i,j)$ is Beta function.

According to integration by parts, the equality (3) and (4) can be obtained.

In the following, we will discuss the closure property of NBUCA (NWUCA) class under Laplace-Stieltjes transform.

Theorem 1 $\{\alpha_n(s), n\geqslant 0\}$ is NBUCA(NWUCA) for all $s\geqslant 0$ if and only if F is NBUCA(NWUCA).

Proof We give the proof only for the case of NBUCA. The case of NWUCA may be discussed similarly.

At first we prove that if F is NBUCA, then $\{\alpha_n(s), n\geqslant 0\}$ is also NBUCA, i. e.,

$$\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\alpha_{i+j}(s) \leqslant \alpha_j(s)\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\alpha_i(s), \quad j,k=0,1,\cdots, \quad s\geqslant 0. \tag{5}$$

The right hand side of (5) equals

$$\alpha_j(s)\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\alpha_i(s) = \alpha_j(s)\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}s\int_0^{\infty}\frac{(vs)^{i-1}}{(i-1)!}\mathrm{e}^{-vs}\bar{F}(v)\,\mathrm{d}v=$$

$$\alpha_j(s)s\sum_{k=0}^{\infty}\int_0^{\infty}\sum_{i=k}^{\infty}\frac{(vs)^{i-1}}{(i-1)!}\mathrm{e}^{-vs}\bar{F}(v)\,\mathrm{d}v=$$

$$\alpha_j(s)s^2\sum_{k=0}^{\infty}\int_0^{\infty}\int_0^{v}\frac{(ws)^{k-2}}{(k-2)!}\mathrm{e}^{-ws}\bar{F}(v)\,\mathrm{d}w\mathrm{d}v=$$

$$s^3\sum_{k=0}^{\infty}\int_0^{\infty}\frac{(us)^{j-1}}{(j-1)!}\mathrm{e}^{-us}\bar{F}(u)\,\mathrm{d}u\int_0^{\infty}\int_w^{\infty}\frac{(ws)^{k-2}}{(k-2)!}\mathrm{e}^{-ws}\bar{F}(v)\,\mathrm{d}v\mathrm{d}w\geqslant$$

$$s^3\sum_{k=0}^{\infty}\int_0^{\infty}\frac{(us)^{j-1}}{(j-1)!}\mathrm{e}^{-us}\int_0^{\infty}\frac{(ws)^{k-2}}{(k-2)!}\mathrm{e}^{-ws}\int_w^{\infty}\bar{F}(u+v)\,\mathrm{d}v\mathrm{d}w\mathrm{d}u=$$

$$s^3\sum_{k=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{v}\frac{(us)^{j-1}}{(j-1)!}\mathrm{e}^{-us}\frac{(ws)^{k-2}}{(k-2)!}\mathrm{e}^{-ws}\bar{F}(u+v)\,\mathrm{d}w\mathrm{d}v\mathrm{d}u=$$

$$s^2\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{(us)^{j-1}}{(j-1)!}\mathrm{e}^{-us}\frac{(vs)^{i-1}}{(i-1)!}\mathrm{e}^{-vs}\bar{F}(u+v)\,\mathrm{d}v\mathrm{d}u=$$

$$s^2\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\int_0^{\infty}\int_v^{\infty}\frac{((t-v)s)^{j-1}}{(j-1)!}\frac{(vs)^{i-1}}{(i-1)!}\mathrm{e}^{-ts}\bar{F}(t)\,\mathrm{d}t\mathrm{d}v=$$

$$s^2\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\int_0^{\infty}\int_0^{t}s^{i+j-2}\frac{(t-v)^{j-1}v^{i-1}}{(j-1)!\,(i-1)!}\mathrm{e}^{-ts}\bar{F}(t)\,\mathrm{d}v\mathrm{d}t=$$

$$s^2\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\int_0^{\infty}s^{i+j-2}\frac{t^{i+j-1}\mathrm{e}^{-st}}{(i+j-1)!}\bar{F}(t)\,\mathrm{d}t=\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\alpha_{i+j}(s).$$

This shows that $\{\alpha_n(s), n\geqslant 0\}$ is NBUCA.

Conversely, if $\{\alpha_n(s), n\geqslant 0\}$ is also NBUCA. From the above proof we known that $\{\alpha_n(s), n\geqslant 0\}$ is also NBUCA is equivalent to

$$\sum_{k=0}^{\infty}\int_0^{\infty}\frac{(us)^{j-1}}{(j-1)!}\int_0^{\infty}\frac{(ws)^{k-1}}{(k-1)!}\mathrm{e}^{-(u+w)s}\int_w^{\infty}\bar{F}(u+v)\,\mathrm{d}v\mathrm{d}w\mathrm{d}u \leqslant \sum_{k=0}^{\infty}\int_0^{\infty}\frac{(us)^{j-1}}{(j-1)!}\int_0^{\infty}\frac{(ws)^{k-1}}{(k-1)!}\mathrm{e}^{-(u+w)s}\int_w^{\infty}\bar{F}(u)\bar{F}(v)\,\mathrm{d}v\mathrm{d}w\mathrm{d}u.$$

Define $G_n=\int_0^u\frac{(vs)^n}{n!}s\mathrm{e}^{-vs}\mathrm{d}v=P\{\sum_{i=1}^{n+1}Y_i\leqslant u\}$, where $Y_1, Y_2, \cdots$ are mutually independent random variables having the same exponential distribution with parameter s. Then the above inequality may be written as

$$\sum_{k=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_w^{\infty}\bar{F}(u+v)\,\mathrm{d}v\mathrm{d}G_{k-1}(w)\,\mathrm{d}G_{j-1}(u)\leqslant\sum_{k=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_w^{\infty}\bar{F}(u)\bar{F}(v)\,\mathrm{d}v\mathrm{d}G_{k-1}(w)\,\mathrm{d}G_{j-1}(u).$$

For any given $x\geqslant 0$, letting $n\to\infty$ and keeping $n/s\to x$, from the law of large numbers it follows that

$$G_n(u)\to I_x(u)=\begin{cases}1, & u>x,\\ 0, & u<x.\end{cases}$$

Taking $k=[sx]$ and $j=[st]$, we have $k/s\to x$ and $j/s\to t$ when $s\to\infty$. Now letting $s\to\infty$ by Helly Theorem we get

$$\int_0^\infty \int_x^\infty \bar{F}(t+v)\,\mathrm{d}v\mathrm{d}x \leqslant \int_0^\infty \bar{F}(t) \int_x^\infty \bar{F}(v)\,\mathrm{d}v\mathrm{d}x$$

at any continuous point t of F, and then at all $t>0$ because the continuous point of F are dense on $[0,+\infty)$. Hence F is NBUCA.

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