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Global Stability of Large Steady-States to a Non-Isentropic Euler-Maxwell System in $\mathbb{R}^3$

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Abstract:This paper is concerned with a stability problem in $\mathbb{R}^3$ for a non-isentropic Euler-Maxwell system without temperature diffusion term. When the initial data are close to the steady states of the system,we show the global existence of smooth solutions which converge toward the steady states as the time tends to infinity. The basic idea is to make a change of unknown variables and choose a non-diagonal symmetrizer of the full Euler equations to get the dissipation estimates. In addition,an induction argument on the order of derivatives of solutions in energy estimates plays a key role in obtaining the stability result.

Key words:Euler-Maxwell system,global smooth solution,stability,steady-states solution,energy estimate

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$\mathbb{R}^3$上非等熵 Euler-Maxwell 系统稳态解的全局稳定性

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[摘要] 本文考虑的是无温度扩散项的非等熵 Euler-Maxwell 系统在$\mathbb{R}^3$上的稳定性问题. 当初值接近系统的稳态时,我们给出光滑解的整体存在性,且当时间趋于无穷大时该光滑解收敛于稳态. 其基本思想是改变未知变量并选取完全 Euler 方程的非对角对称化子来得到耗散估计. 此外,对解的导数的阶的归纳论证在得到稳定性结果中也起着关键作用.

[关键词] Euler-Maxwell 系统,整体光滑解,稳定性,稳态解,能量估计

1 Introduction and Main Results

This paper concerns the global stability of smooth solutions to the Cauchy problem for a non-isentropic Euler-Maxwell system which describe the dynamics of electrons for plasmas. Let $n,\boldsymbol{u}=(u_1,u_2,u_3)^{\mathrm{T}},p,\theta$ be the density,the velocity,the pressure and the absolute temperature of the electrons,respectively. The total energy $\mathcal{E}$ is defined by

$$\mathcal{E}=\frac{1}{2}n|\boldsymbol{u}|^2+n\theta.$$

We denote by $\boldsymbol{E}=(E_1,E_2,E_3)^{\mathrm{T}}$ and $\boldsymbol{B}=(B_1,B_2,B_3)^{\mathrm{T}}$ the electric and magnetic fields of the plasma. The system satisfied by these variables reads(see [1-2])

$$\begin{cases}\partial_t n+\operatorname{div}(n\boldsymbol{u})=0,\\ \partial_t(n\boldsymbol{u})+\operatorname{div}(n\boldsymbol{u}\otimes\boldsymbol{u})+\nabla p=-n(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B})-n\boldsymbol{u},\\ \partial_t\mathcal{E}+\operatorname{div}(\mathcal{E}\boldsymbol{u}+p\boldsymbol{u})\boldsymbol{u}\cdot\nabla\theta=-n\boldsymbol{u}\cdot\boldsymbol{E}-(\mathcal{E}-n\theta_e),\\ \partial_t\boldsymbol{E}-\nabla\times\boldsymbol{B}=n\boldsymbol{u},\operatorname{div}\boldsymbol{E}=b(\boldsymbol{x})-n,\\ \partial_t\boldsymbol{B}+\nabla\times\boldsymbol{E}=0,\operatorname{div}\boldsymbol{B}=0.\end{cases}\tag{1}$$

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for $t\geqslant 0$ and $\boldsymbol{x}=(x_1,x_2,x_3)\in\mathbb{R}^3$, where $\theta_e>0$ is a constant, $b(\boldsymbol{x})$ is the ion density, and $-n(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B})$ stands for the Lorentz force. We remark that b can be large in our stability result and here b is sufficiently smooth with $b\geqslant$ const.>0 in $\mathbb{R}^3$. The system is complemented by the following initial conditions

$$t=0:(n,\boldsymbol{u},\theta,\boldsymbol{E},\boldsymbol{B})=(n_0(\boldsymbol{x}),\boldsymbol{u}_0(\boldsymbol{x}),\theta_0(\boldsymbol{x}),\boldsymbol{E}_0(\boldsymbol{x}),\boldsymbol{B}_0(\boldsymbol{x})),\quad \boldsymbol{x}\in\mathbb{R}^3. \tag{2}$$

We assume that

$$\operatorname{div}\boldsymbol{E}_0=b(\boldsymbol{x})-n_0,\operatorname{div}\boldsymbol{B}_0=0. \tag{3}$$

Then in(1)the constraint equations

$$\operatorname{div}\boldsymbol{E}=b(\boldsymbol{x})-n,\operatorname{div}\boldsymbol{B}=0.$$

hold for all time $t>0$.

For convenience, we consider the case of ideal polytropic gas

$$p=n\theta. \tag{4}$$

By state equation(4), for smooth solutions with $n>0$, the momentum and energy equations in(1)can be written as

$$\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\frac{\theta}{n}\nabla n+\nabla\theta=-(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B})-\boldsymbol{u}, \tag{5}$$

$$\partial_t\theta+\boldsymbol{u}\cdot\nabla\theta+\frac{2}{3}\theta\operatorname{div}\boldsymbol{u}=\frac{1}{3}|\boldsymbol{u}|^2-(\theta-\theta_e). \tag{6}$$

The last terms $-\boldsymbol{u}$ in(5)and $-(\theta-\theta_e)$ in(6)stand for the velocity dissipation and temperature dissipation in energy estimates, respectively.

It is well known that system(1)for variables$(n,\boldsymbol{u},\theta,\boldsymbol{E},\boldsymbol{B})$is symmetrizable hyperbolic when $n>0$. Due to the local existence and uniqueness of smooth solutions(see [3]), Cauchy problem(1)-(2)admits a unique local smooth solution when the initial data are smooth. More precisely, let $s\geqslant 3$ be an integer and$(n_0,\boldsymbol{u}_0,\theta_0,\boldsymbol{E}_0,\boldsymbol{B}_0)\in H^s(\mathbb{R}^3)$ satisfying $n_0\geqslant$ const.>0, then there exist $T_*>0$ and a unique solution$(n,\boldsymbol{u},\theta,\boldsymbol{E},\boldsymbol{B})$to Cauchy problem (1)-(2)such that

$$(n,\boldsymbol{u},\theta,\boldsymbol{E},\boldsymbol{B})\in C([0,T_*];H^s(\mathbb{R}^3))\cap C^1([0,T_*];H^{s-1}(\mathbb{R}^3)),n\geqslant\text{const.}>0.$$

For a multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^3$, we denote

$$\partial_x^\alpha=\frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}},\text{with }|\alpha|=\alpha_1+\alpha_2+\alpha_3.$$

We denote by $\|\cdot\|_s$ the usual norms of Sobolev spaces $H^s(\mathbb{R}^3)$. The inner product and the norm in $L^2(\mathbb{R}^3)$ are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. For any given $T>0$, let $B_{s,T}(\mathbb{R}^3)$be the Banach spaces defined by

$$B_{s,T}(\mathbb{R}^3)=\bigcap_{k=0}^{s}C^k([0,T];H^{s-k}(\mathbb{R}^3)),$$

equipped with the norm

$$\|\boldsymbol{v}\|_{B_{s,T}}=\max_{0\leqslant t\leqslant T}|||\boldsymbol{v}(t,\cdot)|||_s,\quad\forall\boldsymbol{v}\in B_{s,T}(\mathbb{R}^3),$$

where

$$|||\boldsymbol{v}(t,\cdot)|||_s=\Big(\sum_{|\alpha|+k\leqslant s}\|\partial_t^k\partial_x^\alpha\boldsymbol{v}(t,\cdot)\|^2\Big)^{1/2}.$$

By the local well-posedness together with(1), we have$(n,\boldsymbol{u},\theta,\boldsymbol{E},\boldsymbol{B})\in B_{s,T_*}(\mathbb{R}^3)$.

Now consider steady-state solution$(\bar{n},\bar{\boldsymbol{u}},\bar{\theta},\bar{\boldsymbol{E}},\bar{\boldsymbol{B}})$, with zero velocity $\bar{\boldsymbol{u}}=0$. By(1)and(4), we have

$$\begin{cases}\dfrac{\nabla\bar{p}}{\bar{n}}=-\bar{\boldsymbol{E}},\bar{p}=\theta_e\bar{n},\\ \bar{\theta}=\theta_e,\\ \nabla\times\bar{\boldsymbol{B}}=0,\operatorname{div}\bar{\boldsymbol{E}}=b(\boldsymbol{x})-\bar{n},\\ \nabla\times\bar{\boldsymbol{E}}=0,\operatorname{div}\bar{\boldsymbol{B}}=0.\end{cases} \tag{7}$$

The equations for $\bar{\boldsymbol{B}}$ imply that $\bar{\boldsymbol{B}}$ is a constant vector, and the third equations in(7)imply that $\bar{n}$ satisfies an elliptic equation

$$-\theta_e\Delta(\ln n)+n=b(\boldsymbol{x}). \tag{8}$$

Since $n\mapsto\ln n$ is a strictly increasing function, this equation admits a unique solution $\bar{n}$ (see [4-5]). Moreover, $b\geqslant$ const.>0. implies $n\geqslant$ const.>0. Once $\bar{n}$ is known, $\bar{\boldsymbol{E}}$ is given explicitly by

$$\bar{\boldsymbol{E}}=-\nabla(\theta_e\ln\bar{n}).$$

Proposition 1 (see [5]) Let $q\geqslant3$, Assume $\nabla b\in H^{q-1}(\mathbb{R}^3)$, and $b\in L^\infty(\mathbb{R}^3)$ is a smooth function such $b\geqslant$ const.>0. a.e. $\boldsymbol{x}\in\mathbb{R}^3$. Then non-isenteopic Euler-Maxwell system (1) admits a unique smooth steady-state solution $(\bar{n},0,\bar{\theta},\bar{\boldsymbol{E}},\bar{\boldsymbol{B}})$ satisfying $\bar{n}\geqslant$ const.>0, where $\bar{\theta}=\theta_e$ and $\bar{\boldsymbol{B}}$ are constant.

In the recent years, there have been extensive studies on the Euler-Maxwell system because of its physical importance, complexity, rich phenomena, and mathematical challenges. When the background density b is a positive constant or is a small perturbation of a constant for both isentropic and non-isentropic Euler-Maxwell systems, such a stability problem has been investigated in $\mathbb{R}^3$ or in the torus $\mathbb{T}^3=(\mathbb{R}/2\pi)^3$ by many authors (see [6-11]).

When b is large, the above techniques do not work and there are only a few results concerning this topic. For the isentropic Euler-Poisson systems with insulating boundary conditions, Guo and Strauss in [4] utilized an anti-symmetric matrix technique and the energy estimates for the divergence and the rotation of the velocity to obtain the stability. In [12], Peng followed the idea of [4] and further developed an induction argument on the order of the derivatives of the solutions in energy estimates and solved this problem for both the isentropic Euler-Poisson system and Euler-Maxwell system. These results were also extended to the two-fluid isentropic cases in [13]. In a very similar way, Feng et al in [14] obtained the stability of solutions for a non-isentropic Euler-Maxwell system with an additional temperature diffusion term. See also their very recent work [15] for a similar problem of a two-fluid non-isentropic Euler-Maxwell system.

A more interesting problem is that the stability result holds or not, when the temperature diffusion term is absent. To accomplish this, Liu and Peng in [16] introduced new variables $(\ln p,\boldsymbol{u},\theta)$ and chose a non-diagonal symmetrizer for the full Euler equations. This allows to established the desired stability result for non-isentropic Euler-Poisson system and Euler-Maxwell system (see [10-11]). We point out that when b is large, the above results are valid only for bounded domains. More recently, Liu et al in [5] established the global stability of large steady-states for Euler-Maxwell systems in $\mathbb{R}^3$.

The aim of this paper is to generalize the result for isentropic Euler-Maxwell system in [5] to the non-isentropic case, namely, to establish the global existence of smooth solutions to Cauchy problem (1)-(2), when $(n_0,\boldsymbol{u}_0,\theta_0,\boldsymbol{E}_0,\boldsymbol{B}_0)$ is a small perturbation of the steady state $(\bar{n},0,\bar{\theta},\bar{\boldsymbol{E}},\bar{\boldsymbol{B}})$. To this end, we introduce the new variables $(\ln p,\boldsymbol{u},\theta)$, choose a non-diagonal symmetrizer for the full Euler equation (see [11]), and use an induction argument on the order of derivatives of solutions in energy estimates to obtain the stability result.

The main result of this paper is Theorem 1 stated below.

Theorem 1 Let $s\geqslant3$, $q\geqslant3$ be a integer and $(n_0,\boldsymbol{u}_0,\theta_0,\boldsymbol{E}_0,\boldsymbol{B}_0)\in H^s(\mathbb{R}^3)$ satisfying (3). Under the assumptions of Proposition 1, there exist constants $\delta>0$ and $C>0$, such that if

$$\|(n_0-\bar{n},\boldsymbol{u}_0,\theta_0-\theta_e,\boldsymbol{E}_0-\bar{\boldsymbol{E}},\boldsymbol{B}_0-\bar{\boldsymbol{B}})\|_s\leqslant\delta, \tag{9}$$

Cauchy problem (1)-(2) admits a unique global solution $(n,\boldsymbol{u},\theta,\boldsymbol{E},\boldsymbol{B})$ satisfying

$$\begin{aligned}&|||(n(t,\cdot)-\bar{n},\boldsymbol{u}(t,\cdot),\theta(t,\cdot)-\theta_e,\boldsymbol{E}(t)-\bar{\boldsymbol{E}},\boldsymbol{B}(t)-\bar{\boldsymbol{B}})|||_s^2+\int_0^t(|||(n(\tau,\cdot)-\bar{n},\boldsymbol{u}(\tau,\cdot),\theta(\tau,\cdot)-\theta_e)|||_s^2+\\&|||\boldsymbol{E}(\tau,\cdot)-\bar{\boldsymbol{E}}|||_{s-1}^2+|||\partial_\tau\boldsymbol{B}(\tau,\cdot)|||_{s-2}^2+|||\nabla_x\boldsymbol{B}(\tau,\cdot)|||_{s-2}^2)\mathrm{d}\tau\leqslant\\&C\|(n_0-\bar{n},\boldsymbol{u}_0,\theta_0-\theta_e,\boldsymbol{E}_0-\bar{\boldsymbol{E}},\boldsymbol{B}_0-\bar{\boldsymbol{B}})\|_s^2,\quad\forall t\geqslant0.\end{aligned} \tag{10}$$

Moreover,

$$\lim_{t\to\infty}|||(n(t,\cdot)-\bar{n},\boldsymbol{u}(t,\cdot),\theta(t,\cdot)-\theta_e)|||_{s-1}=0, \tag{11}$$

$$\lim_{t\to\infty}|||\boldsymbol{E}(t)-\bar{\boldsymbol{E}}|||_{s-1}=0, \tag{12}$$

$$\lim_{t\to\infty}(|||\partial_t \boldsymbol{B}(t)|||_{s-2}+|||\nabla_x \boldsymbol{B}(t)|||_{s-2})=0,\tag{13}$$

The paper is organized as follows. In the next section, we show the symmetrization of the Euler equations with new variables$(\ln p,\boldsymbol{u},\theta)$. The energy estimates and the proof of Theorem 1 are presented in the last section.

2 Symmetrization of Euler Equations with$(\ln p,\boldsymbol{u},\theta)$

We first introduce new variables. From(1),(4)and(6), it is easy to see the pressure p satisfies the equation

$$\partial_t p+\boldsymbol{u}\cdot\nabla p+\frac{5}{3}p\operatorname{div}\boldsymbol{u}=\frac{p}{3\theta}|\boldsymbol{u}|^2-\frac{p}{\theta}(\theta-\theta_e).$$

Let

$$q=\ln p,\bar{q}=\ln\bar{p}.$$

Obviously, for $p>0$, we have

$$\partial_t q+\boldsymbol{u}\cdot\nabla q+\frac{5}{3}\operatorname{div}\boldsymbol{u}=\frac{1}{3\theta}|\boldsymbol{u}|^2-\frac{1}{\theta}(\theta-\theta_e).\tag{14}$$

Introduce the perturbed variables

$$Q=q-\bar{q},\boldsymbol{\Theta}=\theta-\theta_e,\boldsymbol{F}=\boldsymbol{E}-\overline{\boldsymbol{E}},\boldsymbol{G}=\boldsymbol{B}-\overline{\boldsymbol{B}},U=\begin{pmatrix}Q\\ \boldsymbol{u}\\ \Theta\end{pmatrix},\boldsymbol{Z}=\begin{pmatrix}\boldsymbol{U}\\ \boldsymbol{F}\\ \boldsymbol{G}\end{pmatrix}.\tag{15}$$

Substituting these expressions into(1)and(14), and taking into account(5)-(6), it yields the system satisfied by $\boldsymbol{Z}$:

$$\begin{cases}\partial_t Q+\boldsymbol{u}\cdot\nabla Q+\frac{5}{3}\operatorname{div}\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\bar{q}=\frac{1}{3\theta}|\boldsymbol{u}|^2-\frac{1}{\theta}\boldsymbol{\Theta},\\ \partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\theta\nabla Q+\nabla\bar{q}\boldsymbol{\Theta}=-\boldsymbol{F}-\boldsymbol{u}-\boldsymbol{u}\times(\overline{\boldsymbol{B}}+\boldsymbol{G}),\\ \partial_t\boldsymbol{\Theta}+\boldsymbol{u}\cdot\nabla\boldsymbol{\Theta}+\frac{2}{3}\theta\operatorname{div}\boldsymbol{u}=\frac{1}{3}|\boldsymbol{u}|^2-\boldsymbol{\Theta},\\ \partial_t\boldsymbol{F}-\nabla\times\boldsymbol{G}=n\boldsymbol{u},\operatorname{div}\boldsymbol{F}=-N,\\ \partial_t\boldsymbol{G}+\nabla\times\boldsymbol{F}=0,\operatorname{div}\boldsymbol{G}=0,\end{cases}\tag{16}$$

Where $N=n-\bar{n}$ is regarded as a function of Q and $\boldsymbol{\Theta}$.

$$N=\frac{e^q}{\theta}-\frac{e^{\bar{q}}}{\theta_e}=O(Q)+O(\boldsymbol{\Theta}).\tag{17}$$

The first three equations in(16)are the full Euler equations and can be written in the form

$$\partial_t\boldsymbol{U}+\sum_{j=1}^{d}\boldsymbol{A}_j(\boldsymbol{u},\theta)\partial_{x_j}\boldsymbol{U}+\boldsymbol{L}(\boldsymbol{x})\boldsymbol{U}=\boldsymbol{K}(\boldsymbol{u},\theta,\boldsymbol{F},\boldsymbol{G},\boldsymbol{x}),\tag{18}$$

supplemented by the Maxwell equations

$$\begin{cases}\partial_t\boldsymbol{F}-\nabla\times\boldsymbol{G}=n\boldsymbol{u},\operatorname{div}\boldsymbol{F}=-N,\\ \partial_t\boldsymbol{G}+\nabla\times\boldsymbol{F}=0,\operatorname{div}\boldsymbol{G}=0,\end{cases}\tag{19}$$

where

$$\boldsymbol{A}_j(\boldsymbol{u},\theta)=\begin{pmatrix}u_j & \frac{5}{3}\boldsymbol{e}_j^{\mathrm{T}} & 0\\ \theta\boldsymbol{e}_j & u_j\boldsymbol{I}_3 & 0\\ 0 & \frac{2}{3}\theta\boldsymbol{e}_j^{\mathrm{T}} & u_j\end{pmatrix},\quad j=1,2,3,$$

$$\boldsymbol{L}(\boldsymbol{x})=\begin{pmatrix}0 & (\nabla\bar{q})^{\mathrm{T}} & 0\\ 0 & 0 & \nabla\bar{q}\\ 0 & 0 & 0\end{pmatrix},$$

$$\boldsymbol{K}(\boldsymbol{u},\theta,\boldsymbol{F},\boldsymbol{G},\boldsymbol{x})=\begin{pmatrix}\frac{1}{3\theta}|\boldsymbol{u}|^2-\frac{1}{\theta}\Theta\\ -\boldsymbol{F}-\boldsymbol{u}-\boldsymbol{u}\times(\overline{\boldsymbol{B}}+\boldsymbol{G})\\ \frac{1}{3}|\boldsymbol{u}|^2-\Theta\end{pmatrix},$$

Here $\boldsymbol{I}_3$ is the 3×3 unit matrix, $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$ is the canonical basis of $\mathbb{R}^3$, and $\boldsymbol{e}_j^{\mathrm{T}}$ is the transpose of $\boldsymbol{e}_j$. From (2) and (15), the initial conditions for (18)-(19) is

$$t=0: \boldsymbol{Z}=\boldsymbol{Z}_0 \xlongequal{\text{def.}} (Q_0,\boldsymbol{u}_0,\Theta_0,\boldsymbol{F}_0,\boldsymbol{G}_0)^{\mathrm{T}}(\boldsymbol{x}),\quad \boldsymbol{x}\in\mathbb{R}^3, \tag{20}$$

where

$$Q_0=\ln(n_0\theta_0)-\ln(\bar{n}\theta_e),\ \Theta_0=\theta_0-\theta_e,\ \boldsymbol{F}_0=\boldsymbol{E}_0-\overline{\boldsymbol{E}},\ \boldsymbol{G}_0=\boldsymbol{B}_0-\overline{\boldsymbol{B}}.$$

One can take the non-diagonal symmetrizer $\boldsymbol{A}_0(p,\theta)$ as

$$\boldsymbol{A}_0(p,\theta)=\begin{pmatrix}p & 0 & -\frac{p}{\theta}\\ 0 & \frac{p}{\theta}\boldsymbol{I}_3 & 0\\ -\frac{p}{\theta} & 0 & \frac{5p}{2\theta^2}\end{pmatrix}. \tag{21}$$

Since it is symmetric and positive definite for $p>0$ and $\theta>0$. And

$$\widetilde{\boldsymbol{A}}_j(p,\boldsymbol{u},\theta)\xlongequal{\text{def.}}\boldsymbol{A}_0(p,\theta)\boldsymbol{A}_j(\boldsymbol{u},\theta)=\begin{pmatrix}pu_j & p\boldsymbol{e}_j^{\mathrm{T}} & -\frac{p}{\theta}u_j\\ p\boldsymbol{e}_j & \frac{p}{\theta}u_j\boldsymbol{I}_3 & 0\\ -\frac{p}{\theta}u_j & 0 & \frac{5p}{2\theta^2}u_j\end{pmatrix}$$

is symmetric, system (18) is symmetrizable hyperbolic. Furthermore,

$$\boldsymbol{A}_0(p,\theta)\boldsymbol{L}(\boldsymbol{x})=\begin{pmatrix}0 & p(\nabla\bar{q})^{\mathrm{T}} & 0\\ 0 & 0 & \frac{p}{\theta}\nabla\bar{q}\\ 0 & -\frac{p}{\theta}(\nabla\bar{q})^{\mathrm{T}} & 0\end{pmatrix}.$$

Let us introduce the matrix

$$\boldsymbol{B}(p,\boldsymbol{u},\theta,\boldsymbol{x})=\sum_{j=1}^{3}\partial_{x_j}\widetilde{\boldsymbol{A}}_j(p,u,\theta)-2\boldsymbol{A}_0(p,\theta)\boldsymbol{L}(\boldsymbol{x}).$$

It follows that

$$\boldsymbol{B}(p,\boldsymbol{u},\theta,\boldsymbol{x})=\begin{pmatrix}\operatorname{div}(p\boldsymbol{u}) & (\nabla p)^{\mathrm{T}}-\frac{2p}{\bar{p}}(\nabla\bar{p})^{\mathrm{T}} & \operatorname{div}\left(\frac{p\boldsymbol{u}}{\theta}\right)\\ \nabla p & \operatorname{div}\left(\frac{p\boldsymbol{u}}{\theta}\right)\boldsymbol{I}_3 & -\frac{2p}{\theta\bar{p}}\nabla\bar{p}\\ -\operatorname{div}\left(\frac{p\boldsymbol{u}}{\theta}\right) & -\frac{2p}{\theta\bar{p}}(\nabla\bar{p})^{\mathrm{T}} & \frac{5}{2}\operatorname{div}\left(\frac{p\boldsymbol{u}}{\theta^2}\right)\end{pmatrix}.$$

Therefore,

$$\boldsymbol{B}(p,\boldsymbol{u},\theta,\boldsymbol{x})\mid_{(p,\boldsymbol{u},\theta)=(\bar{p},\boldsymbol{0},\theta_e)}=\begin{pmatrix} 0 & -(\nabla\bar{p})^{\mathrm{T}} & 0 \\ \nabla\bar{p} & 0 & -\dfrac{2}{\theta_e}\nabla\bar{p} \\ 0 & \dfrac{2}{\theta_e}(\nabla\bar{p})^{\mathrm{T}} & 0 \end{pmatrix},$$

is an antisymmetric matrix. These expressions and properties on $\boldsymbol{A}_0,\boldsymbol{A}_j,\widetilde{\boldsymbol{A}}_j$ and $\boldsymbol{B}$ will be useful in the energy estimates of the next section.

3 Energy Estimates and Proof of Theorem 1

Let $T>0$ and $\boldsymbol{Z}$ be a smooth solution of problem(18)-(20) defined on the interval[0,T]. We denote

$$Z_T=\max_{0\leqslant t\leqslant T}|||\boldsymbol{Z}(t,\cdot)|||_s.$$

In this section, we assume $s\geqslant 3$ and Z_T is sufficiently small, so that

$$\frac{\bar{n}}{2}\leqslant n\leqslant\frac{3\bar{n}}{2},\frac{\bar{p}}{2}\leqslant p\leqslant\frac{3\bar{p}}{2},|\boldsymbol{u}|\leqslant\frac{1}{2},\frac{\theta_e}{2}\leqslant\theta\leqslant\frac{3\theta_e}{2}.\tag{22}$$

For $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^3$ and $\beta=(\beta_1,\beta_2,\beta_3)\in\mathbb{N}^3$, $\beta\leqslant\alpha$ stands for $\beta_j\leqslant\alpha_j$, for all $j=1,2,3$, and $\beta<\alpha$ stands for $\beta\leqslant\alpha$ and $\beta\neq\alpha$.

Let $C_0>0, C>0$ be generic constants independent of any time. We want to establish an energy estimate the form

$$|||\boldsymbol{Z}(t,\cdot)|||_s^2+C_0\int_0^t(|||\boldsymbol{U}(\tau,\cdot)|||_s^2+|||\boldsymbol{F}(\tau,\cdot)|||_{s-1}^2+|||\nabla_x\boldsymbol{G}(\tau,\cdot)|||_{s-2}^2+|||\partial_\tau\boldsymbol{G}(\tau,\cdot)|||_{s-2})\mathrm{d}\tau\leqslant C\|\boldsymbol{Z}_0\|_s^2, t\in[0,T].\tag{23}$$

Similar to the periodic case in[10], we can proceed the same procedure to obtain(23) for Cauchy problem through the following lemmas, so we omit the details here.

Lemma 1 For $k\in\mathbb{N}$ and $\alpha\in\mathbb{N}^3$ with $k+|\alpha|\leqslant s$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\langle\boldsymbol{A}_0\boldsymbol{U}_{k,\alpha},\boldsymbol{U}_{k,\alpha}\rangle+\|\boldsymbol{F}_{k,\alpha}\|^2+\|\boldsymbol{G}_{k,\alpha}\|^2)+C_0(\|\boldsymbol{u}_{k,\alpha}\|^2+\|\boldsymbol{\Theta}_{k,\alpha}\|^2)\leqslant C(\|\partial_t^k(\boldsymbol{u},\boldsymbol{\Theta},\boldsymbol{F})\|_{|\alpha|-1}^2+\|\partial_t^kQ\|_{|\alpha|}^2)+C|||N|||_s^2|||\boldsymbol{Z}|||_s+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s.\tag{24}$$

Lemma 2 For all $k\in N$ with $k\leqslant s$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\langle\boldsymbol{A}_0\boldsymbol{U}_{k,0},\boldsymbol{U}_{k,0}\rangle+\|\boldsymbol{F}_{k,0}\|^2+\|\boldsymbol{G}_{k,0}\|^2)+C_0(\|\boldsymbol{u}_{k,0}\|+\|\boldsymbol{\Theta}_{k,0}\|^2)\leqslant C|||N|||_s^2|||\boldsymbol{Z}|||_s+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s.\tag{25}$$

Lemma 3 Let $k\in\mathbb{N}$ and $\alpha\in\mathbb{N}^3$ with $k+|\alpha|\leqslant s$, we have

$$\|\partial_t^kN\|_{|\alpha|-1}^2\leqslant C(\|\partial_t^kQ\|_{|\alpha|-1}^2+\|\partial_t^k\boldsymbol{\Theta}\|_{|\alpha|-1}^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s,\tag{26}$$

$$\|\partial_t^kQ\|_{|\alpha|}^2\leqslant C(\|\partial_t^k(Q,\boldsymbol{u},\boldsymbol{\Theta})\|_{|\alpha|-1}^2+\|\partial_t^{k+1}\boldsymbol{u}\|_{|\alpha|-1}^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s,\tag{27}$$

and

$$\|\partial_t^k\boldsymbol{F}\|_{|\alpha|-1}^2\leqslant C(\|\partial_t^k(Q,\boldsymbol{u},\boldsymbol{\Theta})\|_{|\alpha|-1}^2+\|\partial_t^{k+1}\boldsymbol{u}\|_{|\alpha|-1}^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s.\tag{28}$$

Lemma 4 Let $k\in\mathbb{N}$ with $k\leqslant s-1$, we have

$$\|\partial_t^kQ\|_1^2\leqslant C(\|\partial_t^k(\boldsymbol{u},\boldsymbol{\Theta})\|^2+\|\partial_t^{k+1}\boldsymbol{u}\|^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s,\tag{29}$$

$$\|\partial_t^sQ\|^2\leqslant C\|\partial_t^{s-1}(\boldsymbol{u},\boldsymbol{\Theta})\|_1^2+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s.\tag{30}$$

It follows from Lemma 1-3 that

Lemma 5 For all $k\in\mathbb{N}$ and $\alpha\in\mathbb{N}^3$ with $1\leqslant|\alpha|$ and $k+|\alpha|\leqslant s$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\beta\leqslant\alpha}(\langle\boldsymbol{A}_0\boldsymbol{U}_{k,\beta},\boldsymbol{U}_{k,\beta}\rangle+\|\boldsymbol{F}_{k,\beta}\|^2+\|\boldsymbol{G}_{k,\beta}\|^2)+C_0\|\partial_t^k\boldsymbol{U}\|_{|\alpha|}^2\leqslant C(\|\partial_t^k\boldsymbol{U}\|_{|\alpha|-1}^2+\|\partial_t^{k+1}\boldsymbol{u}\|_{|\alpha|-1}^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s.\tag{31}$$

We infer from Lemma 2 and Lemma 4 that

Lemma 6 When Z_T is sufficiently small, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\langle \boldsymbol{A}_0\partial_t^s\boldsymbol{U},\partial_t^s\boldsymbol{U}\rangle+\|\partial_t^s\boldsymbol{F}\|^2+\|\partial_t^s\boldsymbol{G}\|^2)+C_0\|\partial_t^s\boldsymbol{U}\|^2\leqslant C\|\partial_t^{s-1}\boldsymbol{U}\|_1^2+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s. \tag{32}$$

Proof of Theorem 1 We first prove (23). For any fixed index $k\in\mathbb{N}$ with $k\leqslant s-1$, we carry the induction on $|\alpha|(1\leqslant|\alpha|\leqslant s-k)$ of space derivatives for (31). The step of the induction is increasing from $|\alpha|=1$ to $|\alpha|=s-k$. More specially, for $|\alpha|\geqslant 2$, $\|\partial_t^k\boldsymbol{U}\|_{|\alpha|-1}$ on the right-hand side of (31) can be controlled by $\|\partial_t^k\boldsymbol{U}\|_{|\alpha|}$ in the preceding step on the left-hand side of (31) multiplying an appropriate positive constant. Thus, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\leqslant s-k}a_{k,\alpha}(\langle \boldsymbol{A}_0\boldsymbol{U}_{k,\alpha},\boldsymbol{U}_{k,\alpha}\rangle+\|\boldsymbol{F}_{k,\alpha}\|^2+\|\boldsymbol{G}_{k,\alpha}\|^2)+\|\partial_t^k\boldsymbol{U}\|_{s-k}^2\leqslant$$
$$C(\|\partial_t^k\boldsymbol{U}\|^2+\|\partial_t^{k+1}\boldsymbol{u}\|_{s-k-1}^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s. \tag{33}$$

where $a_{k,\alpha}>0(k\leqslant s-1,1\leqslant|\alpha|\leqslant s-k)$ are constants.

Next, we carry on the induction on k from $k=s$ to $k=0$. The corresponding estimate for $k=s$ is given by (31). For $k=s-1$, (33) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\leqslant 1}a_{s-1,\alpha}(\langle \boldsymbol{A}_0\boldsymbol{U}_{s-1,\alpha},\boldsymbol{U}_{s-1,\alpha}\rangle+\|\boldsymbol{F}_{s-1,\alpha}\|^2+\|\boldsymbol{G}_{s-1,\alpha}\|^2)+\|\partial_t^{s-1}\boldsymbol{U}\|_1^2\leqslant$$
$$C(\|\partial_t^{s-1}\boldsymbol{U}\|^2+\|\partial_t^s\boldsymbol{u}\|^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s. \tag{34}$$

It is clear $\|\partial_t^{s-1}\boldsymbol{U}\|_1^2$ on the right-hand side of (32) can be controlled by the same term on the left-hand side of (34) multiplying an appropriate constant. Similarly, $\|\partial_t^{k+1}\boldsymbol{u}\|_{s-k-1}^2$ can be controlled by $\|\partial_t^k\boldsymbol{U}\|_{s-k}^2$ in the preceding step. In this way, by induction on k, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+|\alpha\leqslant s|}a_{k,\alpha}(\langle \boldsymbol{A}_0\boldsymbol{U}_{k,\alpha},\boldsymbol{U}_{k,\alpha}\rangle+\|\boldsymbol{F}_{k,\alpha}\|^2+\|\boldsymbol{G}_{k,\alpha}\|^2)+\sum_{k=0}^{s}\|\partial_t^k\boldsymbol{U}\|_{s-k}^2\leqslant$$
$$C\sum_{k=0}^{s}(\|\partial_t^k\boldsymbol{U}\|^2+\|\partial_t^{k+1}\boldsymbol{u}\|^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s. \tag{35}$$

where the positive constant $a_{k,\alpha}$ are possibly amended based on (33). Noting the equivalence of $\sum\limits_{k=0}^{s}\|\partial_t^k\boldsymbol{U}\|_{s-k}^2$ and $|||\boldsymbol{U}|||_s^2$, (25), (29) and (35), with a modification again the constants $a_{k,\alpha}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+|\alpha|\leqslant s}a_{k,\alpha}(\langle \boldsymbol{A}_0\boldsymbol{U}_{k,\alpha},\boldsymbol{U}_{k,\alpha}\rangle+\|\boldsymbol{F}_{k,\alpha}\|^2+\|\boldsymbol{G}_{k,\alpha}\|^2)+2|||\boldsymbol{U}(t,\cdot)|||_s^2\leqslant C|||\boldsymbol{U}(t,\cdot)|||_s^2|||\boldsymbol{Z}|||_s.$$

Since Z_T is sufficiently small, we further obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+|\alpha|\leqslant s}a_{k,\alpha}(\langle \boldsymbol{A}_0\boldsymbol{U}_{k,\alpha},\boldsymbol{U}_{k,\alpha}\rangle+\|\boldsymbol{F}_{k,\alpha}\|^2+\|\boldsymbol{G}_{k,\alpha}\|^2)+|||\boldsymbol{U}(t,\cdot)|||_s^2\leqslant 0.$$

Noting the equivalence of $|||\boldsymbol{Z}|||_s^2$ and

$$\sum_{k+|\alpha|\leqslant s}a_{k,\alpha}(\langle \boldsymbol{A}_0\boldsymbol{U}_{k,\alpha},\boldsymbol{U}_{k,\alpha}\rangle+\|\boldsymbol{F}_{k,\alpha}\|^2+\|\boldsymbol{G}_{k,\alpha}\|^2),$$

we get

$$|||\boldsymbol{Z}(t,\cdot)|||_s^2+\int_0^t|||\boldsymbol{U}(\tau,\cdot)|||_s^2\mathrm{d}\tau\leqslant C\|\boldsymbol{Z}_0\|_s^2,t\in[0,T]. \tag{36}$$

It follows from the second equation and Maxwell equation that

$$|||\boldsymbol{F}|||_{s-1}^2\leqslant C|||\boldsymbol{U}|||_s^2+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s. \tag{37}$$

$$|||\partial_t\boldsymbol{G}|||_{s-2}^2+|||\nabla_x\boldsymbol{G}|||_{s-2}^2\leqslant C|||\boldsymbol{U}|||_s^2+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{Z}|||_s. \tag{38}$$

Since Z_T is small enough, (36)-(38) yield (23).

It is obvious that (23) implies (10) and the global existence of smooth solution $(n,\boldsymbol{u},\theta,\boldsymbol{E},\boldsymbol{B})$ to (1)-(2). Finally, for all $k\in\mathbb{N}$ and $\beta\in\mathbb{N}^3$ with $k+|\beta|\leqslant s-1$, from (10), we have

$$\partial_t^k\partial_x^\beta(n-\bar{n},\boldsymbol{u},\theta-\theta_e,\boldsymbol{E}-\bar{\boldsymbol{E}})\in L^2(\boldsymbol{R}^+;L^2(\mathbb{R}^3))\cap W^{1,\infty}(\boldsymbol{R}^+;L^2(\mathbb{R}^3)),$$

which implies (11)-(12). Moreover, if $k+|\beta|\geqslant 1$, noticing $\bar{\boldsymbol{B}}$ is a constant vector, we have

$$\partial_t^k\partial_x^\beta\boldsymbol{B}\in L^2(\boldsymbol{R}^+;L^2(\mathbb{R}^3))\cap W^{1,\infty}(\boldsymbol{R}^+;L^2(\mathbb{R}^3)),$$

which yields (13).

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