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粘性 BBM 型分数阶方程的数值方法

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[**摘要**]　本文构造了两种求解 BBM 型粘性分数阶方程的数值格式,分析了两种格式的稳定性与误差估计,严格证明了两种格式是无条件稳定的,两种格式的收敛都是 $O(\Delta t^{3/2}+N^{1-m})$,数值结果验证了理论分析的准确性.

[**关键词**]　分数阶方程,无条件稳定,误差估计,谱方法

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Numerical Methods for Solving BBM Type Viscous Fractional Equation

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Abstract:In this paper,two numerical schemes for solving BBM type viscous fractional equation are constructed. We analyze the stability and error estimates of the two schemes,a rigorous analysis shows that the proposed schemes are unconditionally stable,and the convergence of two schemes are convergent with order $O(\Delta t^{3/2}+N^{1-m})$,numerical results are consistent with the known theoretical prediction.

Key words:fractional equation,unconditionally stable,error estimates,spectral method

粘性水波方程的研究一直是一个热点问题,近 10 多来年来一直备受关注,这类方程的特点是包含粘性项,它的存在使我们必须考虑耗散机制的影响,因为这会使解变得很复杂. Kakutani,Matsuuchi[1]首先提出一个带有扩散项与色散项的粘性水波模型,接下来 Liu,Orfila[2],Saut,Bona 等人[3],Dutky 和 Dias[4]分别独立地得到了带粘性项的流体边界层水波模型.

Dutykh[5],Chen[6]从流体的自由面问题出发,得到了一个简化的单向波方程

$$\partial_t u+\partial_x u+\beta\partial_t\partial_x^2 u+\frac{\sqrt{v}}{\sqrt{\pi}}\int_0^t\frac{\partial_t u}{\sqrt{t-s}}\mathrm{d}s+u\partial_x u-\alpha\partial_x^2 u=0.$$

对上述方程 Chen[7]提出了一种时间有限差分,空间谱方法的显示格式,但这种格式是条件稳定,Dumont 和 Duval[8]用向后 Euler 方法的 G^{α} 格式来离散分数阶项,虽然数值结果是时间方向是 2 阶,但却没有给出相应的稳定性分析,数值格式的难点在于对分数阶项与非线性项的处理,对于这类方程 Zhang,Xu[9]提出了一种稳定的数值方法,对非线性项半稳化格式,类似可以用来处理这类分数阶方程.

我们构造了两种求解 BBM 型粘性分数阶方程的数值格式,并分析了格式的稳定性,对空间用谱离散,得到了格式的误差估计,数值结果验证了理论分析的准确性.

1　粘性 BBM 型分数阶方程

考虑如下粘性 BBM 型分数阶方程:

$$\partial_t u+\partial_x u-\beta\partial_x^2\partial_t u+\frac{\sqrt{v}}{\sqrt{\pi}}\int_0^t\frac{\partial_t u}{\sqrt{t-s}}\mathrm{d}s+\gamma u\partial_x u-\alpha\partial_x^2 u=0,\quad t\in(0,T],\quad x\in\Lambda,\tag{1}$$

满足下面初值条件:

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$$u(x,0)=u_0(x),\quad x\in\overline{\Lambda},\tag{2}$$

和边界条件:

$$u(x,t)=u(x+L,t),\quad t\in(0,T],x\in\overline{\Lambda},\tag{3}$$

这里 $\Lambda=(0,L)$,$\overline{\Lambda}=[0,L]$,T 表示时间,$\frac{\sqrt{v}}{\sqrt{\pi}}\int_0^t\frac{\partial_t u}{\sqrt{t-s}}\mathrm{d}s$ 是 1/2 阶 Caputo 时间分数阶导数项,其中 α、β、γ、v 为非负常数.

2 格式的构造与稳定性分析

考虑对时间分数阶离散,由 Taylor 展开有:

$$u(t)=u(s)+\partial_s u(s)(t-s)+\int_s^t\partial_\tau^2 u(\tau)(t-\tau)\mathrm{d}\tau,\quad\forall t,s\in(0,T].$$

分别取 $t=t_j,t=t_{j+1}$,我们可以有

$$\partial_s u(x,s)=\frac{u(x,t_{j+1})-u(x,t_j)}{\Delta t}-\frac{1}{\Delta t}\int_s^{t_{j+1}}\partial_\tau^2 u(x,\tau)(t_{j+1}-\tau)\mathrm{d}\tau+\frac{1}{\Delta t}\int_s^{t_j}\partial_\tau^2 u(x,\tau)(t_j-\tau)\mathrm{d}\tau.$$

Lin 等人[10]提出了如下的近似:

$$\frac{1}{\Gamma(1/2)}\int_0^{t_{n+1}}\frac{\partial_s u(s)}{\sqrt{t-s}}\mathrm{d}s=\frac{1}{\Gamma(1/2)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\frac{\partial_s u(s)}{\sqrt{t_{n+1}-s}}\mathrm{d}s=\frac{1}{\Gamma(1/2)}\sum_{j=0}^{n}\frac{u(x,t_{j+1})-u(x,t_j)}{\Delta t}\int_{t_j}^{t_{j+1}}\frac{1}{\sqrt{t_{n+1}-s}}\mathrm{d}s+r^{n+1}=$$

$$\frac{1}{\Gamma(3/2)}\sum_{j=0}^{n}\frac{u(x,t_{n+1-j})-u(x,t_{n-j})}{\Delta t^{1/2}}[(j+1)^{1/2}-j^{1/2}]+r^{n+1}=\frac{1}{\Gamma(3/2)}\sum_{j=0}^{n}a_j\frac{u(x,t_{n+1-j})-u(x,t_{n-j})}{\Delta t^{1/2}}+r^{n+1}=$$

$$\frac{\Delta t^{-1/2}}{\Gamma(3/2)}\left[u(x,t_{n+1})-\sum_{j=0}^{n-1}(a_j-a_{j+1})u(x,t_{n-j})-a_n u(x,t_0)\right]+r^{n+1},$$

这里

$$a_j=(j+1)^{1/2}-j^{1/2},$$

$$r^{n+1}=\frac{1}{\Gamma(1/2)\Delta t}\sum_{j=0}^{n}\left[-\int_{t_j}^{t_{j+1}}\int_s^{t_{j+1}}\partial_\tau u(x,\tau)\frac{t_{k+1}-\tau}{\sqrt{t_{n+1-s}}}\mathrm{d}\tau\mathrm{d}s+\int_{t_j}^{t_{j+1}}\int_s^{t_j}\partial_\tau^2 u(x,\tau)\frac{t_j-\tau}{\sqrt{t_{n+1}-s}}\mathrm{d}\tau\mathrm{d}s\right].$$

Lin 等人[10]证明了 $r^{n+1}=c\Delta t^{3/2}$,c 与 M 与$\partial_t^2 u$ 有关.

(1)基于 BD2 半隐格式:考虑如下基于 BD2 的半隐格式

$$\frac{3u^{n+1}-4u^n+u^{n-1}}{2\Delta t}+\partial_x u^{n+1}+\beta\frac{3\partial_x^2 u^{n+1}-4\partial_x^2 u^n+\partial_x^2 u^{n-1}}{2\Delta t}+\frac{v^{1/2}\Delta t^{-1/2}}{\Gamma(3/2)}\left(u^{n+1}-\sum_{j=0}^{n-1}(a_j-a_{j+1})u^{n-j}-a_n u^0\right)+$$

$$\frac{\gamma}{3}(2\partial_x u^{n+1}(2u^n-u^{n-1})+u^{n+1}\partial_x(2u^n-u^{n-1}))-\alpha\partial_x^2 u^{n+1}=0.\tag{4}$$

对于第一步,

$$\frac{u^1-u^0}{\Delta t}+\partial_x u^1+\frac{\beta}{\Delta t}(\partial_x^2 u^1-\partial_x^2 u^0)+\frac{v^{1/2}\Delta t^{-1/2}}{\Gamma(3/2)}(u^1-a_0 u^0)+\frac{\gamma}{3}(2u^0\partial_x u^1+u^1\partial_x u^0)-\alpha\partial_x^2 u^1=0.\tag{5}$$

(2)基于 Crank-Nicolson 半隐格式:考虑如下基于 Crank-Nicolson 方法的半隐格式:

$$\frac{u^{n+1}-u^n}{\Delta t}+\partial_x u^{n+1/2}-\frac{\beta}{\Delta t}(\partial_x^2 u^{n+1}-\partial_x^2 u^n)+\frac{v^{1/2}\Delta t^{-1/2}}{\Gamma(3/2)}\left(u^{n+1/2}-\sum_{j=0}^{n-1}(a_j-a_{j+1})u^{n-1/2-j}-a_n u^0\right)+$$

$$\frac{\gamma}{6}(2\partial_x u^{n+1/2}(3u^n-u^{n-1})+u^{n+1/2}\partial_x(3u^n-u^{n-1}))-\alpha\partial_x^2 u^{n+1/2}=0,\tag{6}$$

这里 $u^{n+1/2}=\frac{u^{n+1}+u^n}{2}$.

定理 1　式(4)-(5)是无条件稳定的,即:

$$\|u^1\|_1^2+\mu\|u^1\|_0^2\leqslant\|u^0\|_1^2+\mu\|u^0\|_0^2,\tag{7}$$

$$E(u^{n+1})\leqslant E(u^1)+\frac{2v^{1/2}T^{1/2}}{\Gamma(3/2)}\|u^0\|_0^2,n=1,2,\cdots,M-1,\tag{8}$$

这里 $\mu=\dfrac{v^{1/2}\Delta t^{1/2}}{\Gamma(3/2)}$，$E(u^n)=\|u^n\|_1^2+\|2u^n-u^{n-1}\|_1^2+\mu\sum\limits_{j=0}^{n-1}a_j\|u^{n-j}\|_0^2$，$\|u\|_1=\|u\|_0+\beta\|\partial_x u\|_0$.

证明　方程(5)与 $2\Delta t u^1$ 做内积，注意到

$$(2\partial_x u^1u^0+u^1\partial_x u^0,u^1)=(\partial_x u^1u^0+\partial_x(u^1u^0),u^1)=(\partial_x u^1u^0,u^1)-(u^1u^0,\partial_x u^1)=0,$$

$$(u^1-u^0,2u^1)=\|u^1\|_0^2-\|u^0\|_0^2+\|u^1-u^0\|_0^2.$$

则有

$$\|u^1\|_0^2-\|u^0\|_0^2+\|u^1-u^0\|_0^2+\beta(\|\partial_x u^1\|_0^2-\|\partial_x u^0\|_0^2+\|\partial_x u^1-\partial_x u^0\|_0^2)+$$
$$\mu(\|u^1\|_0^2-\|u^0\|_0^2+\|u^1-u^0\|_0^2)+2\Delta t\alpha\|\partial_x u^1\|_0^2=0.$$

整理可得(7)，方程(4)两边与 $4\Delta t u^{n+1}$ 做内积，可得：

$$\|u^{n+1}\|_1^2-\|u^n\|_0^2+\|2u^{n+1}-u^n\|_1^2-\|2u^n-u^{n-1}\|_1^2+\|u^{n+1}-2u^n+u^{n-1}\|_1^2+$$
$$4\mu(u^{n+1}-\sum_{j=0}^{n-1}(a_j-a_{j+1})u^{n-j}-a_nu^0,u^{n+1})+4\Delta t\alpha\|\partial_x u^{n+1}\|_0^2=0.$$

由 Cauchy-Schwarz 不等式，可得：

$$\|u^{n+1}\|_1^2-\|u^n\|_0^2+\|2u^{n+1}-u^n\|_1^2-\|2u^n-u^{n-1}\|_1^2+\|u^{n+1}-2u^n+u^{n-1}\|_1^2+4\mu\|u^{n+1}\|_0^2\leqslant$$
$$2\mu\left(\sum_{j=0}^{n-1}(a_j-a_{j+1})(\|u^{n-j}\|_0^2+\|u^{n+1}\|_0^2)+a_n(\|u^0\|_0^2+\|u^{n+1}\|_0^2)\right)=$$
$$2\mu\sum_{j=0}^{n-1}(a_j-a_{j+1})\|u^{n-j}\|_0^2+2\mu\|u^{n+1}\|_0^2+2\mu a_n\|u^0\|_0^2.$$

因此：

$$\|u^{n+1}\|_1^2+\|2u^{n+1}-u^n\|_1^2+2\mu\sum_{j=0}^{n-1}a_{j+1}\|u^{n-j}\|_0^2+2\mu\|u^{n+1}\|_0^2\leqslant\|u^n\|_1^2+\|2u^n-u^{n-1}\|_1^2+$$
$$2\mu\sum_{j=0}^{n-1}a_j\|u^{n-j}\|_0^2+2\mu a_n\|u^0\|_0^2.$$

注意到：

$$2\mu\sum_{j=0}^{n-1}a_{j+1}\|u^{n-j}\|_0^2+2\mu\|u^{n+1}\|_0^2=2\mu\sum_{j=1}^{n}a_j\|u^{n+1-j}\|_0^2+\mu\|u^{n+1}\|_0^2=2\mu\sum_{j=0}^{n}a_j\|u^{n+1-j}\|_0^2,$$

所以我们有：

$$E(u^{n+1})\leqslant E(u^1)+2\mu\sum_{j=1}^{n-1}a_j\|u^0\|_0^2\leqslant E(u^1)+2\mu n^{1/2}\|u^0\|_0^2\leqslant E(u^1)+2\frac{v^{1/2}T^{1/2}}{\Gamma(3/2)}\|u^0\|_0^2.$$

定理得证.

定理 2　式(6)是无条件稳定的，即：

$$e(u^{n+1})\leqslant e(u^1)+\frac{2v^{1/2}T^{1/2}}{\Gamma(3/2)}\|u^0\|_0^2,n=1,2,\cdots,M-1,\tag{9}$$

这里 $e(u^n)=\|u^n\|_1^2+\mu\sum\limits_{j=0}^{n-1}a_j\|u^{n-1/2-j}\|_0^2$.

证明　方程(6)两边与 $2\Delta t u^{n+1/2}$ 做内积，注意到：

$$(2\partial_x u^{n+1/2}(3u^n-u^{n-1})+u^{n+1/2}\partial_x(3u^n-u^{n-1}),u^{n+1/2})=(\partial_x u^{n+1/2}(3u^n-u^{n-1}),u^{n+1/2})+$$
$$(\partial_x(u^{n+1/2}(3u^n-u^{n-1})),u^{n+1/2})=(\partial_x u^{n+1/2}(3u^n-u^{n-1}),u^{n+1/2})-((3u^n-u^{n-1})u^{n+1/2},\partial_x u^{n+1/2}))=0,$$

可得：

$$\|u^{n+1}\|_1^2-\|u^n\|_1^2+2\mu(u^{n+1/2}-\sum_{j=0}^{n-1}(a_j-a_{j+1})u^{n-1/2-j}-a_nu^0,u^{n+1/2})+2\Delta t\alpha\|\partial_x u^{n+1/2}\|_0^2=0.$$

由 Cauchy-Schwarz 不等式，有：

$$\|u^{n+1}\|_1^2-\|u^n\|_1^2+2\mu\|u^{n+1/2}\|_0^2\leqslant\mu\left(\sum_{j=0}^{n-1}(a_j-a_{j+1})(\|u^{n-1/2-j}\|_0^2+\|u^{n+1/2}\|_0^2+a_n(\|u^0\|_0^2+\|u^{n+1/2}\|_0^2)))\right)=$$
$$\mu\sum_{j=0}^{n-1}(a_j-a_{j+1})\|u^{n-1/2-j}\|_0^2+\mu((1-a_n)+a_n)\|u^{n+1/2}\|_0^2+\mu a_n\|u^0\|_0^2,$$

化简可得:

$$\|u^{n+1}\|_1^2+\mu\sum_{j=0}^{n-1}a_{j+1}\|u^{n-1/2-j}\|_0^2+\mu\|u^{n+1/2}\|_0^2\leqslant\|u^n\|_1^2+\mu\sum_{j=0}^{n-1}a_j\|u^{n-1/2-j}\|_0^2+\mu a_n\|u^0\|_0^2.$$

注意到

$$\mu\sum_{j=0}^{n-1}a_{j+1}\|u^{n-1/2-j}\|_0^2+\mu\|u^{n+1/2}\|_0^2=\mu\sum_{j=1}^{n}a_j\|u^{n+\frac{1}{2}-j}\|_0^2+\mu\|u^{n+1/2}\|_0^2=\mu\sum_{j=0}^{n}a_j\|u^{n+\frac{1}{2}-j}\|_0^2,$$

因此,我们有:

$$\|u^{n+1}\|_1^2+\mu\sum_{j=0}^{n}a_j\|u^{n+\frac{1}{2}-j}\|_0^2\leqslant\|u^n\|_1^2+\mu\sum_{j=0}^{n-1}a_j\|u^{n-1/2-j}\|_0^2+\mu a_n\|u^0\|_0^2.$$

定理得证.

3 空间 Fourier 谱离散

用 Fourier 法对空间进行离散.定义 $S_N=\mathrm{span}\{\exp(-\mathrm{i}2\pi kx/L):-N/2\leqslant k\leqslant N/2-1\}$ 和 L^2 投影算子 π_N: $L^2(\Lambda)\to S_N$,使得:

$$(\pi_N v-v,\psi)=0,\quad\forall\psi\in S_N,$$

和 H^1 投影算子 π_N^1:$H^1(\Lambda)\to S_N$,使得:

$$(\partial_x(\pi_N^1 v-v),\partial_x\psi)=0,(\pi_N^1 v-v,\psi)=0,\quad\forall\psi\in S_N,$$

由文[11-12]知估计式成立:

$$\begin{aligned}&\|u-\pi_N u\|_0\leqslant N^{-m}\|u\|_m,\quad\forall u\in H^m(\Lambda),\quad m>0,\\&\|u-\pi_N^1 u\|_0\leqslant N^{k-m}\|u\|_m,\quad\forall u\in H^m(\Lambda),\quad m>0,k=0,1.\end{aligned}\tag{10}$$

BD2/F-G:半隐格式(4)、(5)的 Fourier 谱方法是:求 $u_N^{n+1}\in S_N$,使得

$$\left(\frac{3u_N^{n+1}-4u_N^n+u_N^{n-1}}{2\Delta t},\varphi_N\right)+(\partial_x u_N^{n+1},\varphi_N)+\beta\left(\frac{3\partial_x^2u_N^{n+1}-4\partial_x^2u_N^n+\partial_x^2u_N^{n-1}}{2\Delta t},\varphi_N\right)+\frac{v^{1/2}\Delta t^{-1/2}}{\Gamma(3/2)}\left(u_N^{n+1}-\sum_{j=0}^{n-1}(a_j-a_{j+1})u_N^{n-j}-a_nu_N^0,\varphi_N\right)+$$
$$\frac{\gamma}{3}((2\partial_x u_N^{n+1}(2u_N^n-u_N^{n-1})+u_N^{n+1}\partial_x(2u_N^n-u_N^{n-1})),\varphi_N)-\alpha(\partial_x^2u_N^{n+1},\varphi_N)=0.\tag{11}$$

对于第一步:

$$\left(\frac{u_N^1-u_N^0}{\Delta t},\varphi_N\right)+(\partial_x u_N^1,\varphi_N)+\frac{\beta}{\Delta t}(\partial_x^2u_N^1-\partial_x^2u_N^0,\varphi_N)+\frac{v^{1/2}\Delta t^{-1/2}}{\Gamma(3/2)}(u_N^1-a_0u_N^0,\varphi_N)+$$
$$\frac{\gamma}{3}(2u_N^0\partial_x u_N^1+u_N^1\partial_x u_N^0,\varphi_N)-\alpha(\partial_x^2u_N^1,\varphi_N)=0.\tag{12}$$

C-N/F-G:半隐格式(6)的 Fourier 谱方法是:求 $u_N^{n+1}\in S_N$,使得

$$\left(\frac{u_N^{n+1}-u_N^n}{\Delta t},\varphi_N\right)+(\partial_x u_N^{n+1/2},\varphi_N)-\frac{\beta}{\Delta t}(\partial_x^2u_N^{n+1}-\partial_x^2u_N^n,\varphi_N)+\frac{v^{1/2}\Delta t^{-1/2}}{\Gamma(3/2)}\left(u_N^{n+1/2}-\sum_{j=0}^{n-1}(a_j-a_{j+1})u_N^{n-1/2-j}-a_nu_N^0,\varphi_N\right)+$$
$$\frac{\gamma}{6}((2\partial_x u_N^{n+1/2}(3u_N^n-u_N^{n-1})+u_N^{n+1/2}\partial_x(3u_N^n-u_N^{n-1})),\varphi_N)-\alpha(\partial_x^2u_N^{n+1/2},\varphi_N)=0.\tag{13}$$

这里 $u_N^n(x)=\sum_{k=-N/2}^{N/2-1}\hat{u}_k^n\exp(-\mathrm{i}2\pi kx/L)$.

引理 1 全离散格式(11)、(12)的解,也无条件稳定的,满足如下的不等式:

$$\|u_N^1\|_1^2+\mu\|u_N^1\|_0^2\leqslant\|u_N^0\|_1^2+\mu\|u_N^0\|_0^2,E(u_N^{n+1})\leqslant E(u_N^1)+\frac{2v^{1/2}T^{1/2}}{\Gamma(3/2)}\|u_N^0\|_0^2,n=1,2,\cdots,M-1.$$

证明 证明过程与定理 1 类似.

引理 2 全离散格式(13)的解,也无条件稳定的,满足如下的不等式:

$$e(u_N^{n+1})\leqslant e(u_N^1)+\frac{2v^{1/2}T^{1/2}}{\Gamma(3/2)}\|u_N^0\|_0^2,n=1,2,\cdots,M-1.$$

证明 证明过程与定理 2 类似.

下面分析格式 BD2/F-G 与 C-N/F-G 的误差估计,定义截断误差:

$$R_1^{n+1}=r_1^{n+1}+\beta r_2^{n+1}+r_3^{n+1},R_2^{n+1}=r_3^{n+1}+r_4^{n+1}+r_5^{n+1},$$

这里

$$r_1^{n+1}=\frac{u(x,t_{n+1})-u(x,t_n)}{\Delta t}-\partial_t u(x,t_{n+1}),$$

$$r_2^{n+1}=\frac{\partial_x^2 u(x,t_{n+1})-\partial_x^2 u(x,t_n)}{\Delta t}-\partial_t\partial_x^2 u(x,t_{n+1}),$$

$$r_3^{n+1}=\frac{v^{1/2}\Delta t^{-1/2}}{\Gamma(3/2)}(u^{n+1}-\sum_{j=0}^{n-1}(a_j-a_{j+1})u^{n-j}-a_n u^0)-\frac{v^{1/2}}{\Gamma(1/2)}\int_0^{t_{n+1}}\frac{\partial_s u(x,s)}{(t_{n+1}-s)^{1/2}}\mathrm{d}s,$$

$$r_4^{n+1}=\frac{3u(x,t_{n+1})-4u(x,t_n)+u(x,t_{n-1})}{2\Delta t}-\partial_t u(x,t_{n+1}),$$

$$r_5^{n+1}=\frac{3\partial_x^2 u(x,t_{n+1})-4\partial_x^2 u(x,t_n)+\partial_x^2 u(x,t_{n-1})}{2\Delta t}-\partial_t\partial_x^2 u(x,t_{n+1}).$$

由 Taylor 展开式可知:

$$\|r_1^{n+1}\|_0^2\leqslant c\Delta t^2,\ \|r_2^{n+1}\|_0^2\leqslant c\Delta t^2,\ \|r_3^{n+1}\|_0^2\leqslant c\Delta t^{9/4},\ \|r_4^{n+1}\|_0^2\leqslant c\Delta t^4,\ \|r_5^{n+1}\|_0^2\leqslant c\Delta t^4. \tag{14}$$

定义误差函数:

$$\tilde{e}_N^n=\pi_N^1 u(t_n)-u_N^n,\bar{e}_N^n=u(t_n)-\pi_N^1 u(t_n),e_N^n=u(t_n)-u_N^n.$$

定理 3　设 $\{u_N^k\}_{k=0}^M$ 为全离散格式(11)、(12)的解,则原函数 m 阶光滑,则有下列估计式:

$$\|u(\cdot,t_k)-u_N^k\|_1\leqslant c(\Delta t^{3/2}+N^{1-m}),k=0,1,2,\cdots,M. \tag{15}$$

证明　方程(1)与 φ_N 做内积,再与(10)做差,可得:

$$\left(\frac{3\tilde{e}_N^{n+1}-4\tilde{e}_N^n+\tilde{e}_N^{n-1}}{2\Delta t},\varphi_N\right)+(\partial_x\tilde{e}_N^{n+1},\varphi_N)+\beta\left(\frac{3\partial_x^2\tilde{e}_N^{n+1}-4\partial_x^2\tilde{e}_N^n+\partial_x^2\tilde{e}_N^{n-1}}{2\Delta t},\varphi_N\right)+\frac{v^{1/2}\Delta t^{-1/2}}{\Gamma(3/2)}(\tilde{e}_N^{n+1}-\sum_{j=0}^{n-1}(a_j-a_{j+1})\tilde{e}_N^{n-j}-a_n\tilde{e}_N^0,\varphi_N)+$$

$$\gamma(u(\cdot,t_{n+1})\partial_x u(\cdot,t_{n+1})-\frac{1}{3}(2\partial_x u_N^{n+1}(2u_N^n-u_N^{n-1})+u_N^{n+1}\partial_x(2u_N^n-u_N^{n-1})),\varphi_N)-\alpha(\partial_x^2\tilde{e}_N^{n+1},\varphi_N)=(R_2^{n+1},\varphi_N)+$$

$$((\pi_N^1-I)\partial_x u(\cdot,t_{n+1}),\varphi_N)+\frac{1}{2\Delta t}((\pi_N^1-I)(3u(\cdot,t_{n+1})-4u(\cdot,t_n)+u(\cdot,t_{n-1})),\varphi_N)+$$

$$\frac{\beta}{2\Delta t}((\pi_N^1-I)(3\partial_x^2 u(\cdot,t_{n+1})-4\partial_x^2 u(\cdot,t_n)+\partial_x^2 u(\cdot,t_{n-1})),\varphi_N)+$$

$$\frac{v^{1/2}\Delta t^{-1/2}}{\Gamma(3/2)}\left((\pi_N^1-I)\left(u(\cdot,t_{n+1})-\sum_{j=0}^{n-1}(a_j-a_{j+1})u(\cdot,t_{n-j})-a_n u(\cdot,t_0)\right),\varphi_N\right),$$

令 $\varphi_N=4\Delta t\tilde{e}_N^{n+1}$,则有:

$$E(\tilde{e}_N^{n+1})-E(\tilde{e}_N^n)+4\Delta tv\|\partial_x\tilde{e}_N^{n+1}\|_0^2\leqslant 4\Delta t\|R_5^{n+1}\|_0\|\tilde{e}_N^{n+1}\|+4\Delta t\|(\pi_N^1-I)\partial_x u(\cdot,t_{n+1})\|_0\|\tilde{e}_N^{n+1}\|_0+$$

$$2\|(\pi_N^1-I)(3u(\cdot,t_{n+1})-4u(\cdot,t_n)+u(\cdot,t_{n-1}))\|_0\|\tilde{e}_N^{n+1}\|_0+2\beta\|(\pi_N^1-I)(3\partial_x^2 u(\cdot,t_{n+1})-$$

$$4\partial_x^2 u(\cdot,t_n)+\partial_x^2 u(\cdot,t_{n-1}))\|_0\|\tilde{e}_N^{n+1}\|_0+4\frac{v^{1/2}\Delta t^{1/2}}{\Gamma(3/2)}\times$$

$$\left\|(\pi_N^1-I)\left(u(\cdot,t_{n+1})-\sum_{j=0}^{n-1}(a_j-a_{j+1})u(\cdot,t_{n-j})-a_n u(\cdot,t_0)\right)\right\|_0\|\tilde{e}_N^{n+1}\|_0+$$

$$4\gamma\Delta t\left\|\frac{2}{3}A_1+\frac{1}{3}A_2\right\|_0\|\tilde{e}_N^{n+1}\|_0,$$

这里

$$A_1=u(\cdot,t_{n+1})\partial_x u(\cdot,t_{n+1})-\partial_x u_N^{n+1}(2u_N^n-u_N^{n-1}),$$

$$A_2=u(\cdot,t_{n+1})\partial_x u(\cdot,t_{n+1})-u_N^{n+1}\partial_x(2u_N^n-u_N^{n-1}).$$

这两项非线性项有如下的估计:

$$\|A_1\|_0^2=\|u(\cdot,t_{n+1})\partial_x u(\cdot,t_{n+1})-(2u(\cdot,t_n)-u(\cdot,t_{n-1}))\partial_x u(\cdot,t_{n+1})+(2u(\cdot,t_n)-u(\cdot,t_{n-1}))\partial_x u(\cdot,t_{n+1})-$$

$$(2u_N^n-u_N^{n-1})\partial_x u(\cdot,t_{n+1})+(2u_N^n-u_N^{n-1})\partial_x u(\cdot,t_{n+1})-(2u_N^n-u_N^{n-1})\partial_x u_N^{n+1}\|_0^2\leqslant 3\|(u(\cdot,t_{n+1})-2u(\cdot,t_n)+u(\cdot,t_{n-1}))\partial_x u(\cdot,t_{n+1})\|_0^2+3\|(e_N^n-e_N^{n-1})\partial_x u(\cdot,t_{n+1})\|_0^2+3\|(2u_N^n-u_N^{n-1})-\partial_x e_N^{n+1}\|_0^2\leqslant c(\Delta t^4+\|2e_N^n-e_N^{n-1}\|_0^2+\|\partial_x e_N^{n+1}\|_0^2),$$

$$A_2=\|u(\cdot,t_{n+1})\partial_x u(\cdot,t_{n+1})-u(\cdot,t_{n+1})\partial_x(2u(\cdot,t_n)-u(\cdot,t_{n-1}))+u(\cdot,t_{n+1})\partial_x(2u(\cdot,t_n)-u(\cdot,t_{n-1}))-u_N^{n+1}\partial_x(2u(\cdot,t_n)-u(\cdot,t_{n-1}))+u_N^{n+1}\partial_x(2u(\cdot,t_n)-u(\cdot,t_{n-1}))-u_N^{n+1}\partial_x(2u_N^n-u_N^{n-1})\|_0^2\leqslant 3\|u(\cdot,t_{n+1})\partial_x u(\cdot,t_{n+1})-2u(\cdot,t_n)+u(\cdot,t_{n-1})\|_0^2+3\|e_N^{n+1}\partial_x(2u(\cdot,t_n)-u(\cdot,t_{n-1}))\|_0^2+3\|u_N^{n+1}\partial_x(2e_N^n-e_N^{n-1})\|_0^2\leqslant c(\Delta t^4+\|\partial_x(2e_N^n-e_N^{n-1})\|_0^2+\|e_N^{n+1}\|_0^2).$$

利用 Cauchy-Schwar 不等式和 Young's 不等式,可得

$$E(\tilde e_N^{n+1})-E(\tilde e_N^n)\leqslant\Delta t\|R_2^{n+1}\|^2+2\Delta t\|\tilde e_N^{n+1}\|_0^2+2\Delta t\|(\pi_N^1-I)\partial_x u(\cdot,t_{n+1})\|_0^2+2\Delta t\|\tilde e_N^{n+1}\|_0^2+2\int_{t_n}^{t_{n+1}}\|(\pi_N^1-I)\partial_t^2 u(\cdot,t)\|_0^2\mathrm{d}t+2\Delta t\|\tilde e_N^{n+1}\|_0^2+8\beta\int_{t_n}^{t_{n+1}}\|(\pi_N^1-I)\partial_x\partial_t^2 u(\cdot,t)\|_0^2\mathrm{d}t+2\beta\Delta t\|\partial_x\tilde e_N^{n+1}\|_0^2+\frac{2v^{1/2}\Delta t}{\Gamma(1/2)}\left\|\int_0^{t_{n+1}}\frac{(\pi_N^1-I)\partial_s u(\cdot,s)}{(t_{n+1}-s)^{1/2}}\mathrm{d}s\right\|_0^2+2v^{1/2}\Delta t\|\partial_x\tilde e_N^{n+1}\|_0^2+c\Delta t(\Delta t^4+\|2e_N^n-e_N^{n-1}\|_1^2+\|e_N^{n+1}\|_1^2+\|\tilde e_N^{n+1}\|_0^2).$$

不等式两边分别对 $n=1,2,\cdots,k$ 求和,由(10)和(14),我们可得:

$$E(\tilde e_N^{n+1})-E(\tilde e_N^n)\leqslant c(\Delta t^{9/4}+N^{2-2m})+c\Delta t\sum_{n=1}^{k+1}(\|2\tilde e_N^n-\tilde e_N^{n-1}\|_0^2+\|\tilde e_N^{n+1}\|_0^2).$$

对于第一步,易知

$$E(\tilde e_N^1)\leqslant c(\Delta t^3+\Delta tN^{2-2m}).$$

由离散的 Gronwall 引理和不等式

$$\|e_N^n\|\leqslant\|\tilde e_N^n\|_1+\|\bar e_N^n\|_1,\ n\geqslant 0,$$

即可得(15).

定理 4　设 $\{u_N^k\}_{k=0}^M$ 为全离散格式(12)、(13)的解,则原函数 m 阶光滑,则有下列估计式:

$$\|u(\cdot,t_k)-u_N^k\|_1\leqslant c(\Delta t^{3/2}+N^{1-m}),\quad k=0,1,2,\cdots,M.$$

证明　证明方法与过程与定理 3 类似,这里略去.

4　数值结果

为了考察算法的有效性,对给定的方程,我们用 Fourire-Galerkin 谱方法,可以得到关于 $\{\hat u_k^{n+1}\}$ 的一系列线性方程组.

BD2/F-G 格式:

$$\frac{1}{2\Delta t}(1+\beta(2\pi k/L)^2)(3\hat u_k^{n+1}-\hat u_k^n+\hat u_k^{n-1})+(\mathrm{i}2\pi k/L+\alpha(2\pi k/L)^2)\hat u_k^{n+1}+\frac{v^{1/2}\Delta t^{-1/2}}{\Gamma(3/2)}\left(\hat u_k^{n+1}-\sum_{j=0}^{n-1}(a_j-a_{j+1})\hat u_k^{n-j}-a_n\hat u_k^0\right)+\frac{\gamma}{3}\{(2u^n-u^{n-1})\partial_x u^{n+1}+u^{n+1}\partial_x(2u^n-u^{n-1})\}_k=0.$$

C-N/F-G 格式:

$$\frac{1}{\Delta t}(\hat u_k^{n+1}-\hat u_k^n)(1+\beta(2\pi k/L)^2)+(\mathrm{i}2\pi k/L+\alpha(2\pi k/L)^2)\hat u_k^{n+1/2}+\frac{v^{1/2}\Delta t^{-1/2}}{\Gamma(3/2)}\left(\hat u_k^{n+1/2}-\sum_{j=0}^{n-1}(a_j-a_{j+1})\hat u_k^{n-1/2-j}-a_n\hat u_k^0\right)+\frac{\gamma}{6}\{2(3u_N^n-u_N^{n-1})\partial_x u_N^{n+1/2}+u_N^{n+1/2}\partial_x(3u_N^n-u_N^{n-1})\}_k=0,$$

这里 $\hat f_k$ 或者 $\{f\}_k$ 表示函数 f 的 k 个 Fourier 系数.

在实际的计算中,方程的准确解很难找到,因此我们考虑用下列近似方法计算收敛阶,定义

$$\mathrm{Rate}=\log_2\left(\frac{\|u_N^{n,2\Delta t}-u_N^{2n,\Delta t}\|_0}{\|u_N^{2n,\Delta t}-u_N^{4n,\Delta t/2}\|_0}\right),$$

这里 $u_N^{n,2\Delta t}$ 为格式在时间步长 Δt 下 $n\Delta t$ 时刻的数值解.

取 $u(x,0)=3\mathrm{sech}^2(0.5^{3/2}(x-x_0))^2$, $x_0=L/2$, $L=400$, $T=10$, $\alpha=\beta=\gamma=1$. 由表可知 $v=0$ 方程不带分数阶项,BD2/F-G 与 C-N/F-G 格式的时间收敛阶是接近 2 阶,当 $v=1$ 时格式中包含分类阶项,分数阶项的误差

会影响整体的误差阶数,所以格式的时间收敛阶是接近 1.5 阶,表中的数据可以证明我们的格式是有效的.

表 1　BD2/F-G 格式解的收敛阶随时间步长的变化情况

Table 1　BD2/F-G, the convergence order of the scheme varies with time

$v/\Delta t$	$\Delta t=0.1$	$\Delta t=0.05$	$\Delta t=0.01$	$\Delta t=0.005$	$\Delta t=0.001$
$v=0$	2.068 7	2.053 4	2.012 2	2.006 3	2.001 3
$v=1$	1.388 9	1.410 7	1.454 4	1.467 4	1.485 3

表 2　C-N/F-G 格式解的收敛阶随时间步长的变化情况

Table 2　C-N/F-G, the convergence order of the scheme varies with time

$v/\Delta t$	$\Delta t=0.1$	$\Delta t=0.05$	$\Delta t=0.01$	$\Delta t=0.005$	$\Delta t=0.001$
$v=0$	2.005 7	2.010 6	2.023 4	2.001 8	2.000 4
$v=1$	1.404 6	1.428 7	1.466 1	1.475 9	1.489 2

5　结语

本文提出了两种求解粘性 BBM 型分数阶方程无条件稳定的数值格式,格式的优点在于每次迭代只需要求解一个线性方程,并且我们给出了格式的误差估计,数值结果验证了格式的准确性,两种格式在时间方向是 1.5 阶,不存在分数阶项时,格式在时间方向是 2 阶,空间方向是谱精度.

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