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广义张量型 Hom-李代数 Kegel 定理

周 璇[1],张学俊[1],杨 涛[2]

(1.江苏第二师范学院数学与信息技术学院,江苏 南京 210013)
(2.南京农业大学数学系,江苏 南京 210095)

[**摘要**] 设(H,α_H)为张量型余三角 Hom-双代数.本文考虑了左(H,α_H)-Hom-余模代数,由此构造得出广义张量型 Hom-李代数,并证明了此情形下的 Kegel 定理.

[**关键词**] 张量型余三角 Hom-双代数,左(H,α_H)-Hom-余模代数,广义张量型 Hom-李代数,Kegel 定理
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Kegel's Theorem for Generalized Monoidal Hom-Lie Algebras

Zhou Xuan[1],Zhang Xuejun[1],Yang Tao[2]

(1.School of Mathematics and Information Technology,Jiangsu Second Normal University,Nanjing 210013,China)
(2.Department of Mathematics,Nanjing Agricultural University,Nanjing 210095,China)

Abstract:In this article,we consider the left(H,α_H)-Hom-comodule algebra for a monoidal cotriangular Hom-bialgebra (H,α_H). By constructing the generalized monoidal Hom-Lie algebra,we obtain the Kegel's theorem in this setting.

Key words:monoidal cotriangular Hom-bialgebras,left(H,α_H)-Hom-comodule algebras,generalized monoidal Hom-Lie algebras,Kegel's theorem

Hom-结构源于李理论中对向量场上的量子离散形变的研究(见文献[1-3]).Hartwig 等在文献[4]引入了 Hom-李代数的概念,并以此分析了 Witt 代数和 Virasoro 代数中的某些结构.Makhlouf 等在文献[5]中给出了 Hom-代数、Hom-余代数、Hom-双代数、Hom-Hopf 代数等概念.近年来,Hom-结构得到了广泛的研究.简言之,Hom-结构就是把原来结构中的恒等映射替换成广义的扭曲映射.最初的 Hom-双代数,是用不同的线性映射 α 和 β 来分别描述扭曲和余扭曲结合条件的,随后,Hom-双代数就分成两类,一类是 $\alpha=\beta$,仍称为 Hom-双代数,另一类是张量型 Hom-双代数,即从张量范畴的观点来阐述 Hom-结构(见文献[6]).关于张量型 Hom-结构的进一步研究,可参考文献[7-8].

经典的 Kegel 定理是指如果一个环可以写成两个幂零子环的和,则它也是幂零的[9].这个结果被推广到结合代数的情形[10]、余三角 Hopf 代数余模范畴中的结合代数的情形[11]、任意 Hopf 代数的 Yetter-Drinfel'd 模范畴中的代数的情形[12-13]以及弱 Hopf 群余代数的情形[14]等.那么,在张量型余三角 Hom-双代数的 Hom-余模范畴中,Kegel 定理是否成立呢? 这正是本文所讨论的问题.本文由张量型余三角 Hom-双代数(H,α_H)出发,通过左(H,α_H)-Hom-余模代数(A,α_A)构造了广义张量型 Hom-李代数,从而证明了此情形下的 Kegel 定理.

1 预备知识

本节将简单回顾相关概念及结论,见文献[6-7,15].设 k 为域,$M_k=(M_k,\otimes,k,a,l,r)$为 k-模范畴,张量型 Hom-范畴 $\widetilde{H}(M_k)=(H(M_k),\otimes,(k,id),\tilde{a},\tilde{l},\tilde{r})$是一个新的张量范畴.$\widetilde{H}(M_k)$中的对象是二元组$(M,\mu)$,$M\in M_k$,$\mu\in Aut_k(M)$.$\widetilde{H}(M_k)$中的态射是 M_k 中态射 $f:(M,\mu)\to(N,\nu)$,且满足 $\nu\circ f=f\circ\mu$.对任意的对象(M,μ),$(N,\nu)\in\widetilde{H}(M_k)$,张量积及单位如下给出:

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通讯联系人:周璇,博士,研究方向:代数学. E-mail:zhouxuanseu@126.com

$$(M,\mu)\otimes(N,\nu)=(M\otimes N,\mu\otimes\nu),(k,id).$$

对任意的$(M,\mu),(N,\nu),(L,\zeta)\in\widetilde{H}(M_k)$,结合性约束 $\tilde{a}$ 及左右单位约束 $\tilde{l}$ 和 $\tilde{r}$,分别为:

$$\tilde{a}_{M,N,L}=a_{M,N,L}\circ((\mu\otimes id)\otimes\zeta^{-1})=(\mu\otimes(id\otimes\zeta^{-1}))\circ a_{M,N,L},$$

$$\tilde{l}_M=\mu\circ l_M=l_M\circ(id\otimes\mu),\quad \tilde{r}_M=\mu\circ r_M=r_M\circ(\mu\otimes id).$$

定义 1　张量型 Hom-结合代数[6]是范畴 $\widetilde{H}(M_k)$ 中的对象(A,α),且存在元素 $1_A\in A$ 和线性映射 $m:A\otimes A\to A,a\otimes b\mapsto ab$,使得对任意的 $a,b,c\in A$,有

$$\alpha(a)(bc)=(ab)\alpha(c),\quad a1_A=1_Aa=\alpha(a),$$

$$\alpha(ab)=\alpha(a)\alpha(b),\quad \alpha(1_A)=1_A.$$

定义 2　张量型 Hom-余结合余代数[6]是范畴 $\widetilde{H}(M_k)$ 中的对象(C,β),且存在线性映射 $\Delta:C\to C\otimes C$, $\Delta(c)=c_1\otimes c_2$,及 $\varepsilon:C\to k$,使得

$$\beta^{-1}(c_1)\otimes\Delta(c_2)=\Delta(c_1)\otimes\beta^{-1}(c_2),\quad c_1\varepsilon(c_2)=\varepsilon(c_1)c_2=\beta^{-1}(c),$$

$$\Delta(\beta(c))=\beta(c_1)\otimes\beta(c_2),\qquad \varepsilon(\beta(c))=\varepsilon(c).$$

定义 3　张量型 Hom-双代数[6] $H=(H,\alpha,m,1_H,\Delta,\varepsilon)$ 是张量范畴 $\widetilde{H}(M_k)$ 中的双代数,也就是说$(H,\alpha,m,1_H)$是张量型 Hom-代数,$(H,\alpha,\Delta,\varepsilon)$是张量型 Hom-余代数,且 Δ 和 ε 是 Hom-代数同态,即,对任意的 $h,g\in H$,有

$$\Delta(hg)=\Delta(h)\Delta(g),\quad \Delta(1_H)=1_H\otimes 1_H,$$

$$\varepsilon(hg)=\varepsilon(h)\varepsilon(g),\quad \varepsilon(1_H)=1_k.$$

定义 4　设(C,β)是张量型 Hom-余代数,左(C,β)-Hom-余模[6]是范畴 $\widetilde{H}(M_k)$ 中对象(M,γ),且存在线性映射 $\rho_M:M\to C\otimes M,\rho_M(m)=m_{(-1)}\otimes m_{(0)}$,使得对任意的 $m\in M$,有

$$\Delta(m_{(-1)})\otimes\gamma^{-1}(m_{(0)})=\beta^{-1}(m_{(-1)})\otimes(m_{(0)(-1)}\otimes m_{(0)(0)}),$$

$$\rho_M(\gamma(m))=\beta(m_{(-1)})\otimes\gamma(m_{(0)}),\varepsilon(m_{(-1)})m_{(0)}=\gamma^{-1}(m).$$

定义 5　设(H,α_H)是张量型 Hom-双代数,(A,α_A)是张量型 Hom-代数,如果(A,α_A)是左(H,α_H)-Hom-余模,且对任意 $a,b\in A$,有 $\rho_A(ab)=a_{(-1)}b_{(-1)}\otimes a_{(0)}b_{(0)},\rho_A(1_A)=1_H\otimes 1_A$,则称$(A,\alpha_A)$为左$(H,\alpha_H)$-Hom-余模代数[7].

定义 6　设(H,α_H)是张量型 Hom-双代数,线性映射 $\sigma:H\otimes H\to k$ 卷积可逆,且对任意的 $h,g,l\in H$,有下列条件成立

$$\sigma(hg,\alpha_H(l))=\sigma(\alpha_H(h),l_1)\sigma(\alpha_H(g),l_2),$$

$$\sigma(\alpha_H(h),gl)=\sigma(h_1,\alpha_H(l))\sigma(h_2,\alpha_H(g)),$$

$$\sigma(h_1,g_1)h_2g_2=g_1h_1\sigma(h_1,g_2),$$

则称(H,α_H)为张量型余拟三角 Hom-双代数. 若 $\sigma^{-1}(h,g)=\sigma(g,h)$,则称$(H,\alpha_H)$为余三角的.若 $\sigma(\alpha_H(h),\alpha_H(g))=\sigma(h,g)$,则称 σ 为 α_H-不变的[15].

2　广义张量型 Hom-李代数

设(H,α_H)为张量型余三角 Hom-双代数,本节将给出广义张量型 Hom-李代数的定义,并证明由左(H,α_H)-Hom-余模代数(A,α_A)可以得到一个广义张量型 Hom-李代数.

定义 7　设(H,α_H)为张量型余三角 Hom-双代数,且 σ 为 α_H-不变的,(L,α_L)是范畴 $\widetilde{H}(M_k)$ 中的对象,同时也是左(H,α_H)-Hom-余模,$[,]:L\otimes L\to L$ 为 $\widetilde{H}(M_k)$ 中态射,同时也是左(H,α_H)-Hom-余模态射.如果任意的 $x,y,z\in L$,有下列条件成立:

σ-Hom-反交换性:

$$[x,y]=-\sigma(y_{(-1)},x_{(-1)})[\alpha_L(y_{(0)}),\alpha_L(x_{(0)})],$$

σ-Hom-Jacobi 恒等式:

$$[\alpha_L(x),[y,z]]+\sigma(y_{(-1)}z_{(-1)},\alpha_H(x_{(-1)}))[\alpha_L^2(y_{(0)}),[\alpha_L(z_{(0)}),\alpha_L(x_{(0)})]]+$$
$$\sigma(\alpha_H(z_{(-1)}),x_{(-1)}y_{(-1)})[\alpha_L^2(z_{(0)}),[\alpha_L(x_{(0)}),\alpha_L(y_{(0)})]]=0,$$

则称(L,α_L)为广义张量型 Hom−李代数.

命题 1　设(H,α_H)为张量型余三角 Hom−双代数，且σ为α_H−不变的，(A,α_A)，(B,α_B)为左(H,α_H)−Hom−余模代数，则$(A\otimes B,\alpha_A\otimes\alpha_B)$为左$(H,\alpha_H)$−Hom−余模代数，其中对任意的$a,c\in A,b,d\in B$，乘法和余模结构分别定义为：

$$(a\otimes b)(c\otimes d)=\sigma(c_{(-1)},b_{(-1)})a\alpha_A(c_{(0)})\otimes\alpha_B(b_{(0)})d,$$
$$\rho_{A\otimes B}(a\otimes b)=a_{(-1)}b_{(-1)}\otimes a_{(0)}\otimes b_{(0)}.$$

定理 1　设(H,α_H)为张量型余三角 Hom−双代数，且σ为α_H−不变的，(A,α_A)为左(H,α_H)−Hom−余模代数，定义

$$[\ ,\ ]:A\otimes A\to A,[a,b]=ab-\sigma(b_{(-1)},a_{(-1)})\alpha_A(b_{(0)})\alpha_A(a_{(0)}),a,b\in A,$$

则(A,α_A)为广义张量型 Hom−李代数.

证明　显然，对任意的$a,b\in A$，$[\alpha_A(a),\alpha_A(b)]=\alpha_A[a,b]$成立. 下证$[\ ,\ ]$为左$(H,\alpha_H)$−Hom−余模态射，事实上，对任意的$a,b\in A$，有

$$\rho_A([a,b])=\rho_A(ab)-\sigma(b_{(-1)},a_{(-1)})\rho_A(\alpha_A(b_{(0)})\alpha_A(a_{(0)}))=a_{(-1)}b_{(-1)}\otimes a_{(0)}b_{(0)}-\sigma(b_{(-1)},a_{(-1)})\times$$
$$\alpha_A(b_{(0)})_{(-1)}\alpha_A(a_{(0)})_{(-1)}\otimes\alpha_A(b_{(0)})_{(0)}\alpha_A(a_{(0)})_{(0)}=a_{(-1)}b_{(-1)}\otimes a_{(0)}b_{(0)}-\sigma(b_{(-1)2},a_{(-1)2})\times$$
$$\alpha_H(a_{(-1)1})\alpha_H(b_{(-1)1})\otimes b_{(0)}a_{(0)}=a_{(-1)}b_{(-1)}\otimes[a_{(0)},b_{(0)}].$$

其次，证明满足σ−Hom−反交换性，对任意的$a,b\in A$，有

$$-\sigma(b_{(-1)},a_{(-1)})[\alpha_A(b_{(0)}),\alpha_A(a_{(0)})]=-\sigma(b_{(-1)},a_{(-1)})\alpha_A(b_{(0)})\alpha_A(a_{(0)})+\sigma(b_{(-1)},a_{(-1)})\times$$
$$\sigma(\alpha_H(a_{(0)(-1)}),\alpha_H(b_{(0)(-1)}))\alpha_A^2(a_{(0)(0)})\alpha_A^2(b_{(0)(0)})=-\sigma(b_{(-1)},a_{(-1)})\times$$
$$\alpha_A(b_{(0)})\alpha_A(a_{(0)})+\sigma(b_{(-1)1},a_{(-1)1})\sigma(a_{(-1)2},b_{(-1)2})\alpha_A(a_{(0)})\times$$
$$\alpha_A(b_{(0)})=-\sigma(b_{(-1)},a_{(-1)})\alpha_A(b_{(0)})\alpha_A(a_{(0)})+ab=[a,b].$$

最后，为证满足σ−Hom-Jacobi 恒等式，分步计算等式中的三部分.对任意的$a,b,c\in A$，计算第一部分，可得

$$[\alpha_A(a),[b,c]]=[\alpha_A(a),bc-\sigma(c_{(-1)},b_{(-1)})\alpha_A(c_{(0)})\alpha_A(b_{(0)})]=\alpha_A(a)(bc)-\sigma(b_{(-1)}c_{(-1)},\alpha_H(a_{(-1)}))\times$$
$$(\alpha_A(b_{(0)})\alpha_A(c_{(0)}))\alpha_A^2(a_{(0)})-\sigma(c_{(-1)},b_{(-1)})\alpha_A(a)(\alpha_A(c_{(0)})\alpha_A(b_{(0)}))+\sigma(c_{(-1)},b_{(-1)})\times$$
$$\sigma(c_{(0)(-1)}b_{(0)(-1)},a_{(-1)})(\alpha_A^2(c_{(0)(0)})\alpha_A^2(b_{(0)(0)}))\alpha_A^2(a_{(0)})=\alpha_A(a)(bc)-\sigma(b_{(-1)}c_{(-1)},\alpha_H(a_{(-1)}))\times$$
$$(\alpha_A(b_{(0)})\alpha_A(c_{(0)}))\alpha_A^2(a_{(0)})-\sigma(c_{(-1)},b_{(-1)})\alpha_A(a)(\alpha_A(c_{(0)})\alpha_A(b_{(0)}))+$$
$$\sigma(c_{(-1)1},b_{(-1)1})\sigma(c_{(-1)2}b_{(-1)2},a_{(-1)})(\alpha_A(c_{(0)})\alpha_A(b_{(0)}))\alpha_A^2(a_{(0)}).$$

计算等式的第二部分，可得

$$\sigma(b_{(-1)}c_{(-1)},\alpha_H(a_{(-1)}))[\alpha_A^2(b_{(0)}),[\alpha_A(c_{(0)}),\alpha_A(a_{(0)})]]=\sigma(b_{(-1)}c_{(-1)},\alpha_H(a_{(-1)}))[\alpha_A^2(b_{(0)}),$$
$$\alpha_A(c_{(0)})\alpha_A(a_{(0)})-\sigma(a_{(0)(-1)},c_{(0)(-1)})\alpha_A^2(a_{(0)(0)})\alpha_A^2(c_{(0)(0)})]=\sigma(b_{(-1)}c_{(-1)},\alpha_H(a_{(-1)}))\times$$
$$\alpha_A^2(b_{(0)})(\alpha_A(c_{(0)})\alpha_A(a_{(0)}))-\sigma(b_{(-1)1}c_{(-1)1},\alpha_H(a_{(-1)1}))\sigma(c_{(-1)2}a_{(-1)2},\alpha_H(b_{(-1)2}))\times$$
$$(\alpha_A(c_{(0)})\alpha_A(a_{(0)}))\alpha_A^2(b_{(0)})-\sigma(b_{(-1)}\alpha_H(c_{(-1)1}),\alpha_H^2(a_{(-1)1}))\sigma(a_{(-1)2},c_{(-1)2})[\alpha_A^2(b_{(0)}),$$
$$\alpha_A(a_{(0)})\alpha_A(c_{(0)})]=\sigma(b_{(-1)}c_{(-1)},\alpha_H(a_{(-1)}))\alpha_A^2(b_{(0)})(\alpha_A(c_{(0)})\alpha_A(a_{(0)}))-\sigma(b_{(-1)1},a_{(-1)1})\times$$
$$\sigma(\alpha_H(c_{(-1)1}),a_{(-1)21})\sigma(c_{(-1)2}\alpha_H(a_{(-1)22}),\alpha_H(b_{(-1)2}))(\alpha_A(c_{(0)})\alpha_A(a_{(0)}))\alpha_A^2(b_{(0)})-$$
$$\sigma(b_{(-1)},\alpha_H(a_{(-1)1}))\sigma(\alpha_H(c_{(-1)1}),a_{(-1)21})\sigma(\alpha_H(a_{(-1)22}),c_{(-1)2})[\alpha_A^2(b_{(0)}),\alpha_A(a_{(0)})\times$$
$$\alpha_A(c_{(0)})]=\sigma(b_{(-1)}c_{(-1)},\alpha_H(a_{(-1)}))\alpha_A^2(b_{(0)})(\alpha_A(c_{(0)})\alpha_A(a_{(0)}))-\sigma(b_{(-1)},a_{(-1)})\alpha_A^2(b_{(0)})\times$$
$$(\alpha_A(a_{(0)})c)-\sigma(b_{(-1)1},a_{(-1)1})\sigma(\alpha_H(c_{(-1)}),b_{(-1)21}a_{(-1)21})\sigma(a_{(-1)22},b_{(-1)22})(\alpha_A(c_{(0)})\alpha_A(a_{(0)}))\times$$
$$\alpha_A^2(b_{(0)})+\sigma(b_{(-1)},a_{(-1)})\sigma(\alpha_H(a_{(0)(-1)})c_{(-1)},\alpha_H^2(b_{(0)(-1)}))(\alpha_A^2(a_{(0)(0)})\alpha_A(c_{(0)}))\alpha_A^3(b_{(0)(0)})=$$
$$\sigma(b_{(-1)}c_{(-1)},\alpha_H(a_{(-1)}))\alpha_A^2(b_{(0)})(\alpha_A(c_{(0)})\alpha_A(a_{(0)}))-\sigma(b_{(-1)},a_{(-1)})\alpha_A^2(b_{(0)})(\alpha_A(a_{(0)})c)-$$
$$\sigma(b_{(-1)11},a_{(-1)1})\sigma(a_{(-1)12},b_{(-1)12})\sigma(c_{(-1)},a_{(-1)2}b_{(-1)2})(\alpha_A(c_{(0)})\alpha_A(a_{(0)}))\alpha_A^2(b_{(0)})+$$
$$\sigma(b_{(-1)1},a_{(-1)1})\sigma(\alpha_H(a_{(-1)2}),b_{(-1)21})\sigma(c_{(-1)},b_{(-1)22})(\alpha_A(a_{(0)})\alpha_A(c_{(0)}))\alpha_A^2(b_{(0)})=$$
$$\sigma(b_{(-1)}c_{(-1)},\alpha_H(a_{(-1)}))\alpha_A^2(b_{(0)})(\alpha_A(c_{(0)})\alpha_A(a_{(0)}))-\sigma(b_{(-1)},a_{(-1)})\alpha_A^2(b_{(0)})(\alpha_A(a_{(0)})c)-$$
$$\sigma(\alpha_H(c_{(-1)}),a_{(-1)}b_{(-1)})(\alpha_A(c_{(0)})\alpha_A(a_{(0)}))\alpha_A^2(b_{(0)})+\sigma(c_{(-1)}),b_{(-1)})(a\alpha_A(c_{(0)}))\alpha_A^2(b_{(0)}).$$

计算等式的第三部分，可得

$$
\begin{gathered}
\sigma(\alpha_H(c_{(-1)}),a_{(-1)}b_{(-1)})[\alpha_A^2(c_{(0)}),[\alpha_A(a_{(0)}),\alpha_A(b_{(0)})]]=\sigma(\alpha_H(c_{(-1)}),a_{(-1)}b_{(-1)})\times \\
[\alpha_A^2(c_{(0)}),\alpha_A(a_{(0)})\alpha_A(b_{(0)})-\sigma(b_{(0)(-1)},a_{(0)(-1)})\alpha_A^2(b_{(0)(0)})\alpha_A^2(a_{(0)(0)})]= \\
\sigma(\alpha_H(c_{(-1)}),a_{(-1)}b_{(-1)})\alpha_A^2(c_{(0)})(\alpha_A(a_{(0)})\alpha_A(b_{(0)}))-\sigma(\alpha_H(c_{(-1)1}),a_{(-1)1}b_{(-1)1})\times \\
\sigma(a_{(-1)2}b_{(-1)2},\alpha_H(c_{(0)(-1)}))(\alpha_A(a_{(0)})\alpha_A(b_{(0)}))\alpha_A^2(c_{(0)})-\sigma(c_{(-1)},a_{(-1)1}b_{(-1)1})\sigma(b_{(-1)2},a_{(-1)2})\times \\
\alpha_A^2(c_{(0)})(\alpha_A(b_{(0)})\alpha_A(a_{(0)}))+\sigma(c_{(-1)},a_{(-1)1}b_{(-1)1})\sigma(b_{(-1)2},a_{(-1)2})\sigma(b_{(0)(-1)}a_{(0)(-1)},\alpha_H(c_{(0)(-1)}))\times \\
(\alpha_A^2(b_{(0)(0)})\alpha_A^2(a_{(0)(0)}))\alpha_A^3(c_{(0)(0)})=\sigma(\alpha_H(c_{(-1)}),a_{(-1)}b_{(-1)})\alpha_A^2(c_{(0)})(\alpha_A(a_{(0)})\alpha_A(b_{(0)}))- \\
(ab)\alpha_A(c)-\sigma(c_{(-1)1},b_{(-1)1})\sigma(c_{(-1)2}b_{(-1)2},a_{(-1)})\alpha_A^2(c_{(0)})(\alpha_A(b_{(0)})\alpha_A(a_{(0)}))+ \\
\sigma(\alpha_H(c_{(-1)1}),b_{(-1)2}\alpha_H(a_{(-1)12}))\sigma(b_{(-1)1},\alpha_H(a_{(-1)11}))\sigma(b_{(0)(-1)}a_{(-1)2},\alpha_H(c_{(-1)2}))\times \\
(\alpha_A^2(b_{(0)(0)})\alpha_A^2(a_{(0)(0)}))\alpha_A^2(c_{(0)})=\sigma(\alpha_H(c_{(-1)}),a_{(-1)}b_{(-1)})\alpha_A^2(c_{(0)})(\alpha_A(a_{(0)})\alpha_A(b_{(0)}))- \\
(ab)\alpha_A(c)-\sigma(c_{(-1)1},b_{(-1)1})\sigma(c_{(-1)2}b_{(-1)2},a_{(-1)})\alpha_A^2(c_{(0)})(\alpha_A(b_{(0)})\alpha_A(a_{(0)}))+ \\
\sigma(c_{(-1)1},b_{(-1)21}a_{(-1)21})\sigma(b_{(-1)1},a_{(-1)1})\sigma(b_{(-1)22}a_{(-1)22},c_{(-1)2})(\alpha_A(b_{(0)})\alpha_A(a_{(0)}))\alpha_A^2(c_{(0)})= \\
\sigma(\alpha_H(c_{(-1)}),a_{(-1)}b_{(-1)})\alpha_A^2(c_{(0)})(\alpha_A(a_{(0)})\alpha_A(b_{(0)}))-(ab)\alpha_A(c)-\sigma(c_{(-1)1},b_{(-1)1})\times \\
\sigma(c_{(-1)2}b_{(-1)2},a_{(-1)})\alpha_A^2(c_{(0)})(\alpha_A(b_{(0)})\alpha_A(a_{(0)}))+\sigma(b_{(-1)},a_{(-1)})(\alpha_A(b_{(0)})\alpha_A(a_{(0)}))\alpha_A(c).
\end{gathered}
$$

综合以上结果可得,

$$
\begin{gathered}
[\alpha_A(a),[b,c]]+\sigma(b_{(-1)}c_{(-1)},\alpha_H(a_{(-1)}))[\alpha_A^2(b_{(0)}),[\alpha_A(c_{(0)}),\alpha_A(a_{(0)})]]+ \\
\sigma(\alpha_H(c_{(-1)}),a_{(-1)}b_{(-1)})[\alpha_A^2(c_{(0)}),[\alpha_A(a_{(0)}),\alpha_A(b_{(0)})]]=0.
\end{gathered}
$$

定理得证.

3　Kegel 定理

设(H,α_H)为张量型余三角 Hom-双代数.本节将讨论可以分解成两个 H-可换 Hom-子代数的和的左(H,α_H)-Hom-余模代数(M,α_M),证明此种情形下的 Kegel 定理,推广了文献[11]中的结果.

定义 8　设(H,α_H)为张量型余三角 Hom-双代数,(A,α_A)为左(H,α_H)-Hom-余模代数,如果$[A,A]=0$,则称(A,α_A)为 H-可换的.

以下给出本文主要结果,即 Kegel 定理.

定理 2　设(H,α_H)为张量型余三角 Hom-双代数,且 σ 为 α_H-不变的,(M,α_M)为左(H,α_H)-Hom-余模代数,$M=Y+X$,其中 X,Y 为(M,α_M)的 Hom-子代数,且均为左(H,α_H)-Hom-余模代数及 H-可换的,则$[M,M][M,M]=0$.

证明　为了证明此定理,先证明如下结论,对任意的 $a,b\in Y,x,y\in X$,记 $a_{(0)}y_{(0)}=\lambda(a_{(0)},y_{(0)})+\theta(a_{(0)},y_{(0)})\in Y+X$,则有

$$
\begin{gathered}
\sigma(x_{(-1)},a_{(-1)})\sigma(y_{(-1)},b_{(-1)})(\alpha_M(x_{(0)})\alpha_M(a_{(0)}))(\alpha_M(y_{(0)})\alpha_M(b_{(0)}))= \\
\sigma(x_{(-1)}b_{(-1)},\alpha_H(a_{(-1)}))\varepsilon(y_{(-1)})\alpha_M^2(x_{(0)})(\alpha_M(b_{(0)})\lambda(a_{(0)},y_{(0)}))+ \\
\sigma(\alpha_H(y_{(-1)1}),x_{(-1)}b_{(-1)})\varepsilon(a_{(-1)})(\theta(a_{(0)},y_{(0)})\alpha_M(x_{(0)}))\alpha_M^2(b_{(0)}).
\end{gathered}
$$

事实上,对任意的 $a,b\in Y,x,y\in X$,有

$$
\begin{gathered}
\sigma(x_{(-1)},a_{(-1)})\sigma(y_{(-1)},b_{(-1)})(\alpha_M(x_{(0)})\alpha_M(a_{(0)}))(\alpha_M(y_{(0)})\alpha_M(b_{(0)}))=\sigma(x_{(-1)},a_{(-1)})\sigma(y_{(-1)},b_{(-1)})\times \\
\alpha_M^2(x_{(0)})(\lambda(a_{(0)}y_{(0)})\alpha_M(b_{(0)}))+\sigma(x_{(-1)},a_{(-1)})\sigma(y_{(-1)},b_{(-1)})\alpha_M^2(x_{(0)})(\theta(a_{(0)},y_{(0)})\alpha_M(b_{(0)}))= \\
\sigma(x_{(-1)},\alpha_H(a_{(-1)1}))\sigma(y_{(-1)1},b_{(-1)1})\sigma(\alpha_H(b_{(-1)2}),a_{(-1)2}y_{(-1)2})\alpha_M^2(x_{(0)})(\alpha_M(b_{(0)})\lambda(a_{(0)},y_{(0)}))+ \\
\sigma(x_{(-1)},a_{(-1)})\alpha(y_{(-1)},b_{(-1)})\sigma(\alpha_H(\theta(a_{(0)},y_{(0)})_{(-1)}),x_{(0)(-1)})(\alpha_M(\theta(a_{(0)},y_{(0)})_{(0)})\alpha_M^2(x_{(0)(0)}))\times \\
\alpha_M^2(b_{(0)})=\sigma(x_{(-1)}b_{(-1)},\alpha_H(a_{(-1)}))\varepsilon(y_{(-1)})\alpha_M^2(x_{(0)})(\alpha_M(b_{(0)})\lambda(a_{(0)},y_{(0)}))+\sigma(\alpha_H(y_{(-1)1}),b_{(-1)})\times \\
\varepsilon(a_{(-1)})\sigma(\alpha_H(y_{(-1)2}),x_{(-1)})(\theta(a_{(0)},y_{(0)})\alpha_M(x_{(0)}))\alpha_M^2(b_{(0)})=\sigma(x_{(-1)}b_{(-1)},\alpha_H(a_{(-1)}))\varepsilon(y_{(-1)})\times \\
\alpha_M^2(x_{(0)})(\alpha_M(b_{(0)})\lambda(a_{(0)},y_{(0)}))+\sigma(\alpha_H(y_{(-1)1}),x_{(-1)}b_{(-1)})\varepsilon(a_{(-1)})(\theta(a_{(0)},y_{(0)})\alpha_M(x_{(0)}))\alpha_M^2(b_{(0)}).
\end{gathered}
$$

由于 X,Y 为 H-可换的,以及$[\,,]$满足 σ-Hom-反交换性,只需证明对任意的 $a,b\in Y,x,y\in X$,$[a,x][b,y]=0$ 成立即可. 令 $xb=c+z$ 其中 $c\in Y,z\in X$,作如下计算:

$[a,x][b,y]=(ax-\sigma(x_{(-1)},a_{(-1)})\alpha_M(x_{(0)})\alpha_M(a_{(0)}))(by-\sigma(y_{(-1)},b_{(-1)})\alpha_M(y_{(0)})\alpha_M(b_{(0)}))=(ax)(by)-$

$$
\begin{gathered}
\sigma(x_{(-1)},a_{(-1)})(\alpha_M(x_{(0)})\alpha_M(a_{(0)}))_{(by)}-\sigma(y_{(-1)},b_{(-1)})(ax)(\alpha_M(y_{(0)})\alpha_M(b_{(0)}))+\sigma(x_{(-1)},a_{(-1)})\times \\
\sigma(y_{(-1)},b_{(-1)})(\alpha_M(x_{(0)})\alpha_M(a_{(0)}))(\alpha_M(y_{(0)})\alpha_M(b_{(0)}))=(a\alpha_M^{-1}(c))\alpha_M(y)+\alpha_M(a)(\alpha_M^{-1}(z)y)- \\
\sigma(x_{(-1)},a_{(-1)})(\alpha_M(x_{(0)})(a_{(0)}\alpha_M^{-1}(b))\alpha_M(y)-\sigma(y_{(-1)},b_{(-1)})\sigma(y_{(0)(-1)},\alpha_H^{-1}(x_{(-1)}))\alpha_M(a)\times \\
((\alpha_M(y_{(0)(0)})x_{(0)})\alpha_M(b_{(0)}))+\sigma(x_{(-1)}b_{(-1)},\alpha_H(a_{(-1)}))\varepsilon(y_{(-1)})\alpha_M(x_{(0)}b_{(0)})\alpha_M(\lambda(a_{(0)},y_{(0)}))+ \\
\sigma(\alpha_H(y_{(-1)1}),x_{(-1)}b_{(-1)})\varepsilon(a_{(-1)})\alpha_M(\theta(a_{(0)},y_{(0)}))\alpha_M(x_{(0)}b_{(0)})=\sigma(c_{(-1)},\alpha_H(a_{(-1)}))(c_{(0)}\alpha_M(a_{(0)}))\alpha_M(y)+ \\
\sigma(\alpha_H(y_{(-1)}),z_{(-1)})\alpha_M(a)(\alpha_M(y_{(0)})z_{(0)})-\sigma(x_{(-1)},\alpha_H(a_{(-1)1}))\sigma(b_{(-1)},\alpha_H(a_{(-1)2}))((x_{(0)}b_{(0)})\alpha_M(a_{(0)}))\times \\
\alpha_M(y)-\sigma(\alpha_H(y_{(-1)}),x_{(-1)}b_{(-1)})\alpha_M(a)(\alpha_M(y_{(0)})(x_{(0)}b_{(0)}))+\sigma(c_{(-1)},\alpha_H(a_{(-1)}))\varepsilon(y_{(-1)})\alpha_M(c_{(0)})\varepsilon(a_{(-1)})\times \\
\alpha_M(\lambda(a_{(0)},y_{(0)}))+\sigma(z_{(-1)},\alpha_H(a_{(-1)}))\varepsilon(y_{(-1)})\alpha_M(z_{(0)})\alpha_M(\lambda(a_{(0)},y_{(0)}))+\sigma(\alpha_H(y_{(-1)1}),z_{(-1)})\times \\
\sigma(z_{(0)(-1)},\theta(a_{(0)},y_{(0)})_{(-1)})\alpha_M^2(z_{(0)(0)})\alpha_M^2(\theta(a_{(0)},y_{(0)}))+\sigma(\alpha_H(y_{(-1)1}),c_{(-1)})\varepsilon(a_{(-1)})\alpha_M(\theta(a_{(0)},y_{(0)}))\times \\
\alpha_M(c_{(0)})=\sigma(c_{(-1)},\alpha_H(a_{(-1)}))(c_{(0)}\alpha_M(a_{(0)}))\alpha_M(y)+\sigma(\alpha_H(y_{(-1)}),z_{(-1)})\alpha_M(a)(\alpha_M(y_{(0)})z_{(0)})- \\
\sigma(c_{(-1)},\alpha_H(a_{(-1)}))(c_{(0)}\alpha_M(a_{(0)}))\alpha_M(y)-\sigma(z_{(-1)},\alpha_H(a_{(-1)}))(z_{(0)}\alpha_M(a_{(0)}))\alpha_M(y)-\sigma(\alpha_H(y_{(-1)}),c_{(-1)})\times \\
\varepsilon(a_{(-1)})(\alpha_M(a_{(0)})\alpha_M(y_{(0)}))\alpha_M(c_{(0)})-\sigma(\alpha_H(y_{(-1)}),z_{(-1)})\alpha_M(a)(\alpha_M(y_{(0)})z_{(0)})+ \\
\sigma(c_{(-1)},\alpha_H(a_{(-1)}))\varepsilon(y_{(-1)})\alpha_M(c_{(0)})\alpha_M(\lambda(a_{(0)},y_{(0)}))+\sigma(z_{(-1)},\alpha_H(a_{(-1)}))\varepsilon(y_{(-1)})\alpha_M(z_{(0)})\times \\
\alpha_M(\lambda(a_{(0)},y_{(0)}))+\sigma(y_{(-1)1},\alpha_H(z_{(-1)1}))\sigma(z_{(-1)22},\alpha_H(a_{(-1)}))\sigma(z_{(-1)21},y_{(-1)2})\alpha_M(z_{(0)})\times \\
\alpha_M(\theta(a_{(0)},y_{(0)}))+\sigma(\alpha_H(y_{(-1)1}),c_{(-1)})\varepsilon(a_{(-1)})\alpha_M(\theta(a_{(0)},y_{(0)}))\alpha_M(c_{(0)})=\sigma(c_{(-1)},\alpha_H(a_{(-1)}))\times \\
(c_{(0)}\alpha_M(a_{(0)}))\alpha_M(y)+\sigma(\alpha_H(y_{(-1)}),z_{(-1)})\alpha_M(a)(\alpha_M(y_{(0)})z_{(0)})-\sigma(c_{(-1)},\alpha_H(a_{(-1)}))(c_{(0)}\alpha_M(a_{(0)}))\times \\
\alpha_M(y)-\sigma(z_{(-1)},\alpha_H(a_{(-1)}))\varepsilon(y_{(-1)})\alpha_M(z_{(0)})\alpha_M(\lambda(a_{(0)},y_{(0)}))-\sigma(z_{(-1)},\alpha_H(a_{(-1)}))\varepsilon(y_{(-1)})\alpha_M(z_{(0)})\times \\
\alpha_M(\theta(a_{(0)},y_{(0)}))-\sigma(\alpha_H(y_{(-1)}),c_{(-1)})\varepsilon(a_{(-1)})\alpha_M(\lambda(a_{(0)},y_{(0)}))\alpha_M(c_{(0)})-\sigma(\alpha_H(y_{(-1)}),c_{(-1)})\varepsilon(a_{(-1)})\times \\
\alpha_M(\theta(a_{(0)},y_{(0)}))\alpha_M(c_{(0)})-\sigma(\alpha_H(y_{(-1)}),z_{(-1)})\alpha_M(a)(\alpha_M(y_{(0)})z_{(0)})+\sigma(c_{(-1)},\alpha_H(a_{(-1)}))\varepsilon(y_{(-1)})\times \\
\alpha_M(c_{(0)})\alpha_M(\lambda(a_{(0)},y_{(0)}))+\sigma(z_{(-1)},\alpha_H(a_{(-1)}))\varepsilon(y_{(-1)})\alpha_M(z_{(0)})\alpha_M(\lambda(a_{(0)},y_{(0)}))=0.
\end{gathered}
$$

定理得证.

当定理 2 中的映射 α_H,α_M 均取恒等映射时,定理 2 即为文献[11]中的结果,即

推论 1 设 H 为余三角双代数,M 为左 H-余模代数,$M=Y+X$,其中 X,Y 为 M 的子代数,且均为左 H-余模代数及 H-可换的,则$[M,M][M,M]=0$.

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