A Remark of Diophantine Equation *ax+by=n*

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Abstract: Let a, b be positive integers such that $(a,b)=1$ and let n be a non-negative integer. Define $D(a,b;n)$ to be the number of non-negative integer solutions (x,y) of the Diophantine equation $ax+by=n$. Tripathi proved that

$$D(a,b;n)=\frac{n}{ab}+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{a}\sum_{j=1}^{a-1}\frac{\zeta_a^{-jn}}{1-\zeta_a^{bj}}+\frac{1}{b}\sum_{k=1}^{b-1}\frac{\zeta_b^{-kn}}{1-\zeta_b^{ak}},$$

where $\zeta_m=e^{2\pi i/m}$. In this note, we put forward a recurrence relation of $D(a,b;n)$, thus giving a new proof of above formula.

Key words: Diophantine equation, generating function, Residue theorem

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丢番图方程*ax+by=n*的一个注记

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[摘要] 令 a,b 为互素的正整数, n 为非负整数. $D(a,b;n)$ 表示不定方程 $ax+by=n$ 的非负整数解 (x,y) 的个数. Tripathi证明了

$$D(a,b;n)=\frac{n}{ab}+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{a}\sum_{j=1}^{a-1}\frac{\zeta_a^{-jn}}{1-\zeta_a^{bj}}+\frac{1}{b}\sum_{k=1}^{b-1}\frac{\zeta_b^{-kn}}{1-\zeta_b^{ak}},$$

其中 $\zeta_m=e^{2\pi i/m}$. 在本文中,我们建立了 $D(a,b;n)$ 的递推关系,从而给出了上述结论的新证明.

[关键词] 丢番图方程,生成函数,留数定理

Let a,b be positive integers such that $(a,b)=1$ and n be a non-negative integer. We denote by

$$D(a,b;n)=\#\{(x,y)|ax+by=n,\ (x,y)\in(\boldsymbol{Z}_{\geqslant 0})^2\},$$

the number of non-negative integer solutions (x,y) of the Diophantine equation

$$ax+by=n.$$

It is well known that

$$D(a,b;n)=\left[\frac{n}{ab}\right]+\gamma,$$

where $\gamma=0$ or $\gamma=1$, see [3,6] for details. We refer the readers to Dickson[1] or Hua[2] for an introduction to this topic and a survey of the literature about related problems.

In 2000, Tripathi[5] proved that

Theorem 1 (Tripathi) Let a, b be positive integers such that $(a,b)=1$ and n be a positive integer. Define

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$D(a,b;n)$ to be the number of non-negative integer solutions (x,y) of the Diophantine equation $ax+by=n$. Then

$$D(a,b;n)=\frac{n}{ab}+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{a}\sum_{j=1}^{a-1}\frac{\zeta_a^{-jn}}{1-\zeta_a^{bj}}+\frac{1}{b}\sum_{k=1}^{b-1}\frac{\zeta_b^{-kn}}{1-\zeta_b^{ak}}, \tag{1}$$

where $\zeta_m=e^{2\pi i/m}$.

The author gave two proofs of the above results. In the first it was determined that the function $D(a,b;n)$ using generating functions, while in the second it was verified that the expression obtained for $D(a,b;n)$ satisfies the properties which uniquely characterize the function $D(a,b;n)$. In this note, a new proof of formula (1) is given.

1 Some Lemmas

To prove Theorem, the following lemmas will be useful.

Lemma 1 (Residue Theorem) Suppose $f(z)$ is analytic inside and on a simple closed contour C except for isolated singularities at $z_1,z_2,\cdots,z_n$, inside C. Then

$$\int_C f(z)\mathrm{d}z=2\pi i\sum_{k=1}^{n}\mathrm{Res}(f(z),z=zk).$$

Proof See p. 310 in [4].

Lemma 2 Let a be a positive integer and $\zeta_a=e^{2\pi i/a}$. Then

$$\sum_{j=1}^{a-1}\frac{1}{1-\zeta_a^j}=\frac{a-1}{2}. \tag{2}$$

Proof First note that (2) is trivially valid for $a=1$. Recall that

$$\sum_{j=0}^{a-1}x^j=\prod_{j=1}^{a-1}(x-\zeta_a^j), \tag{3}$$

if $a\geqslant 2$, it follows from the logarithmic derivative of (3) and taking $x=1$.

Lemma 3 Under the hypothesis of Theorem 1 and $n\geqslant 2$, the following recurrence relation is obtained.

$$D(a,b;n)-2D(a,b;n-1)+D(a,b;n-2)=\frac{1}{a}\sum_{j=1}^{a-1}\frac{(\zeta_a^j-1)^2\zeta_a^{-jn}}{1-\zeta_a^{bj}}+\frac{1}{b}\sum_{k=1}^{b-1}\frac{(\zeta_b^k-1)^2\zeta_b^{-kn}}{1-\zeta_b^{ak}}. \tag{4}$$

Proof On one hand, it is easy to see that

$$\frac{(1-z)^2}{(1-z^a)(1-z^b)}=(1-z)^2\sum_{n\geqslant 0}D(a,b;n)z^n=\sum_{n\geqslant 0}D(a,b;n)z^n-2\sum_{n\geqslant 0}D(a,b;n)z^{n+1}+\sum_{n\geqslant 0}D(a,b;n)z^{n+2}=$$
$$\sum_{n\geqslant 2}(D(a,b;n)-2D(a,b;n-1)+D(a,b;n-2))z^n. \tag{5}$$

On the other hand, by using partial fractions expansion theory, we deduce

$$\frac{(1-z)^2}{(1-z^a)(1-z^b)}=\frac{1}{\prod_{j=1}^{a-1}(1-\zeta_a^jz)\prod_{k=1}^{b-1}(1-\zeta_k^bz)}=\sum_{j=1}^{a-1}\frac{A_j}{1-\zeta_a^jz}+\sum_{k=1}^{b-1}\frac{B_k}{1-\zeta_b^kz}=\sum_{n\geqslant 0}\left(\sum_{j=1}^{a-1}A_j\zeta_a^{jn}+\sum_{k=1}^{b-1}B_k\zeta_b^{kn}\right)z^n, \tag{6}$$

where

$$A_j=\lim_{z\to\zeta_a^{-j}}\frac{(1-z)^2(1-\zeta_a^jz)}{(1-z^a)(1-z^b)}=\frac{(\zeta_a^{-j}-1)^2}{a(1-\zeta_a^{-bj})},$$

$$B_k=\lim_{z\to\zeta_b^{-k}}\frac{(1-z)^2(1-\zeta_a^jz)}{(1-z^a)(1-z^b)}=\frac{(\zeta_a^{-j}-1)^2}{b(1-\zeta_b^{-ak})}.$$

Then (4) follows immediately by comparing the coefficients of z^n in (5) and (6).

2 Proof of Theorem 1

Clearly, $D(1,1;n)=n+1$ and then (1) holds. Without loss of generality, we assume that $1\leqslant a<b$. We now apply the method of induction on n to prove this theorem.

For $n=0$, by using Lemma 2, the proof is clear. We next aim to show that formula (1) is valid for $n=1$.

If $a=1$ and $b>1$, then

$$1=D(a,b;1)=\frac{1}{b}+\frac{1}{2}\left(1+\frac{1}{b}\right)+\frac{1}{b}\sum_{k=1}^{b-1}\frac{\zeta_b^{-k}}{1-\zeta_b^{k}}=\frac{1}{b}+\frac{1}{2}\left(1+\frac{1}{b}\right)+\frac{1}{b}\sum_{k=1}^{b-1}\left(\zeta_b^{-k}+\frac{1}{1-\zeta_b^{k}}\right)=1.$$

If $1<a<b$, the following is desired

$$0=D(a,b;1)=\frac{1}{ab}+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{a}\sum_{j=1}^{a-1}\frac{\zeta_a^{-j}}{1-\zeta_a^{bj}}+\frac{1}{b}\sum_{k=1}^{b-1}\frac{\zeta_a^{-k}}{1-\zeta_b^{ak}}. \tag{7}$$

Let C be a circle centered at the origin having radius $r(>1)$ and consider

$$h(z)=\frac{1-z}{(1-z^a)(1-z^b)z^2}.$$

Indeed, it suffices to observe that $h(z)$ has simple poles inside C at $z=1$, $z=\zeta_a^j\ (j=1,\cdots,a-1)$, $z=\zeta_b^k$ $(k=1,\cdots,b-1)$ and a pole of order 2 at $z=0$ with

$$\mathrm{Res}(h(z),z=1)=\lim_{z\to 1}(z-1)h(z)=-\frac{1}{ab},$$

$$\mathrm{Res}(h(z),z=0)=\lim_{z\to 0}(z^2h(z))'=-1,$$

$$\mathrm{Res}(h(z),z=\zeta_a^j)=\lim_{z\to\zeta_a^j}(z-\zeta_a^j)h(z)=-\frac{\zeta_a^{-j}}{a(1-\zeta_a^{bj})}+\frac{1}{a(1-\zeta_a^{bj})},$$

$$\mathrm{Res}(h(z),z=\zeta_b^k)=\lim_{z\to\zeta_b^k}(z-\zeta_b^k)h(z)=-\frac{\zeta_b^{-k}}{b(1-\zeta_b^{ak})}+\frac{1}{b(1-\zeta_b^{ak})}.$$

Then, by using Lemmas 1 and 2, we obtain

$$\frac{1}{2\pi i}\int_C f(z)\mathrm{d}z=\mathrm{Re}\,s(h(z),z=1)+\mathrm{Re}\,s(h(z),z=0)+\sum_{j=1}^{a-1}\mathrm{Re}\,s(h(z),z=\zeta_a^j)+\sum_{k=1}^{b-1}\mathrm{Re}\,s(h(z),z=\zeta_b^k)=$$

$$-\frac{1}{ab}-1+\frac{1}{a}\sum_{j=1}^{a-1}\frac{1}{1-\zeta_a^{bj}}+\frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{1-\zeta_b^{ak}}-\frac{1}{a}\sum_{j=1}^{a-1}\frac{\zeta_a^{-j}}{1-\zeta_a^{bj}}-\frac{1}{b}\sum_{k=1}^{b-1}\frac{\zeta_b^{-k}}{1-\zeta_b^{ak}}=$$

$$-\frac{1}{ab}-\frac{1}{2}+\left(\frac{1}{a}+\frac{1}{b}\right)-\frac{1}{a}\sum_{j=1}^{a-1}\frac{\zeta_a^{-j}}{1-\zeta_a^{bj}}-\frac{1}{b}\sum_{k=1}^{b-1}\frac{\zeta_b^{-k}}{1-\zeta_b^{ak}}=-D(a,b;1).$$

Moreover,

$$\left|\int_C h(z)\mathrm{d}z\right|=\left|\int_C\frac{1-z}{(1-z^a)(1-z^b)z^2}\mathrm{d}z\right|\leqslant\int_C\left|\frac{1-z}{(1-z^a)(1-z^b)z^2}\right||dz|\leqslant\int_C\frac{|z|+1}{(|z|^a-1)(|z|^b-1)|z|^2}|dz|=$$

$$\int_C\frac{1+r}{(r^a-1)(r^b-1)r^2}\mathrm{d}s=\frac{r+1}{(r^a-1)(r^b-1)r^2}2\pi r,$$

where $ds=\sqrt{(dx)^2+(dy)^2}=|dz|$. It can easily be verified that

$$\lim_{|z|\to\infty}\int_C h(z)\mathrm{d}z=0.$$

Thus (7) follows. So far, we get the case $n=1$.

Now suppose (1) holds for $m\leqslant n-1$. Using induction hypothesis and recurrence relation (4), we finally infer that

$$D(a,b;n)=2D(a,b;n-1)-D(a,b;n-2)=\frac{1}{a}\sum_{j=1}^{a-1}\frac{(\zeta_a^j-1)^2\zeta_a^{-jn}}{1-\zeta_a^{-bj}}+\frac{1}{b}\sum_{k=1}^{b-1}\frac{(\zeta_b^k-1)^2\zeta_b^{-kn}}{1-\zeta_b^{ak}}=\frac{n}{ab}+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)+$$

$$\frac{1}{a}\sum_{j=1}^{a-1}\frac{2\zeta_a^{-j(n-1)}-\zeta_a^{-j(n-2)})+(\zeta_a^j-1)^2\zeta_a^{-jn}}{1-\zeta_a^{-bj}}+\frac{1}{b}\sum_{k=1}^{b-1}\frac{2\zeta_b^{-k(n-1)}-\zeta_b^{-k(n-2)}+(\zeta_b^k-1)^2\zeta_b^{-kn}}{1-\zeta_b^{ak}}=$$

$$\frac{n}{ab}+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{a}\sum_{j=1}^{a-1}\frac{\zeta_a^{-jn}}{1-\zeta_a^{bj}}+\frac{1}{b}\sum_{k=1}^{b-1}\frac{\zeta_b^{-kn}}{1-\zeta_b^{ak}}.$$

This completes the proof of Theorem 1.

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4 Conclusion

In this paper, a projective dynamics is proposed for minimizing general linear programming. The new method is based on the variational inequality properties. We extend the variational inequality method to construct a new ODE system. The new dynamic will be very useful to solve large scale optimization problems.

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