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Two-Dimensional Jet Flow with Gravity in a Semi-Infinitely Long Symmetric Nozzle

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Abstract: The main object of this paper is to investigate the well-posedness theory of the incompressible inviscid jet flow with gravity in an semi-infinitely long symmetric nozzle. The main results read that given a mass flux in the inlet of the nozzle, we established the existence and the uniqueness of the incompressible jet flow problem with gravity in an semi-infinitely long symmetric nozzle, which contain a smooth free surface detaching at the boundary point of the lower nozzle wall.

Key words: existence and uniqueness, free streamline, inviscid, incompressible

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半无穷长对称管道中的二维带重力的喷流

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[摘要] 本文的主要内容是研究半无穷长对称管道中不可压缩、无粘带重力的喷流的适定性. 主要结论是如果在管道入口给定流体的一个质量通量,我们能够建立半无穷长管道中不可压缩带重力喷流问题的存在性和唯一性,该喷流问题还有连接下管道壁边界点的光滑自由边界.

[关键词] 存在性和唯一性,自由流线,无粘,不可压缩

1 Introduction and Main Results

In this paper, we will investigate the well-posedness of the incompressible, inviscid jet flow with gravity in a semi-infinitely long symmetric nozzle. Some existence and uniqueness results are established in this paper.

In the following, we would like to recall some known results about the mathematical results on the cavity and jet flows problem. In 1952, P. R. Garabedian, etc. in[1]investigated the axially symmetric finite cavity problem for Riabouchinsky model by using the variational approach. For general existence results on jet and cavity flows, we can refer to the references[2-4]. In 1981, H.W Alt, etc. developed a new variational approach to obtain the existence and the regularity for a minimum problem with free boundary in their breakthrough work[5]. Based on the work[5], some remarkable results on the existence and uniqueness of axially symmetric jet flow were established in[6], asymmetric jet flow in[7], jet flow with gravity in[8], axially symmetric infinite cavity in[9], and so on.

The motivation in this paper is to investigate the existence and uniqueness of the incompressible jet flow with gravity in a semi-infinitely long symmetric nozzle.

Denote the upper nozzle wall of the symmetric semi-infinitely long nozzle by

$$N_1: y=H_1>0, -\infty<x\leq 0, \tag{1}$$

and the lower nozzle wall by

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$$N_2: x=g(y)\in C^{2,\alpha}(-\infty,0), \tag{2}$$

with $0<\alpha<1$ and satisfying

$$\lim_{y\to H_2} g(y)=-\infty, H_2<H_1, \text{and } g(0)=-a. \tag{3}$$

Denote the symmetric axis of the symmetric nozzle as $T: x=0, -\infty<y\leqslant H_1$ and let the ray $l: x=-a, -\infty<y\leqslant 0$. Next, we introduce the two-dimensional inviscid, incompressible flow with gravity that

$$\begin{cases} \nabla\cdot\boldsymbol{U}=0, \\ (\boldsymbol{U}\cdot\nabla)\boldsymbol{U}+\nabla P=-g\cdot\boldsymbol{e}_2, \end{cases} \tag{4}$$

where $\boldsymbol{U}(x,y)=(u(x,y),v(x,y))$ satisfying the irrotational condition that

$$\nabla\times\boldsymbol{U}=0, \tag{5}$$

$P=P(x,y)$ denote the velocity field and the pressure, respectively, and $\boldsymbol{e}_2=(0,1)$.

Furthermore, we assume that the nozzle wall and the symmetric axis are impermeable, then the flow satisfies the slip boundary condition

$$(u,v)\cdot\boldsymbol{n}=0, \text{on } N_1\cup N_2\cup T\cup l, \tag{6}$$

where $\boldsymbol{n}$ is the unit outward normal to $N_1\cup N_2\cup T\cup l$.

There is an invariance along each streamline for the steady incompressible flow, namely,

$$(u,v)\cdot\nabla\left(\frac{1}{2}(u^2+v^2+P+gy)\right)=0. \tag{7}$$

In this paper, we consider the gravity of the flow in the nozzle, the pressure is assumed to be a constant P_{atm} called atmosphere pressure, then the Bernoulli's law implies that the speed of the fluid is not a constant but a function with respect to y, i.e. $\sqrt{u^2+v^2}=\sqrt{2\lambda-2gy}$ (λ is a constant) on the free boundary. Next, we give the statement of the jet flow with gravity problem in the semi-infinitely long nozzle.

1.1 Statement of the physical problem

Definition 1 (Jet flow with gravity problem) Suppose that the given semi-infinitely long nozzle wall N_1, N_2 satisfy the conditions (1)-(3), given a mass flux $m_0>0$ of the incoming incompressible flow, and the atmospheric pressure $P=P_{\text{atm}}$, does there exist a unique two-dimensional symmetric incompressible jet flow with gravity in the semi-infinitely long nozzle, which has a smooth free streamline leaving the vertex $A=(-a,0)$ of the lower nozzle wall?

Definition 2 (A solution to the jet flow with gravity problem) A vector (u,v,ρ,Γ) is called a solution to the jet flow with gravity problem, provided that

(1) Γ can be expressed by a smooth function $x=f(y)\in C^1(-\infty,0)$, such that

$$\lim_{y\to 0^-} f(y)=g(0)=-a, \lim_{y\to 0^-} f'(y)=g'(0), \tag{8}$$

and $\sqrt{u^2+v^2}=\sqrt{2\lambda-2gy}$ on Γ;

(2) $(u,v,P)\in C^{1,\alpha}(\Omega_0)\cap C^{\alpha}(\overline{\Omega}_0)$ solves the equations (4), where Ω_0 is the flow field bounded by N_1, N_2, T and Γ.

Theorem 1 (Existence of the jet flow with gravity) Assume that the semi-infinitely long nozzle wall N_1 and N_2 satisfy the conditions (1)-(3), for any given mass flux $m_0>0$, then there exist a constant $\lambda>0$ and a solution (u,v,P,Γ) to the jet flow with gravity problem defined in Definition 2.

2 Mathematical Setting on Jet Flow with Gravity Problem

2.1 Stream function setting

In order to solve the jet flow with gravity problem, according to the first equation in (4), set $u=\psi_y, v=-\psi_x$, which combine with the irrotational condition (5) to obtain $\Delta\psi=0$ in the flow field Ω_0. Furthermore, we impose the Dirichlet boundary value conditions as $\psi=m_0$ on $N_1\cup T$, and $\psi=0$ on $N_2\cup l\cup\Gamma$. Thus, the free boundary can be defined by

$$\Gamma=\Omega\cap\partial\{\psi>0\},\tag{9}$$

where Ω is called as the possible flow field bounded by N_1, N_2, l and T, which combining with equation (7) deduces that

$$|\nabla\psi|=\frac{\partial\psi}{\partial\boldsymbol{v}}=\sqrt{2\lambda-2gy}\ \text{ on }\Gamma,\tag{10}$$

where $\boldsymbol{v}$ is the outer unit normal of Γ. Therefore, we formulate the jet flow with gravity problem as the following boundary value problem for the stream function that

$$\begin{cases}\Delta\psi=0\ \text{ in }\Omega\cap\{\psi>0\},\\ \dfrac{\partial\psi}{\partial\boldsymbol{v}}=\sqrt{2\lambda-2gy}\ \text{ on }\Gamma,\\ \psi=0\ \text{ on }N_2\cup l\cup\Gamma,\\ \psi=m_0\ \text{ on }N_1\cup T.\end{cases}\tag{11}$$

2.2 Variational approach and truncation

We first define an admissible set as $K=\{\psi\in H^1_{\mathrm{loc}}(\Omega)\mid \psi=0 \text{ on } N_2\cup l, \psi=m_0 \text{ on } N_1\cup T\}$.

Denote the variational problem (P_λ) as

$$J_\lambda(\psi)=\int_\Omega|\nabla\psi-\sqrt{2\lambda-2gy}I_{\{\psi>0\}\cap\{y<0\}}\boldsymbol{e}_2|^2\mathrm{d}x\mathrm{d}y,\tag{12}$$

where I_A is the characteristic function of a set A and $\boldsymbol{e}_2=(0,1)$.

Since the functional $J_\lambda(\psi)$ is unbounded for any $\psi\in K$, thus we need to truncate the domain Ω, namely, $\Omega_\mu=\Omega\cap\{x\geqslant-\mu\}$. Therefore, the truncated functional is that

$$J_{\lambda,\mu}(\psi)=\int_{\Omega_\mu}|\nabla\psi-\sqrt{2\lambda-2gy}I_{\{\psi>0\}\cap\{y<0\}}\boldsymbol{e}_2|^2\mathrm{d}x\mathrm{d}y.\tag{13}$$

Denote the truncated variational problem ($P_{\lambda,\mu}$) as finding a $\psi_{\lambda,\mu}$ such that

$$J_{\lambda,\mu}(\psi_{\lambda,\mu})=\min_{\psi\in K_{\lambda,\mu}}J_{\lambda,\mu}(\psi),$$

where the corresponding admissible set is that

$$K_\mu=\{\psi\in H^1_{\mathrm{loc}}(\Omega_\mu)\mid 0\leqslant\psi\leqslant m_0, \psi=0\text{ on }N_{2,\mu}\cup l, \psi=m_0\text{ on }N_{1,\mu}\cup T, \psi=\frac{m_0}{H_1-H_{2,-\mu}}\text{on }I_1\},$$

in which $N_{1,\mu}=N_1\cap\{x\geqslant-\mu\}$, $N_{2,\mu}=N_2\cap\{x\geqslant-\mu\}$ and $I_1=\{(-\mu,y)\mid H_{2,-\mu}\leqslant y\leqslant H_1\}$, where $H_{2,-\mu}=\max\{y\mid-\mu=g(y)\}$.

3 Existence of the Jet Flow with Gravity

3.1 Existence of minimizer to the truncated variational problem

Lemma 1 Problem $P_{\lambda,\mu}$ has a solution.

Proof Here, we just need to obtain the boundedness of the functional $J_{\lambda,\mu}$, namely, there exists a function $\psi_0\in K_\mu$, such that the functional $J_{\lambda,\mu}<+\infty$, thus, the variational problem $P_{\lambda,\mu}$ has a minimizer. Take y_0 small, and define

$$\psi_0=\begin{cases}0, & \text{if } x<-\dfrac{m_0}{\sqrt{2\lambda-2gy}},\quad y\leqslant y_0,\\ m_0+\sqrt{2\lambda+2gy}, & \text{if } -\dfrac{m_0}{\sqrt{2\lambda-2gy}}\leqslant x\leqslant0,\quad y\leqslant y_0,\end{cases}$$

then we can extend ψ_0 into the domain $\Omega_\mu\setminus\{y\leqslant y_0\}$ so that it belongs to the admissible set K_μ. Then,

$$J_{\lambda,\mu}(\psi_0)=\int_{\Omega_\mu\cap\{y_0\leqslant y\leqslant H_1\}}\int_{\Omega_\mu}|\nabla\psi_0-\sqrt{2\lambda-2gy}I_{\{\psi_0>0\}\cap\{y<0\}}\boldsymbol{e}_2|^2\mathrm{d}x\mathrm{d}y+$$
$$\int_{\Omega_\mu\cap\{y\leqslant y_0\}}\int_{\Omega_\mu}|\nabla\psi_0-\sqrt{2\lambda-2gy}I_{\{\psi_0>0\}\cap\{y<0\}}\boldsymbol{e}|^2\mathrm{d}x\mathrm{d}y=J_1+J_2.$$

Thanks to the boundedness of the domain $\Omega_\mu \cap \{y_0 \leqslant y \leqslant H_1\}$, then we just need to verify $J_2<+\infty$. According to a series of calculations, we have $J_2<+\infty$. Thus, similar to the argument in [8], there exists a minimizer $\psi_{\lambda,\mu}$ to $J_{\lambda,\mu}(\psi_{\lambda,\mu})=\min\limits_{\psi\in K_\mu} J_{\lambda,\mu}(\psi)$, $\psi_{\lambda,\mu}\in K_\mu$.

For simplicity, we set $\psi=\psi_{\lambda,\mu}$ in the following.

Lemma 2 If ψ is a minimum, then

$$\lim_{\varepsilon\to 0}\int_{\Omega_\mu\cap\partial\{\psi>0\}}(|\nabla\psi|^2-(2\lambda-2gy))\boldsymbol{\eta}\cdot\boldsymbol{v}_\varepsilon \mathrm{d}S=0, \tag{14}$$

for any vector $\boldsymbol{\eta}=(\eta_1,\eta_2)\in(C^1(E))^2$, $\boldsymbol{v}_\varepsilon$ is the unit outward normal to $\Omega_\mu\cap\partial\{\psi\geqslant\varepsilon\}$.

Particularly,

$$\int_{I_0\cap\partial\{\psi>0\}}(|\nabla\psi|^2-(2\lambda-2gy))\boldsymbol{\eta}\cdot\boldsymbol{v}\mathrm{d}S\geqslant 0, \tag{15}$$

for any vector $\boldsymbol{\eta}=(\eta_1,\eta_2)\in(C^1(E))^2$, $\boldsymbol{\eta}=0$ on $\Omega_\mu\backslash I_0$, $\boldsymbol{\eta}\cdot\boldsymbol{v}_\varepsilon\leqslant 0$ on I_0, $I_0=\{\psi=0\}$ and $\boldsymbol{v}$ is the unit outward normal to I_0.

Proof For any real ε, $|\varepsilon|$ small, let $\tau_\varepsilon(x,y)=(x+\varepsilon\eta_1(x,y), y+\varepsilon\eta_2(x,y))$ and define $\psi_\varepsilon(\tau_\varepsilon(x,y))=\psi(x,y)$. Then $\psi_\varepsilon\in K_\mu$ and

$$(D\tau_\varepsilon(r,y))^{-1}=(\boldsymbol{I}+\varepsilon\nabla\cdot\boldsymbol{\eta}\boldsymbol{I}-\varepsilon D\boldsymbol{\eta})(\det D\tau_\varepsilon)^{-1} \text{ and } \det D\tau_\varepsilon=1+\varepsilon\nabla\cdot\boldsymbol{\eta}+o(\varepsilon),$$

where $\boldsymbol{I}$ is the identity matrix.

Hence we have

$$0\leqslant J_{\lambda,\mu}(\psi_\varepsilon)-J_{\lambda,\mu}(\psi)=\varepsilon\int_{\{\psi>0\}\cap E_R}(\operatorname{div}\boldsymbol{\eta}|\nabla\psi|^2-2\nabla\psi\cdot\nabla\boldsymbol{\eta}\cdot\nabla\psi+\operatorname{div}\boldsymbol{\eta}(2\lambda-2gy\eta_2))\mathrm{d}x\mathrm{d}y+o(\varepsilon), \tag{16}$$

in which $E_R=E\cap\{(x,y)\mid x^2+y^2\leqslant R\}$, $R>0$ sufficiently large.

Then, by a series of calculations, we have the following estimates, taking $R\to+\infty$, the linear term of ε in inequality (16) vanishes, thus

$$\begin{aligned}0\leqslant\varepsilon\int_{\{\psi>0\}}&(\operatorname{div}\boldsymbol{\eta}|\nabla\psi|^2-2\nabla\psi\cdot\nabla\boldsymbol{\eta}\cdot\nabla\psi+\operatorname{div}\boldsymbol{\eta}(2\lambda-2gy\eta_2))\mathrm{d}x\mathrm{d}y=\\ &\int_{\partial\{\psi>\varepsilon\}\cap E}(|\nabla\psi|^2\boldsymbol{\eta}-2\nabla\psi(\boldsymbol{\eta}\cdot\nabla\psi)-(2\lambda-2gy\boldsymbol{\eta}))\mathrm{d}S=\\ &\lim_{\varepsilon\to 0^+}\int_{\partial\{\psi>\varepsilon\}\cap E}(2\lambda-2gy-|\nabla\psi|^2\boldsymbol{\eta})\boldsymbol{\eta}\cdot\boldsymbol{v}_\varepsilon\mathrm{d}S,\end{aligned} \tag{17}$$

owing to the facts that $\boldsymbol{v}_\varepsilon$ is parallel to $\nabla\psi$ on the streamlines and the first equation in (11), thus, the proof of (14) is completed.

For any vector $\boldsymbol{\eta}=(\eta_1,\eta_2)\in(C_0^1(E))^2$, let $\varepsilon>0$ in (16) to find the similar arguments in (17) that

$$0=\varepsilon\int_{\{\psi>0\}\cap E}(\operatorname{div}\boldsymbol{\eta}|\nabla\psi|^2-2\nabla\psi\cdot\nabla\boldsymbol{\eta}\cdot\nabla\psi+\operatorname{div}\boldsymbol{\eta}(2\lambda-2gy\eta_2))\mathrm{d}x\mathrm{d}y=\int_{\partial\{\psi>0\}\cap I_0}(2\lambda-2gy-|\nabla\psi|^2\boldsymbol{\eta})\boldsymbol{\eta}\cdot\boldsymbol{v}_\varepsilon\mathrm{d}S,$$

which directly implies the inequality (15).

In fact, Lemma 2 implies that if the free boundary $\partial\{\psi>0\}\in C^1$, and $\psi\in C^1$ up to the free boundary $\Gamma_{\lambda,\mu}$, then $|\nabla\psi|=\sqrt{2\lambda+2gy}$ on $\Gamma_{\lambda,\mu}$. Therefore, we give the definition of the free boundary $\Gamma_{\lambda,\mu}$, for the truncated variational problem $(P_{\lambda,\mu})$ as follows

$$\Gamma_{\lambda,\mu}=\Omega_\mu\cap\{y<0\}\cap\partial\{\psi>0\}. \tag{18}$$

Lemma 3 The minimizer of the functional $J_{\lambda,\mu}(\psi)=\inf\limits_{\min}{}_{v\in K_\mu}J_{\lambda,\mu}(v)$ is unique, and satisfies that $\psi(x_1,y)\geqslant\psi(x_2,y)$ for $x_1>x_2$.

Proof Suppose ψ_1 and ψ_2 are two minimizer and set

$$\psi_1^\varepsilon(x,y)=\psi_1(x+\varepsilon,y),\ 0<\varepsilon<1.$$

Notice that $\psi_1^\varepsilon(x,y)$ is a minimizer the functional $J_{\lambda,\mu}^\varepsilon$ in Ω_μ^ε with the corresponding admissible set K_μ^ε, in which $\Omega_\mu^\varepsilon=\{(x,y)\mid(x-\varepsilon,y)\in\Omega_\mu\}$ and $K_\mu^\varepsilon=\{\psi^\varepsilon(x+\varepsilon,y)\in K_\mu\mid(x+\varepsilon,y)\in\Omega_\mu^\varepsilon\}$. The following, we extend the $\psi_1^\varepsilon(x,y)$ and $\psi_2(x,y)$ to $\Omega_\mu\cup\Omega_\mu^\varepsilon$,

$$\begin{cases}\psi_1^{\varepsilon}=\dfrac{m_0}{H_1-H_{2,\mu}} \text{ for } -\mu\leqslant x\leqslant -\mu+\varepsilon,\\ \psi_2=m_0 \text{ for } 0<x\leqslant\varepsilon,\end{cases}$$

which deduces $v_1=\psi_1^{\varepsilon}\vee\psi_2=\psi_1^{\varepsilon}$ and $v_2=\psi_1^{\varepsilon}\wedge\psi_2=\psi_2$ in $\Omega_\mu\cap\Omega_\mu^{\varepsilon}$, namely, we have

$$\psi_1^{\varepsilon}>\psi_2 \text{ in } \Omega_\mu\cap\Omega_\mu^{\varepsilon}. \tag{19}$$

Then, taking $\varepsilon\to 0$ shows $\psi_1(x,y)\geqslant\psi_2(x,y)$ in Ω_μ. Similarly, we can obtain that $\psi_1(x,y)\leqslant\psi_2(x,y)$ in Ω_μ. Hence, $\psi_1=\psi_2$ in Ω_μ.

Next, set $\psi_1=\psi_2$, combining with inequality (19), to yield that $\psi(x+\varepsilon,y)\geqslant\psi(x,y)$ in Ω_μ. Therefore, we finish the proof of Lemma 3.

3.2 Fundamental properties of the free-boundary

Owing to the condition that the monotonicity of the minimizer $\psi(x,y)$ with respect to x, we set $\Gamma_{\lambda,\mu}: x=f_{\lambda,\mu}(y)$, then

$$\Omega_\mu\cap\{y<0\}\cap\{\psi>0\}=\{(x,y)\mid f_{\lambda,\mu}(y)<x<0,-\infty<y<0\}.$$

Therefore, by using the non-oscillation Lemma 9 in Appendix, we can prove that the free boundary $x=f_{\lambda,\mu}(y)$ is continuous in $(-\infty,0]$, which is similar to the arguments to Lemma 5.4 in [6], thus we omit a part of the proof here.

Proposition 1 $f_{\lambda,\mu}(y)$ is a continuous function in $(-\infty,0)$. Furthermore, $f_{\lambda,\mu}(y)$ is analysis.

Proof In view of Lemma 5.4 in [6], then it suffices to show that $f_{\lambda,\mu}(y+0)=f_{\lambda,\mu}(y-0)=f_{\lambda,\mu}(y)$ for any $y<0$. Firstly, suppose that there exists a point $y_0\in(-\infty,0)$ such that $f_{\lambda,\mu}(y+0)\neq f_{\lambda,\mu}(y_0)$, then without loss of generality, we assume that $f_{\lambda,\mu}(y+0)>f_{\lambda,\mu}(y_0)$. Then there exist some constants $\varepsilon_1,\varepsilon_2$, such that there is a strip that

$$E_{\varepsilon_1,\varepsilon_2}=\{(x,y)\mid f_{\lambda,\mu}(y_0)-\varepsilon_1<x<f_{\lambda,\mu}(y_0)+\varepsilon_1,\quad y_0-\varepsilon_2<y<y_0\}.$$

By virtue of the monotonicity of ψ with respect to x, we deduce that $\{(x,y_0)\mid f_{\lambda,\mu}(y_0)-\varepsilon_1<x<f_{\lambda,\mu}(y_0)+\varepsilon_1\}$ is a part of the free boundary $\Gamma_{\lambda,\mu}$, which yields that

$$\begin{cases}\Delta\psi=0 \text{ in } E_{\varepsilon_1,\varepsilon_2},\\ \psi=0,\quad \dfrac{\partial\psi}{\partial y}=\sqrt{2\lambda-2gy} \text{ on } f_{\lambda,\mu}(y_0)-\varepsilon_1<x<f_{\lambda,\mu}(y_0)+\varepsilon_1.\end{cases}$$

It follows from Cauchy-Kowalewski theorem that there exists a unique solution $\psi_{\lambda,\mu}=-\sqrt{2\lambda-2gy}\,(y-y_0)$, in $E_{\varepsilon_2}\cap\Omega_\mu$, which contradicts to the fact that $\psi=m_0$ on T. Therefore, we finish the proof.

3.3 The continuous fit and the smooth fit of the free boundary

Next, we will consider the continuous fit and the smooth fit conditions of the free boundary. Before that, we need to give some basic and important lemmas. The lemmas are as follows:

Lemma 4 If $\lambda_n\to\lambda$, then $\psi_{\lambda_n,\mu}\to\psi_{\lambda,\mu}$ weakly in $H^1_{\mathrm{loc}}(\Omega_\mu)$ and a.e., and $f_{\lambda_n,\mu}(-\infty,0)\to f_{\lambda,\mu}(-\infty,0)$ for each $y\in(-\infty,0)$.

Proof This proof of this Lemma has been given in section 4.7 in [5], thus, we omit the processes here.

Lemma 5 For any $m_0>0$, if $\lambda>0$ is sufficiently small, then $f_{\lambda,\mu}(0)>-a$.

Proof Suppose that $f_{\lambda,\mu}(0)\leqslant -a$, then set S be a ring centered at $\bar{B}=(f_{\lambda,\mu}(0),0)$ with some suitable radius R_1 and R_2 which is independent of λ and $R_1<R_2$, such that $\Gamma_{\lambda,\mu}\cap S\cap\{y<0\}$ and $T\cap S\cap\{y<0\}$ is nonempty. It follows from Lemma 10 in Appendix that there exists a constant C (independent of λ), such that $|D\psi|\leqslant C\max\limits_{S\cap\{y<0\}}\sqrt{2\lambda-2gy}$.

Choosing $X_1\in\Gamma_{\lambda,\mu}\cap S\cap\{y<0\}$, $X_2\in T\cap S\cap\{y<0\}$ with $|X_1-\bar{B}|=|X_2-\bar{B}|$, shows that

$$m_0=|\psi(X_1)-\psi(X_2)|\leqslant|D\psi||X_1-X_2|\leqslant C\max_{S\cap\{y<0\}}\sqrt{2\lambda-2gy}\leqslant C(R_2)\sqrt{\lambda},$$

which implies that it is impossible for sufficiently small λ. Hence, we complete the proof of this Lemma.

Lemma 6 For any $m_0>0$, if $\lambda>0$ is sufficiently large, then $f_{\lambda,\mu}(0)<-a$.

Proof Suppose that $f_{\lambda,\mu}(0)\geqslant -a$, then there exist a free boundary point $X_0=(x_0,y_0)\in\Gamma_{\lambda,\mu}$ with $-a\leqslant x\leqslant \frac{f_{\lambda,\mu}+a}{2}$ and $y_0<0$, choosing a fixed $0<r<\frac{y_0}{2}$ (independent of λ) such that either $B_{\frac{r}{2}(X_0)}\subset\Omega_\mu$. According to the non-degeneracy Lemma 8 in Appendix, we have

$$2m_0c^*\geqslant\frac{2m_0}{r}\geqslant\frac{2}{r}⨍_{\partial B_{\frac{r}{2}(X_0)}}\psi\mathrm{d}S\geqslant c^*\min_{B_{\frac{r}{2}(X_0)}}\sqrt{2\lambda-2gy}\geqslant c^*\sqrt{2\lambda},$$

which leads a contradiction for sufficiently large λ, where $r=\frac{1}{c^*\sqrt[4]{\lambda}}$.

Fix the above λ as λ_μ, combining with Lemma 4, Lemma 5 and Lemma 6, we can give the continuous fit condition $f_{\lambda_\mu,\mu}(0)=-a$. Finally, we will check that the smooth fit condition $f'_{\lambda_\mu,\mu}=g'(0)$ indeed holds, and the fact can be obtained along the similar argument in [5] and [6], hence we omit it here.

Proposition 2 $f_{\lambda_\mu,\mu}(0)=-a$ holds, and then $N_2\cap\Gamma_{\lambda_\mu,\mu}$ continuously differentiable in a neighborhood of A. Furthermore, $\psi_{\lambda_\mu,\mu}$ is continuously differentiable in $\{\psi_{\lambda_\mu,\mu}>0\}\cap B_\delta(A)$, for some $\delta>0$.

4 The Existence and Uniqueness of the Jet Flow with Gravity Problem

4.1 The existence of the jet flow with gravity problem

To establish the existence of the solution to jet flow with gravity problem, we will take limit $\mu\to\infty$ to the solution $\psi_{\lambda_\mu,\mu}$ to the truncated variational problem $(P_{\lambda_\mu,\mu})$ and show the limit ψ_λ is indeed a solution to the variational problem P_λ(12). By virtue of the uniform gradient estimate $|\nabla\psi_{\lambda_{\mu_n},\mu_n}|\leqslant C$ in any compact subset of Ω which can be referred on Lemma 10 in Appendix, it follows from the similar arguments in Section 12 in Chapter 3 in [3] that there exists a subsequence $\{\psi_{\lambda_{\mu_n},\mu_n}\}$ and $\{\lambda_{\mu_n}\}$, such that $\lambda_{\mu_n}\to\lambda$ and $\psi_{\lambda,\mu_n}\to\psi_\lambda$ weakly in $H^1_{loc}(\Omega)$ and uniformly in any compact subset of Ω, as $n\to\infty$. In the following, we will verify that ψ_λ is in fact a minimizer to the variational problem (P_λ), and solves the boundary value problem (11). In particular, the continuous fit and smooth fit conditions are fulfilled.

Since $\psi_{\lambda_{\mu_n},\mu_n}$ is a minimizer to the function $J_{\lambda_{\mu_n},\mu_n}$, then $J_{\lambda_{\mu_n},\mu_n}(\psi_{\lambda_{\mu_n},\mu_n})\leqslant J_{\lambda_{\mu_n},\mu_n}(\psi_{\mu_n})$ for any $\psi_{\mu_n}\in K_{\mu_n}$. For any bounded domain $D\subset\Omega$, we can choose a sufficiently large μ_n, such that $D\subset\Omega_{\mu_n}$. Choosing $\psi_{\mu_n}=\psi_{\lambda_{\mu_n},\mu_n}$ on ∂D, and extending ψ_{μ_n} with $\psi_{\lambda_{\mu_n},\mu_n}$ outside D, hence we can conclude

$$J_D(\psi_{\lambda_{\mu_n},\mu_n})\leqslant J_D(\psi_{\mu_n}).\tag{20}$$

Thus for any $\psi\in K$ with $\psi-\psi_\lambda\in H^1_0(D)$ and $\eta\in C^1_0(D)$, $0\leqslant\eta\leqslant1$, set $\bar{\psi}_{\mu_n}=\psi+(1-\eta)(\psi_{\lambda_{\mu_n},\mu_n}-\psi_\lambda)$, it is easy to check that $\bar{\psi}_{\mu_n}\in K_{\mu_n}$ and $\bar{\psi}_{\mu_n}=\psi_{\lambda_{\mu_n},\mu_n}$ on ∂D. It follows from (20) that $J_D(\psi_{\lambda_{\mu_n},\mu_n})\leqslant J_D(\bar{\psi}_{\mu_n})$.

Therefore, similar to the proof in Lemma 5.4 in [5] and taking $n\to\infty$, one has $J_D(\psi_\lambda)\leqslant J_D(\psi)$, for any $\psi\in K$ with $\psi=\psi_\lambda$ on ∂D, namely, ψ_λ is a minimizer to the variational problem (P_λ).

Moreover, $\psi_\lambda(x,y)$ is monotonic increasing with respect to x. In fact, for any (x_1,y), $(x_2,y)\in\Omega$ with $x_1>x_2$, there exists a compact subset G of Ω, such that (x_1,y), $(x_2,y)\in G$. For sufficiently large n, we have $G\subset\Omega_{\mu_n}$, and it follows from Lemma 3 that

$$\psi_{\lambda_{\mu_n},\mu_n}(x_1,y)\geqslant\psi_{\lambda_{\mu_n},\mu_n}(x_2,y).\tag{21}$$

Since $\psi_{\lambda_{\mu_n},\mu_n}$ converges to ψ_λ uniformly in any compact subset of Ω, then there exists a subsequence $\psi_{\lambda_{\mu_n},\mu_n}$ and letting $n\to+\infty$ in (21), one has $\psi_\lambda(x_1,y)\geqslant\psi_\lambda(x_2,y)$, for any (x_1,y), $(x_2,y)\in\Omega$ with $x_1>x_2$. The monotonicity of $\psi_\lambda(x,y)$ with respect to x implies that the free boundary is y-graph. Then it follows that there exists a continuous function $f_\lambda(y)$, such that the free boundary Γ_λ of ψ_λ can be described as

$$\Gamma_\lambda: x=f_\lambda(y)\ \text{for}\ y\in(-\infty,0].$$

Furthermore, it follows from the similar arguments in Lemma 4 that $f_\lambda(y)=\lim_{n\to\infty} f_{\lambda_{\mu_n},\mu_n}(y)$, for $y\in(-\infty,0]$. Consequently, one has $f_\lambda(0)=-a$, which is the continuous fit condition to the jet flow problem with gravity.

The smoothness near the detachment point A implies that $N_2\cup\Gamma_\lambda$ are C^1, which can be obtained similarly to the argument in [5]. By using the similar arguments in Lemma 2, we can conclude that $|\nabla\psi_\lambda|=-\sqrt{2\lambda-2gy}$ on Γ_λ. Therefore, we can obtain the solution (u,v,ρ,Γ) to the two-dimensional symmetric incompressible jet flow problem with satisfying the conditions (1)-(3) mentioned in Definition 2.

4.2 Uniqueness of the jet flow with gravity problem

This part shows the uniqueness of the two-dimensional symmetric jet flow with gravity problem.

Theorem 2 For any $m_0>0$, then there exists a unique λ such that the solution (u,v,ρ,Γ) established in Theorem 1 is unique.

Proof Suppose that there exist two different λ and $\tilde\lambda$ corresponding to two different solutions to the boundary value problem (11), i.e., (ψ_λ,P,Γ) and $(\psi_{\tilde\lambda},\tilde P,\tilde\Gamma)$.

Without loss of generality, we assume that $\lambda\geqslant\tilde\lambda$. Choosing any $0<k<1$ and $\sigma>0$, consider

$$\psi_\lambda^\sigma(x,y)=\psi_\lambda\left(\frac{x-\sigma}{k},y+\frac{y-y_0}{k}\right).$$

Then, its free boundary Γ_λ^σ is given by

$$x=kf_\lambda\left(y_0+\frac{y-y_0}{k}\right)+\sigma,\ for\ y<k(y-y_0)+y_0,$$

thus,

$$kf_\lambda\left(y_0+\frac{y-y_0}{k}\right)+\sigma=-\frac{k^{\frac{3}{2}}m_0(1+\delta_1(y))}{(-y+y_0(1-k))^{1/2}}+\sigma>-\frac{k^{1/2}m_0(1+\delta)}{|y|^{1/2}}>\frac{k^{1/2}m_0(1+\delta_2(y))}{|y|^{1/2}}>f_{\tilde\lambda}(y),\tag{22}$$

where $k=\dfrac{1}{1-\delta}$, and $k\uparrow 1$, as $\delta\uparrow 0$. Thus, the free boundary of Γ_λ^σ of ψ_λ^σ lies above the free boundary of $\Gamma_{\tilde\lambda}$ of $\psi_{\tilde\lambda}$ in the region $\{y<y_0\}$.

It follows from (22) that $kf_\lambda\left(y_0+\dfrac{y-y_0}{k}\right)+\sigma\to 0$, as $\sigma\to 0$, uniformly for $y<y_1$, for any $y_1<k(y-y_0)+y_0$, which directly yields that $\Gamma_\lambda^\sigma\cup N_2^\sigma$ lies above $\Gamma_{\tilde\lambda}\cup\tilde N_2$, if σ is large enough.

Thus there exists a smallest σ_0 such that the above condition holds. By the maximum principle, we have

$$\psi_\lambda^{\sigma_0}\leqslant\psi_{\tilde\lambda},\ \text{in}\ \Omega\cap\{\psi_\lambda^{\sigma_0}>0\},\tag{23}$$

which implies that there exists a point $X_2=(x_2,y_2)$ such that $\psi_\lambda^{\sigma_0}(X_2)=\psi_{\tilde\lambda}(X_2)$.

If $\sigma_0>c_0>0$ for all k near 1, thus the point X_2 cannot belong to $N_2^{\sigma_0}$ and $X_2\in\Gamma_\lambda^{\sigma_0}\cap\Gamma_{\tilde\lambda}$, by the maximum principle, we deduce that

$$|\nabla\psi_\lambda^{\sigma_0}|=\sqrt{2\lambda-2g\left(y_0+\frac{y_2-y_0}{k}\right)}\leqslant|\nabla\psi_{\tilde\lambda}|=\sqrt{2\tilde\lambda-2gy_2}\ \text{at}\ X_2,$$

which implies that $\tilde\lambda>\lambda$. It is a contradiction. Hence, $\sigma_0\to 0$, for a subsequence $k\uparrow 1$, and we acquire from (23) that $\psi_\lambda(x,y)\leqslant\psi_{\tilde\lambda}$. Suppose that $\psi_\lambda(x,y)\neq\psi_{\tilde\lambda}$ we have for some $r>0$ and $\varepsilon>0$ such that $(1+\varepsilon)\psi_\lambda\leqslant\psi_{\tilde\lambda}$, on the boundary of $B_r(A)\cap\{\psi_\lambda>0\}$, similarly, we deduce that $(1+\varepsilon)\sqrt{\lambda}=-(1+\varepsilon)|\nabla\psi_\lambda|_A\leqslant-(1+\varepsilon)|\nabla\psi_\lambda|_A=\sqrt{\tilde\lambda}$, which also makes a contradiction. Hence, we finish the proof.

5 Appendix

In this part, we will mention some Lemmas that has been proved in references[6], which is important for the proof of the Theorem 1.

Lemma 7 There exists a enough large positive constant C independent of μ and λ, such that if ψ is a minimum, then for any ball $B_R(X^0)\in\Omega_\mu$ with $X^0=(x_0,y_0)$,

$$\frac{1}{r}⨍_{\partial B_R(X^0)}\psi_{\lambda,\mu}\mathrm{d}S\geqslant C\min_{B_r(X^0)}\sqrt{2\lambda-2gy},$$

implies that $\psi_{\lambda,\mu}>0$ in $B_R(X^0)$.

Lemma 8 For any small $0<k<1$, there is a universal constant $c>0$ such that for all balls $B_R(X^0)$ with center $X^0\in\Omega_\mu$ and r small, the following holds

$$\frac{1}{r}⨍_{\partial B_r(X^0)}\psi_{\lambda,\mu}\mathrm{d}S\leqslant C\min_{B_r(X^0)}\sqrt{2\lambda-2gy},$$

implies $\psi_{\lambda,\mu}=0$ in $B_{\kappa r}(X^0)$.

Lemma 9 Let G be a domain in Ω_μ bounded by two disjointed arcs γ_1,γ_2 of the free boundary, $y=\beta_1$, $y=\beta_2$. Suppose that the acrs $\gamma_i(i=1,2)$ lie in $\{\beta_1<y<\beta_2\}$ with the endpoints(α_i,β_1) and (ζ_i,β_2). Suppose the distant $d=\mathrm{dist}(G,A)>0$, then

$$|\beta_2-\beta_1|\leqslant C\max\{|\alpha_1-\alpha_2|,|\zeta_1-\zeta_2|\},$$

where C is a constant depending only on λ, d and m_0.

Lemma 10 Let X_0 be a free boundary point in G and $G\in\Omega_\mu$, then there exists a constant $C>0$ depending only on λ, G such that for any minimizer $\psi_{\lambda,\mu}$,

$$|\nabla\psi|\leqslant C\max_G\sqrt{2\lambda-2gy}\text{ in } G.$$

[参考文献]

[1] GARABEDIAN P R, LEWY H, SCHIFFER M. Axially symmetric cavitational flow[J]. Annals of mathematics, 1952, 56: 560-602.

[2] BIRKHOFF G, ZARANTONELLO E H. Jets, wakes, and cavities[M]. New York: Academic Press, 1957.

[3] FRIEDMAN A. Variational principles and free-boundary problems[C]//Pure and Applied Mathematics. New York: Wiley, 1982.

[4] GILBARG D, TRUDINGER N S. Elliptic partial differential equations of second order[C]//Classic in Mathematics. Berlin: Springer-Verlag, 2001.

[5] ALT H W, CAFFARELLI L A. Existence and regularity for a minimum problem with free boundary[J]. Journal fur die reine und angewandte mathematik, 1981, 325: 105-144.

[6] ALT H W, CAFFARELLI L A, FRIEDMAN A. Axially symmetric jet flows[J]. Archive for rational mechanics and analysis, 1983, 81: 97-150.

[7] ALT H W, CAFFARELLI L A, FRIEDMAN A. Asymmetric jet flows[J]. Communication on pure and applied mathematics, 1982, 35: 29-68.

[8] ALT H W, CAFFARELLI L A, FRIEDMAN A. Jet flow with gravity[J]. Journal fur die reine und angewandte mathematik, 1982, 331: 58-103.

[9] CAFFARELLI L A, FRIEDMAN A. Axially symmetric infinite cavities[J]. Indiana university mathematics journal, 1982, 30: 135-160.

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