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Backward Bifurcation in an Epidemic Model

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Abstract:In this paper,we formulate a SIVS epidemic model with special recovery rate to study the impact of limited medical resource on the transmission dynamics of diseases with vaccination. The basic investigation of the model has been finished. The backward bifurcation has been proved precisely. It is shown that limited medical resource leads to vital dynamics,such as bistability. Backward bifurcation implies that even if the basic reproduction number is smaller than unity,there may be a stable endemic equilibrium and the basic reproductive number itself is not enough to describe whether a disease will prevail or not and we should pay more attention to the initial conditions. It is also shown that sufficient medical services and medicines are very important for the disease control and eradication. Besides,the impact of vaccination has been explored too.

Key words:vaccination,epidemic model,medical resource,equilibrium,stability,backward bifurcation

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一个传染病模型中的后向分支问题

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[**摘要**] 为了研究在考虑免疫接种情况下有限的医疗资源对疾病传播的影响,我们建立了一个带有特殊恢复率的 SIVS 传染病模型,研究了模型的基本动力学性质并对后向分支进行了详细的证明. 结果表明,有限的医疗资源会导致重要的动力学性质,比如双稳现象等. 后向分支意味着,即使基本再生数小于 1 模型依然可能会有稳定的地方病平衡点,基本再生数不能完全反映疾病流行与否. 此时,人们应该注意疾病爆发时的初始状态. 研究结果同时表明,充足的医疗资源和服务对于疾病的消除与控制非常重要. 另外,文章也分析了免疫接种的影响.

[**关键词**] 免疫接种,传染病模型,医疗资源,平衡点,稳定性,后向分支

Recently,attention has been given to vaccination and treatment policies in terms of the different vaccine classes,efficacy,treatment resource and associated costs([1-12],etc.).

In[2],Kribs-Zaleta et al. introduced a vaccination compartment with temporarily immune state and set up a SIV model with general incidence rate. Their analysis indicated that when the vaccine for all population is not totally effective,the basic reproduction number R_0 is no longer a threshold for the spread of diseases and the model will exhibit multiple endemic states. In[9],Shan and Zhu took the per capita recovery rate as a function of the number of hospital beds. Their analysis indicated that the system could undergo backward bifurcation,saddle-node bifurcation,Hopf bifurcation and cusp type of Bogdanov-Takens bifurcation within different conditions. In [11],Xiao and Tang analyzed a SIV epidemic model with nonlinear incidence rate. The main result shows that the system undergoes forward bifurcation with hysteresis except for the backward bifurcation. Besides,Erika et al. ([7])studied the dynamics of a SIR epidemic model with nonlinear incidence rate,vertical transmission vaccina-

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tion for the newborns and the capacity of treatment, that takes into account the limitedness of the medical resources and the efficiency of the supply of available medical resources. Under some conditions, they proved that the existence of backward bifurcation, the stability and the direction of Hopf bifurcation. They also explored how the mechanism of backward bifurcation affects the control of the infectious disease.

In order to consider the impact of limited medical resources and vaccination on the transmission dynamics of infection diseases more precisely, we formulate a SIVS epidemic model with human population demography and vaccinated individuals.

The organizations of this paper are as follow. Firstly, we will introduce our SIVS model in section 2. In section 3, we will analyze the existence of equilibria. Stability of equilibria and backward bifurcation analysis will be given in Section 4. Some discussion will be given in Section 5.

1 Model

We classify the population in a given region/area into three categories: susceptible, infective and vaccinated. Let $S(t)$, $I(t)$ and $V(t)$ denote the number of susceptible, infective, vaccinated individuals at time t, respectively. Based on standard SIS model with the incidence of mass action, we can construct a model

$$\begin{cases} \dfrac{dS}{dt}=\Lambda-\beta SI-\varphi S+\mu(b,I)I+\theta V-dS, \\ \dfrac{dI}{dt}=\beta SI+\beta\sigma VI-\mu(b,I)I-dI, \\ \dfrac{dV}{dt}=\varphi S-\beta\sigma VI-\theta V-dV, \end{cases} \tag{1}$$

with initial data $S(0)\geqslant 0, I(0)\geqslant 0, V(0)\geqslant 0, S(0)+I(0)+V(0)\leqslant \Lambda/d$, where all parameters listed in Table 1 are positive.

Table 1　Parameters involved in the model

Description	Parameter
Recruitment rate	Λ
Contact transmission rate	β
The vaccination rate of susceptible individuals	φ
The per capita natural death rate	d
The rate at which the vaccination wears off	θ
The per capita recovery rate	μ
The efficiency of the vaccine, 0 = completely effective, 1 = useless	$0<\sigma<1$

In classical epidemic models, the per capita recovery rate is assumed to be a constant. Nevertheless, in general, the recovery rate depends on the resources of the health system available to the public, particularly the capacity of the hospital settings and efficiency of the treatment. There are many factors determining the recovery rate. The significant factor is the number of the hospital beds and medicines are another significant factors which are essential for safe and effective prevention, diagnosis and treatment of illness. We take the per capita recovery rate as $\mu=\mu_0+(\mu_1-\mu_0)\dfrac{b}{b+I}$ which is used in[9]. Therefore, the recovery rate in a unit time μI is a that can describe the impact of limited medical resource.

It is not difficult to prove the following theorem:

Theorem 1　With an initial value condition in(1), there is a unique solution, and the solution remains positive and bounded for any finite time $t\geqslant 0$.

Therefore, Model(1) is mathematically well-defined and biologically reasonable.

Summing up the system(1), then $\dfrac{dN}{dt}=\Lambda-dN$. So $N(t)$ tends to Λ/d as t increases to infinity. Therefore, we can reduce the size of the model by letting $S=\Lambda/d-I-V$. Now the model becomes

$$\begin{cases} \dfrac{dI}{dt}=\beta\left(\dfrac{\Lambda}{d}-I-V\right)I+\beta\sigma VI-\mu(b,I)I-dI, \\ \dfrac{dV}{dt}=\varphi\left(\dfrac{\Lambda}{d}-I-V\right)-\beta\sigma VI-\theta V-dV. \end{cases} \tag{2}$$

2 Existence of Equilibria

Let the right-hand side of (2) to be zero. One can verify that the model (2) has one disease free equilibrium at $E_0=\left(0,\dfrac{\varphi\Lambda}{d(d+\varphi+\theta)}\right)$. The local stability of E_0 can be obtained through a straightforward calculation for the eigenvalues.

It follows from ([13]) that for the compartmental models, the local stability of the disease-free equilibrium is governed by the reproduction number of the model. If we use the notation in ([13]), then we have

$$\boldsymbol{F}=\begin{pmatrix}\beta(\Lambda/d-I-V)I+\beta\sigma VI\\ 0\end{pmatrix},\quad \boldsymbol{V}=\begin{pmatrix}u(b,I)+\mathrm{d}I\\ \sigma\beta VI+\theta V+\mathrm{d}V-\varphi(\Lambda/d-I-V)\end{pmatrix}.$$

The infected compartment is I, hence a straightforward calculation gives

$$\tilde{F}(E_0)=\left(\frac{\beta\Lambda(\theta+d+\sigma\varphi)}{d(\mu_1+d)(\varphi+\theta+d)}\right),\quad \tilde{\nu}(E_0)=(\mu_1+d),$$

and

$$\tilde{F}\tilde{V}^{-1}=\left(\frac{\beta\Lambda(\theta+d+\sigma\varphi}{d(\mu_1+d)(\varphi+\theta+d)}\right).$$

Hence the reproduction number is given by $\rho(FV^{-1})$, and

$$R_0=\frac{\beta\Lambda(\theta+d+\sigma\varphi)}{d(\mu 1+d)(\varphi+\theta+d)}.$$

Remark 1 According to a straightforward calculation, R_0 is a monotone decreasing function with respect to the vaccination rate φ.

Let the right hand side of (2) be zero, then the endemic equilibrium $E(I,V)$ satisfies

$$V(I)=\frac{\varphi(\Lambda-Id)}{d(d+\theta+\varphi+\sigma\beta I)} \tag{3}$$

and I must satisfy the following equation:

$$F(I)=\beta^2\sigma dI^3+BI^2+CI+D, \tag{4}$$

where

$B=-\beta[\beta\sigma(\Lambda-bd)-d(\theta+d+\sigma\varphi+\mu_0\sigma+d\sigma)]$,

$C=-\beta^2\Lambda\sigma b+a\beta+d(\mu_0+d)(\varphi+\theta+d)$, $a=-\Lambda(d+\theta+\sigma\varphi)+db(\theta+d+\sigma\varphi+\mu_0\sigma+d\sigma)+\sigma bd(\mu_1-\mu_0)$,

$D=bd(d+\mu_1)(d+\theta+\varphi)(1-R_0)$.

Because it is complicated to discuss the root of function $F(I)$, we will study the number of the root from the geometry.

Let the right-hand side of (2) be zero, then

$$f(I)=g(I), \tag{5}$$

where

$$f(I)=\frac{\varphi(\Lambda-dI)}{d(d+\theta+\varphi+\sigma\beta I)},$$

$$g(I)=\frac{\beta I^2+(\beta b+d-\frac{\beta\Lambda}{d}+\mu_0)I+b(\mu_1+d-\frac{\beta\Lambda}{d})}{\beta(b+I)(\sigma-1)}.$$

One can easily verify that

$$f(0)=\frac{\varphi\Lambda}{d(d+\theta+\varphi)}, f(\frac{\Lambda}{d})=0, g(0)=\frac{\mu 1+d-\frac{\beta\Lambda}{d}}{\beta(\sigma-1)}, g(\frac{\Lambda}{d})=\frac{\Lambda+\mu_0\frac{\Lambda}{d}+b\mu_1+bd}{\beta\left(b+\frac{\Lambda}{d}\right)(\sigma-1)}<0,$$

and

$$
\begin{aligned}
f'(I)&=\frac{-\varphi[d(d+\theta+\varphi)+\sigma\beta\Lambda]}{d(d+\theta+\varphi+\sigma\beta I)^2}<0,\\
f''(I)&=\frac{2\sigma\beta\varphi[d(d+\theta+\varphi)+\sigma\beta\Lambda]}{d(d+\theta+\varphi+\sigma\beta I)^3}>0,\\
g'(I)&=\frac{\beta I^2+2\beta bI+b(\beta b+\mu_0-\mu_1)}{\beta(b+I)^2(\sigma-1)},\\
g''(I)&=\frac{-2b(\mu_0-\mu_1)}{\beta(b+I)^3(\sigma-1)}<0.
\end{aligned}
\tag{6}
$$

Therefore, $f(I)$ is a monotonous decreasing concave function whereas $g(I)$ is a convex function on the interval $[0,\Lambda/d]$. Besides,

$$
\begin{aligned}
f(0)-g(0)&=\frac{\varphi\Lambda}{d(d+\theta+\varphi)}-\frac{\mu_1+d-\frac{\beta\Lambda}{d}}{\beta(\sigma-1)}=\frac{\varphi\Lambda\beta(\sigma-1)-d(d+\theta+\varphi)(\mu_1+d-\frac{\beta\Lambda}{d})}{\beta d(\sigma-1)(d+\theta+\varphi)}=\\
&\frac{\beta\Lambda(d+\theta+\sigma\phi)-d(d+\theta+\phi)(\mu_1+d)}{\beta d(\sigma-1)(d+\theta+\phi)}=\frac{d(d+\theta+\phi)(\mu_1+d)(R_0-1)}{\beta d(\sigma-1)(d+\theta+\phi)},
\end{aligned}
\tag{7}
$$

which implies that

$$
\begin{aligned}
&f(0)<g(0)\quad R_0>1,\\
&f(0)=g(0)\quad R_0=1,\\
&f(0)>g(0)\quad R_0<1.
\end{aligned}
\tag{8}
$$

Now we can discuss the number of the equilibrium points in three cases.

(1) If $R_0>1$, then $f(0)<g(0)$, and $g(\Lambda/d)<0$ always exists. Thus there is only one positive intersection which gives one endemic equilibrium (See Fig. 1(a)).

(2) If $R_0=1$, then $f(0)=g(0)$, and $g(\Lambda/d)<0$ always exists. Thus $f(I)$ and $g(I)$ will intersect one point if and only if $f^0(0)<g^0(0)$ (See Fig. 1(b)).

(3) If $R_0<1$, then $f(0)>g(0)$, and $g(\Lambda/d)0$ always exists. Thus $f(I)$ and $g(I)$ will intersect with each

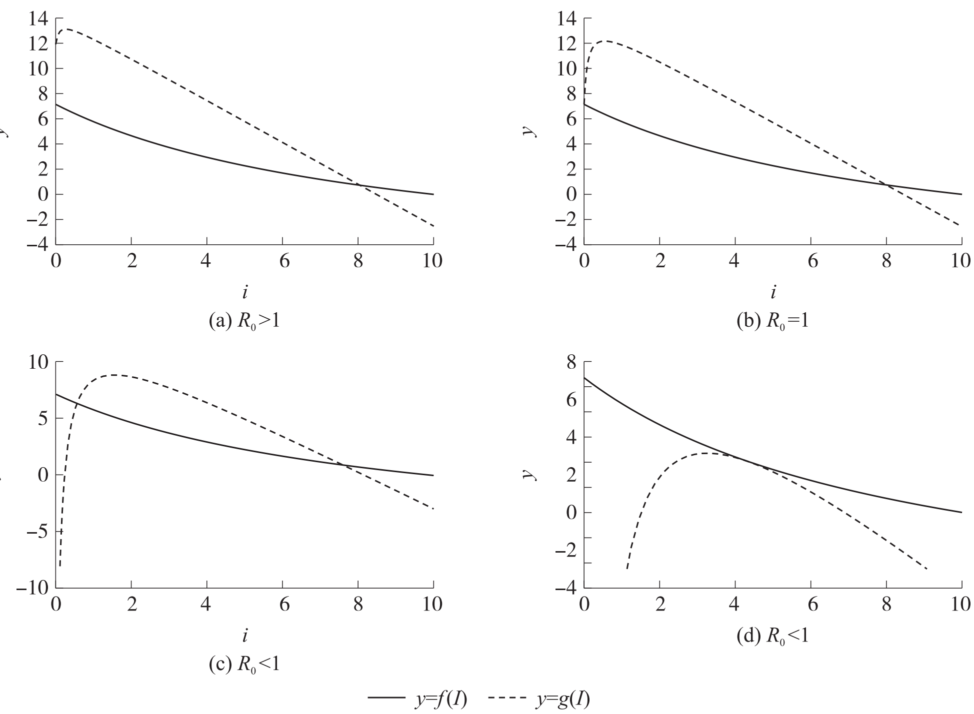

Fig. 1　The existence of endemic equilibria

other at two points which gives two endemic equilibrium if and only if $f'(0)<g'(0)$ and there exists $\hat{I}$ such that $f(\hat{I})<g(\hat{I})$; The two points coalesce with each other if $f(\hat{I})=g(\hat{I})$ and $f'(\hat{I})=g'(\hat{I})$ (See Fig. 1(c)(d)).

As a result, we have the following theorem:

Theorem 2 For the system(2).

(1) The disease-free equilibrium E_0 always exists.

(2) If $R_0>1$, there exists only one endemic equilibrium $E_1(I_2, V_2)$.

(3) If $R_0=1$, there exists one endemic equilibrium if and only if $g^0(0)>f^0(0)$. Otherwise, there is no endemic equilibrium.

(4) If $R_0<1$, there exist two endemic equilibrium $E_1(I_1, V_1)$ and $E_2(I_2, V_2)$ if and only if $f'(0)<g'(0)$ and there exists $\hat{I}$ such that $f(\hat{I})<g(\hat{I})$, the two equilibria will coalesce if $f(\hat{I})=g(\hat{I})$ and $f'(\hat{I})=g'(\hat{I})$.

Fix parameters $\Lambda=1, d=0.1, \beta=0.2, \theta=0.1, \sigma=0.4, b=0.1, \mu_0=0.2, \varphi=0.5, R_0=2, 1, 0.2, 0.047$ respectively, we will get the Fig. 1, respectively.

3 Stability of the Equilibria

Firstly, we will discuss stability of the disease-free equilibrium.

The Jacobian matrix of system(2)

$$\boldsymbol{J}(E)=\begin{pmatrix}\beta(\Lambda/d-2I-V)+\sigma\beta V-\mu(\dot{b}, I)-\mu(b, I)-d & -\beta I+\sigma\beta I\\ -\varphi-\sigma\beta V & -\varphi-\sigma\beta I-\theta-d\end{pmatrix} \tag{9}$$

So we will get

$$\boldsymbol{J}(E_0)=\begin{pmatrix}(\mu 1+d)(\mathrm{R}_0-1) & 0\\ -\varphi-\dfrac{\sigma\beta\varphi\Lambda}{d(\varphi+\theta+d)} & -\varphi-\theta-d\end{pmatrix} \tag{10}$$

we have the following theorem:

Theorem 3 When $R_0<1$, E_0 is locally asymptotically stable; when $R_0>1$, E_0 is unstable; when $R_0=1$, E_0 is a saddle-node.

Proof One can verify that system at E_0 has an eigenvalue, $-(\varphi+\theta+d)<0$. The other eigenvalue is $(\mu_1+d)(R_0-1)<0$. The result from the fact that $(\mu_1+d)(R_0-1)<0$ is equivalent to $R_0<1$ and $(\mu_1+d)(R_0-1)>0$ is equivalent to $R_0>1$. Hence, E_0 is locally asymptotically stable if $R_0<1$ and E_0 is unstable if $R_0>1$. Besides, obviously E_0 is a Lyapunov singularity when $R_0=1$. A straightforward calculation gives that one eigenvalue of Jacobian matrix of system at E_0 is zero when $R_0=1$, the other is $-(\varphi+\theta+d)$. The transformation $I'=I$, $V'=V+\dfrac{\varphi\Lambda}{d(\varphi+\theta+d)}$ brings E_0 to the origin. Then the system in a neighborhood of the origin becomes

$$\begin{cases}\dfrac{\mathrm{d}I'}{\mathrm{d}t}=\beta\left(\dfrac{\Lambda}{d}-I'-V'-\dfrac{\varphi\Lambda}{d(\varphi+\theta+d)}\right)I'+\beta\sigma I'\left(V'+\dfrac{\varphi\Lambda}{d(\varphi+\theta+d)}\right)-\mu(b, I')I'-\mathrm{d}I'\\ \dfrac{\mathrm{d}V'}{\mathrm{d}t}=\varphi\left(\dfrac{\Lambda}{d}-I'-V'-\dfrac{\varphi\Lambda}{d(\varphi+\theta+d)}\right)-(\sigma\beta I'+\theta+d)\left(V'+\dfrac{\varphi\Lambda}{d(\varphi+\theta+d)}\right).\end{cases} \tag{11}$$

Simplifying the system, we will get

$$\begin{cases}\dfrac{\mathrm{d}I'}{\mathrm{d}t}=\beta\left(\dfrac{\Lambda}{d}-I'-V'\right)I'+\dfrac{\Lambda\varphi\beta(\sigma-1)I'}{d(\varphi+\theta+d)}+\sigma\beta I'V'-\mu(b, I')I'-\mathrm{d}I'\\ \dfrac{\mathrm{d}V'}{\mathrm{d}t}=\varphi(-I'-V')-\dfrac{\sigma\beta^2\varphi\Lambda I'}{d(\varphi+\theta+d)}-\sigma\beta V'I'+\theta V'-\mathrm{d}V'.\end{cases} \tag{12}$$

Linearizing the above system and still substituting $I=I'$, $V=V'$. By straightforward calculating, we get

$$\begin{cases}\dfrac{\mathrm{d}I}{\mathrm{d}t}=\psi(I, V)\\ \dfrac{\mathrm{d}I}{\mathrm{d}t}=\left(-\varphi-\dfrac{\sigma\beta^2\varphi\Lambda}{d(\varphi+\theta+d)}\right)I-(\varphi+\theta+d)V+\varphi(I, V)\end{cases} \tag{13}$$

where

$$\psi(I,V)=\left(-2\beta+\frac{2(\mu_1-\mu_0)}{b}\right)I^2+2(-\beta+\sigma\beta)IV,$$
$$\varphi(I,V)=-2\sigma\beta IV.$$

Let the right-hand side of the second equation be zero. According to implicit function, one can obtain a function for V in terms of I. Suppose $V(I)=a_1I+a_2I^2+O(I^3)$ with $V(0)=0$. Substituting $V(I)$ into $\psi(I,V)$,

$$a_1\frac{\mathrm{d}I}{\mathrm{d}t}+2a_2I\frac{\mathrm{d}I}{\mathrm{d}t}+O(I^3)=\left(-\varphi-\frac{\sigma\beta^2\varphi\Lambda}{d(\varphi+\theta+d)}\right)I-(\varphi+\theta+d)V-2\sigma\beta IV, \tag{14}$$

$$a_1\left[\left(-2\beta+\frac{2(\mu_1-\mu_0)}{b}\right)I^2+2(-\beta+\sigma\beta)I(a_11+a_21^2+O(I^3))\right]+$$
$$2a_2I\left[\left(-2\beta+\frac{2(\mu_1-\mu_0)}{b}\right)I^2+2(-\beta+\sigma\beta)I(a_11+a_21^2+O(I^3))\right]=$$
$$\left(-\varphi-\frac{\sigma\beta^2\varphi\Lambda}{d(\varphi+\theta+d)}\right)I-(\varphi+\theta+d)(a_1I+a_2I^2+O(I)^3)-2\sigma\beta I(a_1I+a_2I^2+O(I^3)). \tag{15}$$

Comparing the coefficients of the same powers gives that $a_1=\frac{-\varphi}{(\varphi+\theta+d)}-\frac{\sigma\beta^2\varphi\Lambda}{d(\varphi+\theta+d)^2}$. Thus,

$$V(I)=\left(\frac{-\varphi}{(\varphi+\theta+d)}-\frac{\sigma\beta^2\varphi\Lambda}{d(\varphi+\theta+d)^2}\right)I+O(I^2) \tag{16}$$

and

$$\psi(I,V)=(-2\beta+\frac{2(\mu_1-\mu_0)}{b})I^2+2(-\beta+\sigma\beta)I\left[\left(\frac{-\varphi}{(\varphi+\theta+d)}-\frac{\sigma\beta^2\varphi\Lambda}{d(\varphi+\theta+d)^2}\right)I+O(I^2)\right]]=$$
$$\left[-2\beta+\frac{2(\mu_1-\mu_0)}{b})+2(-\beta+\sigma\beta)\left(\frac{-\varphi}{(\varphi+\theta+d)}-\frac{\sigma\beta^2\varphi\Lambda}{d(\varphi+\theta+d)^2}\right)\right]I^2+O(I^3)). \tag{17}$$

Clearly, $m=2$ which implies that E_0 is half saddle node. Let

$$C=\left[-2\beta+\frac{2(\mu_1-\mu_2)}{b}+2(-\beta+\sigma\beta)\left(\frac{-\varphi}{(\varphi+\theta+d)}-\frac{\sigma\beta^2\varphi\Lambda}{d(\varphi+\theta+d)^2}\right)\right],$$

Then, E_0 is left saddle and right node if $C<0$ while E_0 is right saddle and left node if $C>0$.

In the following, we will discuss the stability of the endemic equilibrium $E^-(I^-,V^-)$.

The Jacobian matrix of the model(2) at $E(I^-,V^-)$ gives

$$\boldsymbol{J}(\bar{E})=\begin{pmatrix}-\beta\bar{I}+\frac{(\mu_1-\mu_0)b\bar{I}}{(b+I)^2} & \beta\bar{I}(\sigma-1)\\ -\varphi-\sigma\beta\bar{V} & -(\varphi+\theta+d)-\sigma\beta\bar{I}\end{pmatrix} \tag{18}$$

and the corresponding characteristic equation is given by

$$\lambda^2-\mathrm{tr}(I)\lambda+\det(I)=0, \tag{19}$$

where

$$\mathrm{tr}(\bar{I})=\frac{-H(\bar{I})}{(b+\bar{I})^2},$$
$$H(\bar{I})=\beta(\sigma+1)\bar{I}^3+[2b(\beta\sigma+\beta)+\varphi+\theta+d]\bar{I}^2+b[b(\beta+\sigma\beta)+2(\varphi+\theta+d)+\mu_0-\mu_1]\bar{I}+b^2(\varphi+\theta+d), \tag{20}$$
$$\det(\bar{I})=\frac{IF'(\bar{I})}{d(b+\bar{I})}.$$

Obviously, $F'(I)$ and $H(I)$ determine the eigenvalues of the matrix $J(E^-)$. It is easy to know $F'(I_1)>0$, $F'(I_2)<0$. So $E_2(I,V)$ is a hyperbolic saddle and is always unstable, and $E_1(I,V)$ is a anti-saddle. Furthermore, if $H(I)>0$, $E_1(I,V)$ is locally asymptotically stable; If $H(I)=0$, $E_2(I,V)$ is a weak focus or center; If $H(I)<0$, $E_1(I,V)$ is a unstable node or focus. Thus we have the following theorem.

Theorem 4 When $R_0>1$, the system exists unique endemic equilibrium and it's an anti-saddle; when $R_0<1$, and the two endemic equilibria exist, $E_1(I,V)$ is always unstable, $E_2(I,V)$ is locally asymptotically stable if $H(I)>0$ and unstable if $H(I)<0$.

Lemma 1 (Theorem 3 in[14]) Assume

A_1: $\boldsymbol{A}=D_x f(0,0)=\left(\dfrac{\partial f_i(0,0)}{\partial x_j}\right)$ is the linearization matrix of system around the equilibrium 0 with φ evaluated at 0. Zero is a simple eigenvalue of $\boldsymbol{A}$ and all other eigenvalues of $\boldsymbol{A}$ have negative real parts;

A_2: Matrix $\boldsymbol{A}$ has a nonnegative right eigenvector w and a left eigenvector v corresponding to the zero eigenvalue.

Let f_k be the kth component of f and

$$a=\sum u_k w_i w_j \frac{\partial^2 f_k(0,0)}{\partial x_i \partial x_j}, b=\sum u_k w_i \frac{\partial^2 f_k(0,0)}{\partial x_i \partial \varphi}, \tag{21}$$

The local dynamics of system around 0 are totally determined by a and b.

(1) $a>0, b>0$. When $\varphi<0$ with $|\varphi|\ll 1$. 0 is locally asymptotically stable. and there exists a positive unstable equilibrium; when $0<\varphi\ll 1$, 0 is unstable and there exists a negative and locally asymptotically stable equilibrium;

(2) $a<0, b<0$. When $\varphi<0$ with $|\varphi|\ll 1$. 0 is unstable; When $0<\varphi\ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium.

(3) $a>0, b<0$. When $\varphi<0$ with $|\varphi|\ll 1$. 0 is unstable; When $0<\varphi\ll 1$, 0 is stable, and there exists a locally asymptotically stable negative equilibrium, and a positive unstable equilibrium appears.

(4) $a<0, b>0$. When $\varphi<0$ changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly a negative unstable equilibrium becomes positive and locally asymptotically stable.

Theorem 5 When $R_0=1$, $b<\dfrac{d(\mu_1-\mu_0)(d+\theta+\varphi)^2}{\beta[d\varphi(\varphi+\theta+d)+\sigma\beta\varphi\Lambda+d(\varphi+\theta+d)^2]}$, the system will undergo backward bifurcation.

Proof Let $\bar{\mu}_1=\dfrac{\beta\Lambda(d+\theta+\sigma\varphi)}{d(d+\theta+\varphi)}-d$. Then $R_0=1$ if and only if $\mu_1=\bar{\mu}_1$.

To apply Lemma 1, we essentially have to compute two quantities, labeled $\mathscr{A}$ and $\mathscr{B}$, which depend on the higher order terms in the Taylor expansion of system, and require, for their computation, a change of coordinates involving the right and left eigenvectors of the Jacobian $J(E_0)$ associated with the eigenvalue $\lambda=0$. we will express $\mathscr{A}$ and $\mathscr{B}$, in terms of parameters. The right and left eigenvectors of the Jacobian $J(E_0)$ are

$$\boldsymbol{w}=\left(1, \frac{-\varphi}{\varphi+\theta+d}-\frac{\sigma\beta\varphi\Lambda}{d(\varphi+\theta+d)^2}\right), \quad \boldsymbol{v}=(1,0)),$$

respectively. In order to follow the notations introduced in Theorem 3 of[18], we let $x_1=I$, $x_2=V$ and $\varphi=\bar{\mu}_1-\mu_1$. Then the Taylor expansion system are represented by the $f_i(x,\varphi)$, $(i=1,2)$, and we have

$$\begin{aligned}
&\frac{\partial^2 f_1(0,0)}{\partial x_1^2}=(-2\beta+\frac{2(\mu_1-\mu_0)}{b}),\\
&\frac{\partial^2 f_1(0,0)}{\partial x_1 \partial x_2}=2(-\beta+\sigma\beta),\\
&\frac{\partial^2 f_2(0,0)}{\partial x_1 \partial x_2}=-2\sigma\beta,\\
&\frac{\partial^2 f_1(0,0)}{\partial x_1 \partial \varphi}=1-R_0+\frac{\beta\Lambda(d+\theta+\sigma\varphi)}{d(d+\theta+\varphi)(d+\mu_1)}.
\end{aligned} \tag{22}$$

All other derivatives equal to zero. Consequently, we can readily compute the following quantity

$\mathscr{A}:=\sum u_k w_i w_j \dfrac{\partial^2 f_1(0,0)}{\partial x_i \partial x_j}=\left(-2\beta+\dfrac{2(\mu_1-\mu_0)}{b}\right)+2(-\beta+\sigma\beta)\left(\dfrac{-\varphi}{(\varphi+\theta+d)}-\dfrac{\sigma\beta\varphi\Lambda}{d(\varphi+\theta+d)^2}\right)=2(-\beta+\sigma\beta)(f'(0)-g'(0))$. Note that $\dfrac{\partial^2 f_1(0,0)}{\partial x_1 \partial\varphi}=1-R_0+\dfrac{\beta\Lambda(d+\theta+\sigma\varphi)}{d(d+\theta+\varphi)(d+\mu_1)}$ and all other derivative $\dfrac{\partial^2 f_k(0,0)}{\partial x_i \partial\varphi}$ equal to zero. so we can calculate $\mathscr{B}$ by substituting the vector v and w and the respective partial derivatives into the expression $\beta=\sum u_k w_i \dfrac{\partial^2 f_k(0,0)}{\partial x_i \partial\varphi}=1-R_0+\dfrac{\beta\Lambda(d+\theta+\sigma\varphi)}{d(d+\theta+\varphi)(d+\mu_1)}$.

So we conclude that when $R_0=1$ and $f^0(0)<g^0(0)$, $A>0$ and $B>0$, which is the defining condition for a backward bifurcation[14].

4 Discussion

In this paper, we formulate a SIVS epidemic model with special recovery rate to study the impact of limited medical resource on the transmission dynamics of diseases with vaccination.

In Model(2), the disease-free equilibrium always exists which is locally asymptotically stable if $R_0<1$ and unstable if $R_0>1$. There is a unique endemic equilibrium if $R_0>1$. Furthermore, on the one hand, if the medical resource in a given region is not sufficient enough, according to Theorem 5, backward bifurcation will occur and there are at most two endemic equilibria even if $R_0<1$. One(E_1) is a hyperbolic saddle and the other(E_2) is a anti-saddle. Furthermore, E_2 is a stable anti-saddle if $H(I)>0$. Backward bifurcation implies that the basic reproductive number itself is not enough to describe whether the disease will prevail or not and we should pay more attention to the initial value. On the other hand, if the medical resource is not less than the threshold value of b described in Theorem 5, R_0 can act as a critical value and one just need to control R_0 less than unity in order to control a disease. It is also shown that sufficient medical services and medicines are very important for the disease control and eradication.

According to Remark 1, the basic reproduction number is monotone decreasing function with respect to the vaccination rate φ, which implies that vaccination may help to control a disease, especially if the medical resource is not sufficient. By vaccination, one can decrease the basic reproduction number to small enough value to guarantee there is no endemic equilibrium.

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