On a New Proof of Wittich Theorem of Complex Difference Equations

Xu Xinping，Zhang Jianjun

（School of Mathematics and Information Technology，Jiangsu Second Normal University，Nanjing 210013，China）

Abstract：The main purpose of this paper is to give a new proof of Wittich theorem of complex difference equations，which don't depend on the result obtained by Laine I and Yang C C[1] in 2007.

Key words：Wittich theorem，difference equations，growth order，Meromorphic functions

CLC number：O174.5 **Document code**：A **Article ID**：1001-4616（2015）04-0022-06

关于复差分方程Wittich定理的一个新证明

徐新萍，张建军

（江苏第二师范学院数学与信息技术学院，江苏 南京 210013）

［摘要］ 本文不依赖于Laine I和Yang C C[1]在2007年获得的一个结果，给出了复差分方程Wittich定理的一个新证明.

［关键词］ Wittich定理，差分方程，增长级，亚纯函数

1 Introduction and Result

Let $f(z)$ be a function meromorphic in the complex plane. We assume that the reader is familiar with the standard notations and results in Nevanlinna's value distribution theory of meromorphic functions such as the characteristic function $T(r,f)$，proximity function $m(r,f)$，counting function $N(r,f)$ etc.[2-4]. The notation $\rho(f)$ denotes the order of $f(z)$. $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f)=o(T(r,f))$ as $r\to\infty$ possibly outside an exceptional set of r of finite logarithmic measure. A meromorphic function $a(z)$ is called a small function of $f(z)$ or a small function relative to $f(z)$ if and only if $T(r,a(z))=S(r,f)$.

In general，a nonlinear difference equation has always the form

$$P[z,f]=P(z,f,f(z+c_1),\cdots,f(z+c_n))=0, \tag{1}$$

where $c_1,c_2,\cdots,c_n$ are distinct，nonzero complex numbers，P is a polynomial in f and its shifts with meromorphic coefficients. One can rewrite equation（1）in the form

$$P[z,f]=\sum_{\lambda\in I}\alpha_\lambda(z)f^{\lambda_0}f(z+c_1)^{\lambda_1}\cdots f(z+c_n)^{\lambda_n}=0, \tag{2}$$

where I is a finite set of multi-indices $(\lambda_0,\lambda_1,\cdots,\lambda_n)=\lambda$ and $\alpha_\lambda(z)$ is a meromorphic function. We define a difference monomial in f as

$$M_\lambda[z,f]=\alpha_\lambda(z)f^{\lambda_0}f(z+c_1)^{\lambda_1}\cdots f(z+c_n)^{\lambda_n}.$$

Received data：2014-10-12.

Foundation item：Supported by the NSFC（11426118、11171160），Natural Science Foundation of Jiangsu Province（(BK20140767)，Natural Science Foundation of the Jiangsu Higher Education Institutions（14KJB110004），Qing Lan Project of Jiangsu Province and Colonel-level Topics（JSNU-ZY-01）.

Corresponding author：Zhang Jianjun，Ph. D，associate professor，majored in complex analysis. E-mail：zhangjianjun1982@163.com

The degree γ_{M_λ} of M_λ is defined by

$$\gamma_{M_\lambda} := \lambda_0 + \lambda_1 + \cdots + \lambda_n.$$

Thus the left hand side of (2) can be expressed as a finite sum of difference monomials and which will be called a difference polynomial in f, that is

$$P[z,f] = \sum_{\lambda \in I} \alpha_\lambda(z) f^{\lambda_0} f(z+c_1)^{\lambda_1} \cdots f(z+c_n)^{\lambda_n} = \sum_{\lambda \in I} M_\lambda[z,f].$$

The degree γ_P of P is defined by

$$\gamma_P = \max_{\lambda \in I} \gamma_{M_\lambda}.$$

If $\gamma_{M_\lambda} = \gamma_P$, then we call the term $M_\lambda[z,f]$ is a dominant term of $P[z,f]$. Obviously, a difference polynomial may have not only one dominant term.

We say that a meromorphic solution f of equation (2) is admissible, if $T(r,\alpha) = S(r,f)$ holds for all $\alpha = \alpha_\lambda(z)$, $\lambda \in I$.

Recently, Zhang and Li[5] gave the following Wittich theorem of complex difference equations.

Theorem If the difference equation

$$P[z,f] = 0,$$

where $P[z,f]$ is a difference polynomial in f with meromorphic coefficient, has only one dominant term, then equation (2) has no admissible meromorphic solution of finite order ρ and satisfying $N(r,f) = S(r,f)$.

In [5], the author gave some examples to show that the conditions $P[z,f]$ has only one dominant term, $N(r,f) = S(r,f)$ and the restriction condition on the growth order of the solution in Theorem are unmovable. In the proof of Theorem, a result obtained by Laine I and Yang C C[1] in 2007 played a very important role. However, the proof of this result was very complex. In this paper, we will give a new proof of Wittich theorem of complex difference equations, which don't depend on Laine I and Yang C C's results.

2 Some Lemmas

The following lemmas will be needed in the proof of our results.

Lemma 1 (see [4]) Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,

$$R(z,f) = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_{i=0}^{p} a_i(z) f^i}{\sum_{j=0}^{q} b_j(z) f^j},$$

such that the meromorphic coefficient $a_i(z), b_j(z)$ satisfy

$$\begin{cases} T(r,a_i) = S(r,f), & i = 0,1,\cdots,p, \\ T(r,b_j) = S(r,f), & j = 0,1,\cdots,q, \end{cases}$$

we have

$$T(r,R(z,f)) = \max\{p,q\} \cdot T(r,f) + S(r,f).$$

Lemma 2 (see [4, 6]) Let f be admissible relative to the coefficients of $P(z,f) = a_0(z) + a_1(z) f + \cdots + a_n(z) f^n$. Then

$$N(r,P(z,f)) = nN(r,f) + S(r,f).$$

Lemma 3 (see [7]) Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let ρ be the order of $f(z)$, then for each $\varepsilon > 0$, we have

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\rho-1+\varepsilon}).$$

Lemma 4 (see [7]) Let $f(z)$ be a meromorphic function with order $\rho = \rho(f)$, $\rho < +\infty$, and c be a fixed nonzero complex number, then for each $\varepsilon > 0$, we have

$$T(r,f(z+c))=T(r,f)+O(r^{\rho-1+\varepsilon})+O(\log r).$$

Lemma 5 (see[8])Let $f(z)$ be a transcendental meromorphic solution of finite order ρ of a difference equation of the form

$$f^{n}P[z,f]=Q[z,f],$$

where $P[z,f]$ and $Q[z,f]$ are difference polynomials in $f(z)$ and its shifts. If the degree of $Q[z,f]$ in $f(z)$ and its shifts $\deg Q[z,f]\leqslant n$. Then for each $\varepsilon>0$,

$$m(r,P[z,f])=O(r^{\rho-1+\varepsilon})+S(r,f).$$

possibly outside of an exceptional set of finite logarithmic measure.

3 Proof of Theorem

Proof We rewrite equation(2)in the following form

$$P(z,f,f(z+c_1),\cdots,f(z+c_n))=P(z,f,0,\cdots,0)+\sum_{\lambda\in J}\alpha_\lambda(z)f^{\lambda_0}f(z+c_1)^{\lambda_1}\cdots f(z+c_n)^{\lambda_n}=-Q(z,f)+\Omega[z,f]=0.$$

where $\Omega[z,f]$, resp. $Q(z,f)$, is a difference polynomial, resp. a polynomial in f, with meromorphic coefficients, $J\subseteq I$ is a finite index set.

Thus, equation(2)changes into the following form

$$\Omega[z,f]=Q(z,f). \tag{3}$$

Since(3)has only one dominant term, we may denote

$$\Omega[z,f]=\sum_{\lambda\in J}\alpha_\lambda(z)f^{\lambda_0}f(z+c_1)^{\lambda_1}\cdots f(z+c_n)^{\lambda_n},\quad \gamma_\Omega\geqslant 1,$$

$$Q(z,f)=\beta_q(z)f^{q}+\beta_{q-1}(z)f^{q-1}+\cdots+\beta_0(z),\quad q\neq\gamma_\Omega,$$

where $\beta_q(z)\not\equiv 0$, and $T(r,\alpha_\lambda)=S(r,f)$, $T(r,\beta_j)=S(r,f)$ for $\lambda\in J, j=0,\cdots,q$.

Next, we discuss four cases separately.

Case 1 $q\geqslant\gamma_\Omega+1$. We rewrite equation(3)in the following form

$$Q[z,f]:=\Omega[z,f]-(\beta_{q-1}(z)f^{q-1}+\cdots+\beta_0(z))=\beta_q(z)f^{q},$$

where $Q[z,f]$ is a differential polynomial in f with meromorphic coefficients and the total degree of $Q[z,f]$ is $\leqslant q-1$, thus by lemma 5, we obtain $m(r,f)=O(r^{\rho-1+\varepsilon})+S(r,f)$. Therefore, $T(r,f)=m(r,f)+N(r,f)=O(r^{\rho-1+\varepsilon})+S(r,f)$, this is a contradiction. So we may assume that $q\leqslant\gamma_\Omega-1$ in next three cases.

Case 2 $\gamma_\Omega\geqslant 3$ and $q\geqslant 1$.

Now, writing $Q(z,f)=\sum_{j=0}^{\gamma_\Omega-1}\beta_j(z)f^{j}$. we may assume below that either $\beta_{\gamma_\Omega-1}(z)\equiv 0$ or $\beta_{\gamma_\Omega-2}(z)\equiv 0$. In fact, if $\beta_{\gamma_\Omega-1}(z)\equiv 0$, we may make a preliminary transformation

$$f=u-\frac{\beta_{\gamma_\Omega-2}(z)}{(\gamma_\Omega-1)\beta_{\gamma_\Omega-1}(z)},$$

substituting into(3), we find that the coefficients of term $u^{\gamma_\Omega-2}$ vanish identically by a simple calculation.

Next, we see that outside of a finite exceptional set of τ values,

$$\begin{cases} m\left(r,\dfrac{1}{f-\tau}\right)=O(r^{\rho-1+\varepsilon})+S(r,f),\\ Q(z,\tau)\neq 0. \end{cases} \tag{4}$$

hold simultaneously.

In fact, denoting $\mu_j=\max\{\lambda_j|(\lambda_0,\lambda_1,\cdots,\lambda_n)=\lambda\in J\}\ (j=0,1,\cdots,n)$. Substituting $f=\tau+\frac{1}{u}\ (\tau\neq 0)$ into(3), since $\Omega[z,f]$ has only one dominant term and $q<\gamma_\Omega$, we have

$$\left(\left(\sum_{\lambda\in J}\alpha_\lambda\tau^{\lambda_0+\lambda_1+\cdots+\lambda_n}\right)-\left(\beta_q\tau^q+\cdots+\beta_1\tau+\beta_0\right)\right)u^{q+\mu_0}u(z+c_1)^{\mu_1}\cdots u(z+c_n)^{\mu_n}=S_{q+\mu_0+\mu_1+\cdots+\mu_n-1}(u)\ , \tag{5}$$

where $S_{q+\mu_0+\mu_1+\cdots+\mu_n-1}(u)$ is a difference polynomial in u of total degree at most $q+\mu_0+\mu_1+\cdots+\mu_n-1$. Obviously, u is an admissible solution of (5). Choosing $\tau\in\mathbb{C}$ to satisfy $\left(\sum_{\lambda\in J}\alpha_\lambda\tau^{\lambda_0+\lambda_1+\cdots+\lambda_n}\right)-(\beta_q\tau^q+\cdots+\beta_1\tau+\beta_0)\neq 0$, which may fail for at most finitely many values of τ only, in fact, the number of such τ is at most γ_Ω. By lemma 5 and $\rho(u)=\rho(f)=\rho$, we have $m(r,u)=O(r^{\rho-1+\varepsilon})+S(r,u)$. Therefore, (4) follows.

Take now distinct complex constants $\tau_1,\cdots,\tau_{\gamma_\Omega}$ to satisfy (4). The constants $\tau_1,\cdots,\tau_{\gamma_\Omega}$ will be specified later on. It is immediate to deduce, for each $\gamma_{M_\lambda}\leqslant\gamma_\Omega(\lambda\in J)$, that

$$m\left(r,\frac{M_\lambda[z,f]}{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}\right)=m\left(r,\frac{\alpha_\lambda(z)f^{\lambda_0}f(z+c_1)^{\lambda_1}\cdots f(z+c_n)^{\lambda_n}}{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}\right)=O(r^{\rho-1+\varepsilon})+S(r,f). \tag{6}$$

In fact, we may write

$$\frac{\alpha_\lambda(z)f^{\lambda_0}f(z+c_1)^{\lambda_1}\cdots f(z+c_n)^{\lambda_n}}{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}=\alpha_\lambda(z)\frac{f}{f-\tau_1}\cdots\frac{f}{f-\tau_{\lambda_0}}\frac{f(z+c_1)}{f-\tau_{\lambda_0+1}}\cdots\frac{f(z+c_1)}{f-\tau_{\lambda_0+\lambda_1}}\cdots\frac{f(z+c_n)}{f-\tau_{\gamma_{M_\lambda}}}\cdots\frac{1}{f-\tau_{\gamma_{M_\lambda}}+1}\cdots\frac{1}{f-\tau_{\gamma_\Omega}}\ ,$$

Since

$$m\left(r,\frac{f}{f-\tau_\mu}\right)=m\left(r,1+\frac{\tau_\mu}{f-\tau_\mu}\right)\leqslant m\left(r,\frac{1}{f-\tau_\mu}\right)+O(1)=O(r^{\rho-1+\varepsilon})+S(r,f), \tag{7}$$

for $\mu=1,\cdots,\gamma_\Omega$ and by lemma 3, we have

$$m\left(r,\frac{f(z+c_i)}{f-\tau_j}\right)\leqslant m\left(r,\frac{f(z+c_i)}{f}\right)+m\left(r,\frac{f}{f-\tau_j}\right)=O(r^{\rho-1+\varepsilon})+S(r,f)\ . \tag{8}$$

for $i-1,2,\cdots,n$ and $j=1,2,\cdots,\gamma_\Omega$. Therefore, (6) follows by (4), (7) and (8).

Let us now define

$$F\left(z,\tau_j\right):=\frac{\Omega\left[z,f\right]-Q(z,\tau_j)}{f-\tau_j},j=1,2,\cdots,\gamma_\Omega\ . \tag{9}$$

Obviously, $\Omega\left[z,f\right]-Q(z,\tau_j)=Q(z,f)-Q(z,\tau_j)$ is divisible by $f-\tau_j$, and therefore (9) implies

$$N(r,F(z,\tau_j))=S(r,f),j=1,2,\cdots,\gamma_\Omega\ . \tag{10}$$

by lemma 2. We consider

$$h(z):=\sum_{j=1}^{\gamma_\Omega}A_jF\left(z,\tau_j\right)=\Omega\left[z,f\right]\sum_{j=1}^{\gamma_\Omega}\frac{A_j}{f-\tau_j}-\sum_{j=1}^{\gamma_\Omega}\frac{A_jQ(z,\tau_j)}{f-\tau_j}\ , \tag{11}$$

for some complex constants $A_1,\cdots,A_{\gamma_\Omega}$ to be specified immediately. By (10), it is clearly that $N(r,h)=S(r,f)$, and by (4), we obtain

$$m\left(r,\sum_{j=1}^{\gamma_\Omega}\frac{A_jQ(z,\tau_j)}{f-\tau_j}\right)=O(r^{\rho-1+\varepsilon})+S(r,f)\ . \tag{12}$$

Now let us choose the constants $A_1,\cdots,A_{\gamma_\Omega}$ to satisfy

$$\sum_{j=1}^{\gamma_\Omega}\frac{A_j}{f-\tau_j}=\frac{1}{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}\ , \tag{13}$$

which results from the elementary partial fractional representation. By (13) and (6) it is clear that

$$m\left(r,\Omega\left[z,f\right]\sum_{j=1}^{\gamma_\Omega}\frac{A_j}{f-\tau_j}\right)=m\left(r,\frac{\Omega\left[z,f\right]}{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}\right)=O(r^{\rho-1+\varepsilon})+S(r,f)\ . \tag{14}$$

Combining(12)and(14), we get that $m(r,h)=O(r^{\rho-1+\varepsilon})+S(r,f)$ and so $T(r,h)=O(r^{\rho-1+\varepsilon})+S(r,f)$.

Assuming now that h doesn't vanish identically, we obtain by(11)and(13)that

$$\Omega\left[z,f\right]=h(z)\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)+\sum_{j=1}^{\gamma_\Omega}\left(A_jQ\left(z,\tau_j\right)\prod_{i=1,i\neq j}^{\gamma_\Omega}(f-\tau_i)\right). \tag{15}$$

namely,

$$Q\left(z,f\right)=h(z)\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)+\sum_{j=1}^{\gamma_\Omega}\left(A_jQ\left(z,\tau_j\right)\prod_{i=1,i\neq j}^{\gamma_\Omega}(f-\tau_i)\right). \tag{16}$$

From (16), we conclude by lemma 1 that $qT(r,f)=\gamma_\Omega T(r,f)+S(r,f)$, thus $\gamma_\Omega T(r,f)=qT(r,f)+S(r,f)\leqslant(\gamma_\Omega-1)T(r,f)+S(r,f)$, therefore we get the contradiction $T(r,f)=S(r,f)$.

Next we may assume that h vanish identically. Using(11)and(13)again, we have

$$\Omega\left[z,f\right]=\sum_{j=1}^{\gamma_\Omega}\left(A_jQ\left(z,\tau_j\right)\prod_{i=1,i\neq j}^{\gamma_\Omega}(f-\tau_i)\right).$$

Comparing with the original differential equation, we obtain the identity

$$\sum_{j=1}^{\gamma_\Omega}\left(A_jQ\left(z,\tau_j\right)\prod_{i=1,i\neq j}^{\gamma_\Omega}(f-\tau_i)\right)=\sum_{j=0}^{\gamma_\Omega-1}\alpha_j(z)f^j. \tag{17}$$

We may equate the coefficients on both sides of(17), making use of the fact that either $\alpha_{\gamma_\Omega-1}(z)\equiv0$ or $\alpha_{\gamma_\Omega-2}(z)\equiv0$. In the case of $\alpha_{\gamma_\Omega-1}(z)\equiv0$, we get

$$\sum_{j=1}^{\gamma_\Omega}A_jQ(z,\tau_j)\equiv0. \tag{18}$$

Selecting now $z_1,\cdots,z_{\gamma_\Omega}$ and specifying $\tau_1,\cdots,\tau_{\gamma_\Omega}$ such that the determinant of

$$\begin{pmatrix} Q\left(z_1,\tau_1\right) & Q\left(z_1,\tau_2\right) & \cdots & Q\left(z_1,\tau_{\gamma_\Omega}\right)\\ Q\left(z_2,\tau_1\right) & Q\left(z_2,\tau_2\right) & \cdots & Q\left(z_2,\tau_{\gamma_\Omega}\right)\\ \vdots & \vdots & \ddots & \vdots\\ Q\left(z_{\gamma_\Omega},\tau_1\right) & Q\left(z_{\gamma_\Omega},\tau_2\right) & \cdots & Q\left(z_{\gamma_\Omega},\tau_{\gamma_\Omega}\right)\end{pmatrix}$$

is not equal to zero, thus we obtain $A_1=A_2=\cdots=A_{\gamma_\Omega}=0$ from(18), which contradicts our assumption. If then $\alpha_{\gamma_\Omega-2}(z)\equiv0$, we obtain

$$\sum_{j=1}^{\gamma_\Omega}\left(A_jQ(z,\tau_j)\sum_{i=1,i\neq j}^{\gamma_\Omega}\tau_i\right)\equiv0. \tag{19}$$

A similar reasoning results the same contradiction $A_1=A_2=\cdots=A_{\gamma_\Omega}=0$ from(19).

Case 3　$\gamma_\Omega=2$ and $q=1$.

In this case, the equation(3)becomes one of the following two forms

$$ff(z+c_k)+D_1(f)=\beta_1(z)f+\beta_0(z), \tag{20}$$

or

$$f(z+c_m)f(z+c_l)+D_2(f)=\beta_1^*(z)f+\beta_0^*(z), \tag{21}$$

where c_k, c_m, c_l are nonzero complex numbers, $\beta_1(z)$, $\beta_1^*(z)\neq0$, $D_1(f)$ and $D_2(f)$ are finite sums in the shifts of f with meromorphic coefficients and $\gamma_{D_1},\gamma_{D_2}\leqslant1$.

Squaring both sides of equations(20)and(21), we have

$$f^2\left(f(z+c_k)\right)^2+D_1^*(f)=\beta_1^{\ 2}f^2+2\beta_1\beta_0f+\beta_0^{\ 2}, \tag{22}$$

or

$$\left(f(z+c_m)\right)^2\left(f(z+c_l)\right)^2+D_2^*(f)=\beta_1^{*2}f^2+2\beta_1^*\beta_0^*f+\beta_0^{*2}, \tag{23}$$

where $D_1^*(f)$, $D_2^*(f)$ are polynomial in f and its shifts with meromorphic coefficients, and $\gamma_{D_1^*},\gamma_{D_2^*}\leqslant3$. Thus the case 2 changes into case 1 and satisfies the assumption in Theorem. According to case 1, equations(22)and

(23) have no admissible meromorphic solutions. Therefore, equations (20) and (21) also have no admissible meromorphic solutions.

Case 4 $q=0$. In this case, equation (3) reduces into the following form

$$\Omega[z,f]=\beta_0(z). \tag{24}$$

Since $q\leqslant\gamma_\Omega-1$, we have $\gamma_\Omega\geqslant 1$. If $\gamma_\Omega=1$, then equation (3) must be the following form

$$\alpha(z)f(z+c)=\beta_0(z), \tag{25}$$

where $\alpha(z)\neq 0$ is a meromorphic function satisfying $T(r,\alpha(z))=S(r,f)$ and c is a complex number. By (25), we have $T(r,f(z+c))=S(r,f)$ thus $T(r,f)=S(r,f)$ by lemma 4, this is a contradiction. Therefore, $\gamma_\Omega\geqslant 2$.

Subcase 1 $\beta_0(z)\neq 0$. Multiplying f on both sides of equation (24), then we have the following new equation

$$\Delta[z,f]:=\sum_{\lambda\in J}\alpha_\lambda(z)f^{\lambda_0+1}f(z+c_1)^{\lambda_1}\cdots f(z+c_n)^{\lambda_n}=\beta_0(z)f, \tag{26}$$

Since $\gamma_\Delta\geqslant 3$, case 4 changes into case 2, thus (26) has no admissible meromorphic solutions with $N(r,f)=S(r,f)$. Therefore, equation (24) also has no admissible meromorphic solutions with $N(r,f)=S(r,f)$..

Subcase 2 $\beta_0(z)=0$. We may make a transformation $f=g+\gamma(z)$, where $\gamma(z)$ is a meromorphic function satisfying $T(r,\gamma(z))=S(r,f)$. Substituting into (3), we obtain the following difference equation

$$N[z,g]:=\sum_{\mu\in K}a_\mu(z)g^{\mu_0}\big(g(z+c_1)\big)^{\mu_1}\cdots\big(g(z+c_n)\big)^{\mu_n}=b(z). \tag{27}$$

where $N[z,g]$ is a difference polynomial in g with meromorphic coefficients $a_\mu(z)$ satisfying $T\big(r,a_\mu(z)\big)=S(r,g)$, $N(z,0,\cdots,0)=0$, K is a finite set of multi- indices $(\mu_0,\mu_1,\cdots,\mu_n)=\mu$, $b(z)=-\sum_{\lambda\in J}\alpha_\lambda(z)\gamma(z)^{\lambda_0}\gamma\big(z+c_1\big)^{\lambda_1}\cdots\gamma\big(z+c_n\big)^{\lambda_n}$. We may choose $\gamma(z)$, such that $b(z)\neq 0$. By lemma 4, we have $T(r,b(z))=S(r,f)=S(r,g)$. Obviously, $\gamma_N=\gamma_\Omega\geqslant 2$ and (27) has only one dominant term. Thus, this case changes into one of the cases of case 2, case 3 and subcase 1. Thus, (27) has no admissible meromorphic solutions with $N(r,g)=S(r,g)$. Hence, we conclude that (24) has no admissible meromorphic solutions with $N(r,f)=S(r,f)$. The proof of Theorem is completed.

[参考文献]

[1] CHERRY W, YE Z. Nevanlinna's theory of value distribution[M]. Berlin: Springer-Verlag, 2001.

[2] HAYMAN W K. Meromorphic functions[M]. Oxford: Clarendon Press, 1964.

[3] LAINE I. Nevanlinna theory and complex differential equations[M]. Berlin: Walter de Gruyter, 1993.

[4] ZHANG J J, LI J. Some results on complex difference and q-difference equations[J]. Journal of nanjing normal university (natural science edition), 2013, 36: 30-35.

[5] LAINE I, YANG C C. Clunie theorems for difference and q-difference polynomials[J]. J London Math Soc, 2007, 76: 556-566.

[6] ISHIZAKI K. On a conjecture of Gackstatter and Laine for some differential equations[J]. Proc Japan Acad, 1991, 67: 270-273.

[7] CHIANG Y M, FENG S J. On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane[J]. Ramanujan journal, 2008, 16: 105-129.

[8] HALBURD R G, KORHONEN R J. Difference analogue of the lemma on the logarithmic derivative with applications to difference equations[J]. J Math Anal Appl, 2006, 314: 477-487.

[责任编辑:黄　敏]