doi:10.3969/j.issn.1001-4616.2021.03.003

Ornstein-Uhlenbeck 随机波动率模型下蝶式期权的定价

田　凡[1],李美红[2],刘国祥[1],张昀菡[1],黄凤云[1],尤　磊[1]

(1.南京师范大学数学科学学院,江苏 南京 210023)
(2.南京市第十二初级中学,江苏 南京 210009)

[摘要]　本文将考虑标的资产价格服从均值回复 Ornstein-Uhlenbeck 随机波动率模型,分别采用随机偏微分方程方法和鞅方法探讨蝶式期权的定价公式.

[关键词]　蝶式期权,Ornstein-Uhlenbeck 随机波动率模型,随机偏微分方程方法,鞅方法
[中图分类号]O211.9　[文献标志码]A　[文章编号]1001-4616(2021)03-0014-06

Pricing of Butterfly Option in the Ornstein-Uhlenbeck Stochastic Volatility Model

Tian Fan[1],Li Meihong[2],Liu Guoxiang[1],Zhang Yunhan[1],Huang Fengyun[1],You Lei[1]

(1.School of Mathematical Sciences,Nanjing Normal University,Nanjing 210023,China)
(2.Nanjing No.12 Middle School,Nanjing 210009,China)

Abstract:In this article,we will consider the underlying asset is in the mean-reverting Ornstein-Uhlenbeck stochastic volatility model,and apply the stochastic partial differential equation pricing method and the martingale method respectively to discuss the pricing formula of butterfly option.

Key words:butterfly option,Ornstein-Uhlenbeck stochastic volatility model,stochastic partial differential equation pricing method,martingale method

蝶式期权(butterfly option)是由具有相同到期日,3 种不同执行价格的 4 份期权组成的一种奇异期权. 最为常见的一种蝶式期权的构成为:买入执行价格较高与较低的两份欧式看涨期权,同时卖出执行价格居中两份欧式看涨期权. 若记期权的到期日为 T,3 个执行价格为:K_1,K_2,K_3,则:$K_1<K_2<K_3$, $K_2=(K_1+K_3)/2$,且一般来说,K_2 接近标的资产的当前价格,那么这样一个蝶式期权在到期日 T 的收益函数可以表示为:

$$h=(S_T-K_1)^+-2(S_T-K_2)^++(S_T-K_3)^+. \tag{1}$$

在经典的 Black-Scholes 定价模型中[1],我们假设标的资产价格的波动率为常数,这种假设过于理想化,计算出的期权价格往往与实际有着较大差异,并且具有“微笑”波动率效应,为了使模型更加切合实际,人们改进为:将波动率设为一个与标的资产价格相关的随机过程,也就是我们说的随机波动率模型.

1987 年,Hull 等研究了 Hull-White 随机波动率模型的期权定价问题[2].随之,Scott[3]、Heston[4]利用此模型,得到了关于欧式期权定价的一系列理论. 1998 年,Rainer Schobel 等考虑标的资产价格服从均值回复 Ornstein-Uhlenbeck 随机波动率过程,并得到了欧式期权价格的解析解[5].

近年来,我国学者对于随机波动率下的美式期权也进行了大量研究. 邓国和等研究了随机波动率与双指数跳扩组合模型的美式期权定价问题[6]. 梅正阳等研究了一类随机波动率的美式期权定价[7]. 此外,一些奇异期权如:亚式期权、障碍期权、两值期权等的定价问题也得到了广泛关注.

在本文中,我们将主要探究收益函数由式(1)给出的蝶式期权,并在均值回复 Ornstein-Uhlenbeck 随机波动率模型下,考虑该类奇异期权的定价问题.

收稿日期:2020-06-19.
基金项目:国家自然科学基金项目(61374080).
通讯作者:刘国祥,教授,研究方向:金融数学. E-mail:gxliu63@163.com

1　Ornstein-Uhlenbeck 随机波动率模型下蝶式期权的定价

假定在风险中性概率测度 Q 下,标的资产价格 S 及其波动率 Y 遵从下述随机微分方程组:

$$\begin{cases} \mathrm{d}S_t = rS_t\mathrm{d}t + Y_tS_t\mathrm{d}W_t^*, \\ \mathrm{d}Y_t = \kappa(\theta - Y_t)\mathrm{d}t + \sigma\mathrm{d}Z_t, \end{cases} \tag{2}$$

式中,θ 表示波动率 Y_t 的长期均值,κ 代表均值回复速度,r,κ,θ,σ 均为常数,$(W_t^*, 0\leqslant t\leqslant T)$,$(Z_t, 0\leqslant t\leqslant T)$ 为风险中性概率测度 Q 下的标准 Brown 运动,$\mathrm{cov}(\mathrm{d}W_t^*, \mathrm{d}Z_t) = \rho\mathrm{d}t$.

1.1　随机偏微分方程方法定价

首先,我们利用随机偏微分方程方法求解.

设 $W_t^* = \rho Z_t + \sqrt{1-\rho^2}\,W_t$,其中,$Z_t$ 与 W_t 为风险中性概率测度 Q 下相互独立的标准 Brown 运动,则式(2)变为:

$$\begin{cases} \mathrm{d}S_t = rS_t\mathrm{d}t + Y_tS_t\rho\mathrm{d}Z_t + Y_tS_t\sqrt{1-\rho^2}\,\mathrm{d}W_t, \\ \mathrm{d}Y_t = \kappa(\theta - Y_t)\mathrm{d}t + \sigma\mathrm{d}Z_t. \end{cases} \tag{3}$$

以下在风险中性概率测度 Q 下,考虑包含一份金融衍生产品 V_t 和 Δ_1 份标的资产 S_t 的投资组合,为了消除随机波动率所引发的风险,我们再引入 Δ_2 份具有不同到期日和不同执行价格的任一未定权益 V_{2t},其中 V_t,V_{2t}与 t,Y,S 有关,Y,S 满足式(3). 这样构成的一个投资组合 Π,它的价值为:

$$\Pi_t = V(t,S_t,Y_t) - \Delta_1S_t - \Delta_2V_2(t,S_t,Y_t).$$

由 Ito 公式,我们有:

$$\mathrm{d}\Pi_t = \mathrm{d}V(t,S_t,Y_t) - \Delta_1\mathrm{d}S_t - \Delta_2\mathrm{d}V_2(t,S_t,Y_t) = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}Y_t^2S_t^2 + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2}\sigma^2 + \frac{\partial^2 V}{\partial S\partial Y}\rho\sigma Y_tS_t\right)\mathrm{d}t -$$

$$\Delta_2\left(\frac{\partial V_2}{\partial t} + \frac{1}{2}\frac{\partial^2 V_2}{\partial S^2}Y_t^2S_t^2 + \frac{1}{2}\frac{\partial^2 V_2}{\partial Y^2}\sigma^2 + \frac{\partial^2 V_2}{\partial S\partial Y}\rho\sigma Y_tS_t\right)\mathrm{d}t + \left(\frac{\partial V}{\partial S} - \Delta_1 - \Delta_2\frac{\partial V_2}{\partial S}\right)\mathrm{d}S_t + \left(\frac{\partial V}{\partial Y} - \Delta_2\frac{\partial V_2}{\partial Y}\right)\mathrm{d}Y_t, \tag{4}$$

为了消除随机因素,需要满足:

$$\Delta_1 = \frac{\partial V}{\partial S} - \frac{\frac{\partial V}{\partial Y}}{\frac{\partial V_2}{\partial Y}}\frac{\partial V_2}{\partial S}, \quad \Delta_2 = \frac{\frac{\partial V}{\partial Y}}{\frac{\partial V_2}{\partial Y}}.$$

为使投资组合 Π 无风险,即

$$\mathrm{d}\Pi_t = r\Pi_t\mathrm{d}t = r(V_t - \Delta_1S_t - \Delta_2V_{2t})\mathrm{d}t.$$

将我们解得的 Δ_1,Δ_2 代入上式,可得:

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}Y_t^2S_t^2 + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2}\sigma^2 + \frac{\partial^2 V}{\partial S\partial Y}\rho\sigma Y_tS_t\right)\mathrm{d}t - \frac{\frac{\partial V}{\partial Y}}{\frac{\partial V_2}{\partial Y}}\left(\frac{\partial V_2}{\partial t} + \frac{1}{2}\frac{\partial^2 V_2}{\partial S^2}Y_t^2S_t^2 + \frac{1}{2}\frac{\partial^2 V_2}{\partial Y^2}\sigma^2 + \frac{\partial^2 V_2}{\partial S\partial Y}\rho\sigma Y_tS_t\right)\mathrm{d}t =$$

$$r\left(V_t - \left(\frac{\partial V}{\partial S} - \frac{\frac{\partial V}{\partial Y}}{\frac{\partial V_2}{\partial Y}}\frac{\partial V_2}{\partial S}\right)S_t - \frac{\frac{\partial V}{\partial Y}}{\frac{\partial V_2}{\partial Y}}V_{2t}\right)\mathrm{d}t,$$

整理可得等式:

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}Y_t^2S_t^2 + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2}\sigma^2 + \frac{\partial^2 V}{\partial S\partial Y}\rho\sigma Y_tS_t + r\frac{\partial V}{\partial S}S_t - rV_t}{\frac{\partial V}{\partial Y}} = \frac{\frac{\partial V_2}{\partial t} + \frac{1}{2}\frac{\partial^2 V_2}{\partial S^2}Y_t^2S_t^2 + \frac{1}{2}\frac{\partial^2 V_2}{\partial Y^2}\sigma^2 + \frac{\partial^2 V_2}{\partial S\partial Y}\rho\sigma Y_tS_t + r\frac{\partial V_2}{\partial S}S_t - rV_{2t}}{\frac{\partial V_2}{\partial Y}}. \tag{5}$$

由文献[8]可知,由未定权益 V_{2t}的任意性,式(5)是一个仅依赖于 Y,S,t,而与 V,V_2 无关的函数,我们把这个函数记为:

$$-(\mu_1(t,Y)-\lambda_1(t,S,Y)\hat{\sigma}_1(t,Y)),$$

我们称 $\lambda_1(t,S,Y)$为波动率风险的市场价格.

则由式(5),我们有微分方程:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}Y_t^2S_t^2+\frac{1}{2}\frac{\partial^2 V}{\partial Y^2}\sigma^2+\frac{\partial^2 V}{\partial S\partial Y}\rho\sigma Y_tS_t+r\frac{\partial V}{\partial S}S_t+(\mu_1-\lambda_1\hat{\sigma}_1)\frac{\partial V}{\partial Y}=rV_t.$$

由此,我们便可以得到一个到期日为 T,标的资产价格服从式(3)的均值回复 Ornstein-Uhlenbeck 过程,且收益函数 h 由式(1)给出的蝶式期权,它的价格是下列偏微分方程的解:

$$\begin{cases}\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}Y_t^2S_t^2+\frac{1}{2}\frac{\partial^2 V}{\partial Y^2}\sigma^2+\frac{\partial^2 V}{\partial S\partial Y}\rho\sigma Y_tS_t+r\frac{\partial V}{\partial S}S_t+(\mu_1-\lambda_1\hat{\sigma}_1)\frac{\partial V}{\partial Y}-rV_t=0,\\ V(T,S_T,Y_T)=(S_T-K_1)^+-2(S_T-K_2)^++(S_T-K_3)^+,\\ V(t,0,Y_t)=0.\end{cases}\tag{6}$$

注意到,此式(6)是没有显示解的,即在随机偏微分方程方法下,我们无法得到蝶式期权价格的显示公式.

1.2　鞅方法定价

下面,我们将利用鞅方法,结合特征函数以及傅里叶逆变换[9-11],重新考虑蝶式期权的定价问题.

令 $X_t=\log(S_t)$,则在风险中性概率测度 Q 下,我们的式(2)变为:

$$\begin{cases}\mathrm{d}X_t=\left(r-\frac{1}{2}Y_t^2\right)\mathrm{d}t+Y_t\mathrm{d}W_t^*,\\ \mathrm{d}Y_t=\kappa(\theta-Y_t)\mathrm{d}t+\sigma\mathrm{d}Z_t,\end{cases}\tag{7}$$

式中,$\mathrm{cov}(\mathrm{d}W_t^*,\mathrm{d}Z_t)=\rho\mathrm{d}t$.

记 C^O 为 O-U 随机波动率模型下蝶式期权的价格,则由资产定价的基本定理,有:

$$\begin{aligned}C^O(t,S_t,Y_t)&=E^Q(\mathrm{e}^{-r(T-t)}[(S_T-K_1)^+-2(S_T-K_2)^++(S_T-K_3)^+])=E^Q(\mathrm{e}^{-r(T-t)}S_TI_{\{X_T>\log(K_1)\}})-\\&\mathrm{e}^{-r(T-t)}K_1E^Q(I_{\{X_T>\log(K_1)\}})-2\{E^Q(\mathrm{e}^{-r(T-t)}S_TI_{\{X_T>\log(K_2)\}})-\mathrm{e}^{-r(T-t)}K_2E^Q(I_{\{X_T>\log(K_2)\}})\}+\\&E^Q(\mathrm{e}^{-r(T-t)}S_TI_{\{X_T>\log(K_3)\}})-\mathrm{e}^{-r(T-t)}K_3E^Q(I_{\{X_T>\log(K_3)\}}).\end{aligned}\tag{8}$$

定理 1　到期日为 T,标的资产价格服从 O-U 随机波动率模型式(7),且收益函数 h 由式(1)给出的蝶式期权,它的价格 C^O 为:

$$\begin{aligned}C^O(t,S_t,Y_t)=&S_tF^{Q_1}(S_T>K_1)-\mathrm{e}^{-r(T-t)}K_1F^{Q_2}(S_T>K_1)-2[S_tF^{Q_1}(S_T>K_2)-\mathrm{e}^{-r(T-t)}K_2F^{Q_2}(S_T>K_2)]+\\&S_tF^{Q_1}(S_T>K_3)-\mathrm{e}^{-r(T-t)}K_3F^{Q_2}(S_T>K_3),\end{aligned}\tag{9}$$

式中,$\frac{\mathrm{d}Q_1}{\mathrm{d}Q}=\frac{\mathrm{e}^{-r(T-t)}S_T}{S_t}$,$\frac{\mathrm{d}Q_2}{\mathrm{d}Q}=1$,$F^{Q_1}$,$F^{Q_2}$分别表示 Q_1,Q_2 的概率分布.

证明　由文献[12],我们有:若无股息的资产 X 是一个在任意 $t\in[0,T]$时刻都几乎必然为正的随机过程,那么在可测空间$(\Omega,\mathbb{F})$上,存在一个新的概率测度 Q_X,使得任意资产的价格与 X_t 的比值是一个 Q_X-鞅.

根据上述理论,我们设 $P(t,T)$为到期日 T 时刻的价值为一元的零息债券在 t 时刻的价格,即 $P(T,T)=1$. 现在分别以 S_t 和 $P(t,T)$作为计价单位,定义两个新的鞅测度 Q_1,Q_2,

$$\frac{\mathrm{d}Q_1}{\mathrm{d}Q}=\frac{\mathrm{e}^{-r(T-t)}S_T}{S_t},\frac{\mathrm{d}Q_2}{\mathrm{d}Q}=\frac{\mathrm{e}^{-r(T-t)}P(T,T)}{P(t,T)}=1.\tag{10}$$

用 Q_1,Q_2 改写式(9),则有:

$$\begin{aligned}C^O(t,S_t,Y_t)=&S_tE^{Q_1}(I_{\{X_T>\log(K_1)\}})-\mathrm{e}^{-r(T-t)}K_1E^{Q_2}(I_{\{X_T>\log(K_1)\}})-2[S_tE^{Q_1}(I_{\{X_T>\log(K_2)\}})-\\&\mathrm{e}^{-r(T-t)}K_2E^{Q_2}(I_{\{X_T>\log(K_2)\}})]+S_tE^{Q_1}(I_{\{X_T>\log(K_3)\}})-\mathrm{e}^{-r(T-t)}K_3E^{Q_2}(I_{\{X_T>\log(K_3)\}})=\\&S_tF^{Q_1}(S_T>K_1)-\mathrm{e}^{-r(T-t)}K_1F^{Q_2}(S_T>K_1)-2[S_tF^{Q_1}(S_T>K_2)-\mathrm{e}^{-r(T-t)}K_2F^{Q_2}(S_T>K_2)]+\\&S_tF^{Q_1}(S_T>K_3)-\mathrm{e}^{-r(T-t)}K_3F^{Q_2}(S_T>K_3).\end{aligned}$$

由此,定理得证.

下一步,我们想要求得概率测度 Q_1,Q_2 的概率分布函数. 为此定义特征函数:

$$f_j(u)=E^{Q_j}(\mathrm{e}^{\mathrm{i}uX_T})\quad j=1,2.$$

定理 2　Q_1,Q_2 的特征函数分别为:

$$f_1(u)=\mathrm{e}^{\mathrm{i}u[r(T-t)+X_t]-\frac{1}{2}\rho(1+\mathrm{i}u)[\sigma^{-1}Y_t^2+\sigma(T-t)]+\frac{1}{2}D(t,T;s_1,s_3)Y_t^2+B(t,T;s_1,s_2,s_3)Y_t+C(t,T;s_1,s_2,s_3)},\tag{11}$$

$$f_2(u)=\mathrm{e}^{\mathrm{i}u[r(T-t)+X_t]-\frac{1}{2}\mathrm{i}u\rho[\sigma^{-1}Y_t^2+\sigma(T-t)]+\frac{1}{2}D(t,T;\hat{s}_1,\hat{s}_3)Y_t^2+B(t,T;\hat{s}_1,\hat{s}_2,\hat{s}_3)Y_t+C(t,T;\hat{s}_1,\hat{s}_2,\hat{s}_3)},\tag{12}$$

式中,$s_1=-\frac{1}{2}(1+\mathrm{i}u)^2(1-\rho^2)+\frac{1}{2}(1+\mathrm{i}u)(1-2\kappa\rho\sigma^{-1})$,$s_2=(1+\mathrm{i}u)\kappa\theta\rho\sigma^{-1}$,$s_3=\frac{1}{2}(1+\mathrm{i}u)\rho\sigma^{-1}$,$\hat{s}_1=\frac{1}{2}u^2(1-\rho^2)+\frac{1}{2}\mathrm{i}u(1-2\kappa\rho\sigma^{-1})$,$\hat{s}_2=\mathrm{i}u\kappa\theta\rho\sigma^{-1}$,$\hat{s}_3=\frac{1}{2}\mathrm{i}u\rho\sigma^{-1}$,

$$D(t,T;s_1,s_3)=\frac{1}{\sigma^2}\left(\kappa-\gamma_1\frac{\sinh[\gamma_1(T-t)]+\gamma_2\cosh[\gamma_1(T-t)]}{\cosh[\gamma_1(T-t)]+\gamma_2\sinh[\gamma_1(T-t)]}\right),$$

$$B(t,T;s_1,s_2,s_3)=\frac{1}{\sigma^2\gamma_1}\left(\frac{(\kappa\theta\gamma_1-\gamma_2\gamma_3)+\gamma_3(\sinh[\gamma_1(T-t)]+\gamma_2\cosh[\gamma_1(T-t)])}{\cosh[\gamma_1(T-t)]+\gamma_2\sinh[\gamma_1(T-t)]}-\kappa\theta\gamma_1\right),$$

$$\begin{aligned}C(t,T;s_1,s_2,s_3)=&-\frac{1}{2}\log(\cosh[\gamma_1(T-t)]+\gamma_2\sinh[\gamma_1(T-t)]+\\&\frac{1}{2}\kappa(T-t)+\frac{\kappa^2\theta^2\gamma_1^2-\gamma_3^2}{2\sigma^2\gamma_1^3}\left(\frac{\sinh[\gamma_1(T-t)]}{\cosh[\gamma_1(T-t)]+\gamma_2\sinh[\gamma_1(T-t)]}-\gamma_1(T-t)\right)+\\&\frac{(\kappa\theta\gamma_1-\gamma_2\gamma_3)\gamma_3}{\sigma^2\gamma_1^3}\left(\frac{\cosh[\gamma_1(T-t)]-1}{\cosh[\gamma_1(T-t)]+\gamma_2\sinh[\gamma_1(T-t)]}\right),\end{aligned}$$

$$D(t,T;\hat{s}_1,\hat{s}_3)=\frac{1}{\sigma^2}\left(\kappa-\hat{\gamma}_1\frac{\sinh[\hat{\gamma}_1(T-t)]+\hat{\gamma}_2\cosh[\hat{\gamma}_1(T-t)]}{\cosh[\hat{\gamma}_1(T-t)]+\hat{\gamma}_2\sinh[\hat{\gamma}_1(T-t)]}\right),$$

$$B(t,T;\hat{s}_1,\hat{s}_2,\hat{s}_3)=\frac{1}{\sigma^2\hat{\gamma}_1}\left(\frac{(\kappa\theta\hat{\gamma}_1-\hat{\gamma}_2\hat{\gamma}_3)+\hat{\gamma}_3(\sinh[\hat{\gamma}_1(T-t)]+\hat{\gamma}_2\cosh[\hat{\gamma}_1(T-t)])}{\cosh[\hat{\gamma}_1(T-t)]+\hat{\gamma}_2\sinh[\hat{\gamma}_1(T-t)]}-\kappa\theta\hat{\gamma}_1\right),$$

$$\begin{aligned}C(t,T;\hat{s}_1,\hat{s}_2,\hat{s}_3)=&-\frac{1}{2}\log(\cosh[\hat{\gamma}_1(T-t)]+\hat{\gamma}_2\sinh[\hat{\gamma}_1(T-t)]+\frac{1}{2}\kappa(T-t)+\frac{\kappa^2\theta^2\hat{\gamma}_1^2-\hat{\gamma}_3^2}{2\sigma^2\hat{\gamma}_1^3}\times\\&\left(\frac{\sinh[\hat{\gamma}_1(T-t)]}{\cosh[\hat{\gamma}_1(T-t)]+\hat{\gamma}_2\sinh[\hat{\gamma}_1(T-t)]}-\hat{\gamma}_1(T-t)\right)+\frac{(\kappa\theta\hat{\gamma}_1-\hat{\gamma}_2\hat{\gamma}_3)\hat{\gamma}_3}{\sigma^2\hat{\gamma}_1^3}\left(\frac{\cosh[\hat{\gamma}_1(T-t)]-1}{\cosh[\hat{\gamma}_1(T-t)]+\hat{\gamma}_2\sinh[\hat{\gamma}_1(T-t)]}\right).\end{aligned}$$

这里,$\gamma_1=\sqrt{2\sigma^2s_1+\kappa^2}$,$\gamma_2=\frac{1}{\gamma_1}(\kappa-2\sigma^2s_3)$,$\gamma_3=\kappa^2\theta-s_2\sigma^2$,$\hat{\gamma}_1=\sqrt{2\sigma^2\hat{s}_1+\kappa^2}$,$\hat{\gamma}_2=\frac{1}{\hat{\gamma}_1}(\kappa-2\sigma^2\hat{s}_3)$,$\hat{\gamma}_3=\kappa^2\theta-\hat{s}_2\sigma^2$.

证明　设 $W_t^*=\rho Z_t+\sqrt{1-\rho^2}W_t$,其中 Z_t 与 W_t 为相互独立的标准 Brown 运动.

由式(9),$f_1(u)$在风险中性概率测度 Q 下可表示为:

$$\begin{aligned}f_1(u)=&E^{Q_1}(\mathrm{e}^{\mathrm{i}uX_T})=E^Q\left(\frac{\mathrm{e}^{-r(T-t)}S_T}{S_t}\mathrm{e}^{\mathrm{i}uX_T}\right)=E^Q(\mathrm{e}^{-r(T-t)-X_t+(1+\mathrm{i}u)X_T})=E^Q\left(\mathrm{e}^{-r(T-t)-X_t+(1+\mathrm{i}u)\left(X_t+\int_t^T\left(r-\frac{1}{2}Y_u^2\right)\mathrm{d}u+\int_t^TY_u\mathrm{d}W_u^*\right)}\right)=\\&\mathrm{e}^{\mathrm{i}u[r(T-t)+X_t]}E^Q\left(\mathrm{e}^{(1+\mathrm{i}u)\left[-\frac{1}{2}\int_t^TY_u^2\mathrm{d}u+\int_t^TY_u\mathrm{d}W_u^*\right]}\right)=\mathrm{e}^{\mathrm{i}u[r(T-t)+X_t]}E^Q\left(\mathrm{e}^{(1+\mathrm{i}u)\left[-\frac{1}{2}\int_t^TY_u^2\mathrm{d}u+\rho\int_t^TY_u\mathrm{d}Z_u+\sqrt{1-\rho^2}\int_t^TY_u\mathrm{d}W_u\right]}\right).\end{aligned}$$

由于 $\mathrm{d}Z_t$ 与 $\mathrm{d}W_t$ 是相互独立的,且 $\int_t^TY_u\mathrm{d}W_u$ 服从均值为 0,方差为$\frac{1}{2}\int_t^TY_u^2\mathrm{d}u$ 的正态分布,所以:

$$\begin{aligned}f_1(u)=&\mathrm{e}^{\mathrm{i}u[r(T-t)+X_t]}E^Q\left(\mathrm{e}^{(1+\mathrm{i}u)\left[-\frac{1}{2}\int_t^TY_u^2\mathrm{d}u+\rho\int_t^TY_u\mathrm{d}Z_u\right]+\frac{1}{2}(1+\mathrm{i}u)^2(1-\rho^2)\int_t^TY_u^2\mathrm{d}u}\right)=\\&\mathrm{e}^{\mathrm{i}u[r(T-t)+X_t]}E^Q\left(\mathrm{e}^{\frac{1}{2}(1+\mathrm{i}u)[(1+\mathrm{i}u)(1-\rho^2)-1]\int_t^TY_u^2\mathrm{d}u+(1+\mathrm{i}u)\rho\int_t^TY_u\mathrm{d}Z_u}\right)=\\&\mathrm{e}^{\mathrm{i}u[r(T-t)+X_t]}E^Q\left(\mathrm{e}^{\frac{1}{2}(1+\mathrm{i}u)[(1+\mathrm{i}u)(1-\rho^2)-1]\int_t^TY_u^2\mathrm{d}u+(1+\mathrm{i}u)\frac{\rho}{2\sigma}\left(Y_T^2-Y_t^2-\sigma^2(T-t)-2\kappa\theta\int_t^TY_u\mathrm{d}u+2\kappa\int_t^TY_u^2\mathrm{d}u\right)}\right)=\\&\mathrm{e}^{\mathrm{i}u[r(T-t)+X_t]}E^Q\left(\mathrm{e}^{\frac{1}{2}(1+\mathrm{i}u)\left[(1+\mathrm{i}u)(1-\rho^2)-1+\frac{2\kappa\rho}{\sigma}\right]\int_t^TY_u^2\mathrm{d}u-(1+\mathrm{i}u)\frac{\rho}{2\sigma}(Y_t^2+\sigma^2(T-t))+(1+\mathrm{i}u)\frac{\rho}{2\sigma}Y_T^2-(1+\mathrm{i}u)\frac{\kappa\theta\rho}{\sigma}\int_t^TY_u\mathrm{d}u}\right)=\\&\mathrm{e}^{\mathrm{i}u[r(T-t)+X_t]-(1+\mathrm{i}u)\frac{\rho}{2\sigma}(Y_t^2+\sigma^2(T-t))}E^Q\left(\mathrm{e}^{-s_1\int_t^TY_u^2\mathrm{d}u-s_2\int_t^TY_u\mathrm{d}u+s_3Y_T^2}\right),\end{aligned}$$

式中,$s_1=-\frac{1}{2}(1+\mathrm{i}u)^2(1-\rho^2)+\frac{1}{2}(1+\mathrm{i}u)(1-2\kappa\rho\sigma^{-1})$,$s_2=(1+\mathrm{i}u)\kappa\theta\rho\sigma^{-1}$,$s_3=\frac{1}{2}(1+\mathrm{i}u)\rho\sigma^{-1}$.

下面计算期望值,记

$$y(Y,t,T)=E^Q\left(\mathrm{e}^{-s_1\int_t^T Y_u^2\mathrm{d}u-s_2\int_t^T Y_u\mathrm{d}u+s_3Y_T^2}\right)=E^Q\left(\mathrm{e}^{\int_t^T(-s_1Y_u^2-s_2Y_u)\mathrm{d}u}\mathrm{e}^{s_3Y_T^2}\right).$$

对于任意的复数 s_1,s_2 和 s_3,$-s_1Y_u^2-s_2Y_u$ 有下界,根据 Feynman-Kac 公式,y 需满足下列带边界条件的微分方程:

$$\begin{cases}\frac{1}{2}\sigma^2\frac{\partial^2 y}{\partial Y^2}+\kappa(\theta-Y)\frac{\partial y}{\partial Y}-(s_1Y^2+s_2Y)y+\frac{\partial y}{\partial t}=0,\\ y(Y,T,T)=\mathrm{e}^{s_3Y^2}.\end{cases}\tag{13}$$

且上述微分方程的解为:

$$y(Y,t,T)=\mathrm{e}^{\frac{1}{2}A(t,T)Y_t^2+B(t,T)Y_t+C(t,T)+s_3Y_t^2}=\mathrm{e}^{\frac{1}{2}(A(t,T)+2s_3)Y_t^2+B(t,T)Y_t+C(t,T)}=\mathrm{e}^{\frac{1}{2}D(t,T)Y_t^2+B(t,T)Y_t+C(t,T)},\tag{14}$$

式中,$D(t,T)=A(t,T)+2s_3$.

将式(14)代入上述微分方程式(13),我们可以得到决定 $D(t,T)$,$B(t,T)$,$C(t,T)$ 的 3 个常微分方程:

$$D_t=-\sigma^2D^2+2\kappa D+2s_1,$$

$$B_t=(\kappa-\sigma^2D)B-\kappa\theta D+s_2,$$

$$C_t=-\frac{1}{2}\sigma^2B^2-\kappa\theta B-\frac{1}{2}\sigma^2D,$$

式中,$D(T,T)=2s_3$,$B(T,T)=C(T,T)=0$. 求解上述常微分方程组,我们有:

$$D(t,T)=\frac{1}{\sigma^2}\left(\kappa-\gamma_1\frac{\sinh[\gamma_1(T-t)]+\gamma_2\cosh[\gamma_1(T-t)]}{\cosh[\gamma_1(T-t)]+\gamma_2\sinh[\gamma_1(T-t)]}\right),$$

$$B(t,T)=\frac{1}{\sigma^2\gamma_1}\left(\frac{(\kappa\theta\gamma_1-\gamma_2\gamma_3)+\gamma_3(\sinh[\gamma_1(T-t)]+\gamma_2\cosh[\gamma_1(T-t)])}{\cosh[\gamma_1(T-t)]+\gamma_2\sinh[\gamma_1(T-t)]}-\kappa\theta\gamma_1\right),$$

$$\begin{aligned}C(t,T)=&-\frac{1}{2}\log(\cosh[\gamma_1(T-t)]+\gamma_2\sinh[\gamma_1(T-t)]+\frac{1}{2}\kappa(T-t)+\\&\frac{\kappa^2\theta^2\gamma_1^2-\gamma_3^2}{2\sigma^2\gamma_1^3}\left(\frac{\sinh[\gamma_1(T-t)]}{\cosh[\gamma_1(T-t)]+\gamma_2\sinh[\gamma_1(T-t)]}-\gamma_1(T-t)\right)+\\&\frac{(\kappa\theta\gamma_1-\gamma_2\gamma_3)\gamma_3}{\sigma^2\gamma_1^3}\left(\frac{\cosh[\gamma_1(T-t)]-1}{\cosh[\gamma_1(T-t)]+\gamma_2\sinh[\gamma_1(T-t)]}\right),\end{aligned}$$

式中,$\gamma_1=\sqrt{2\sigma^2s_1+\kappa^2}$,$\gamma_2=\frac{1}{\gamma_1}(\kappa-2\sigma^2s_3)$,$\gamma_3=\kappa^2\theta-s_2\sigma^2$.

得到了与时间独立的函数 $D(t,T)$,$B(t,T)$,$C(t,T)$,现在我们便可得到 $f_1(u)$ 的闭解形式:

$$f_1(u)=\mathrm{e}^{\mathrm{i}u[r(T-t)+X_t]-\frac{1}{2}\rho(1+\mathrm{i}u)[\sigma^{-1}Y_t^2+\sigma(T-t)]+\frac{1}{2}D(t,T;s_1,s_3)Y_t^2+B(t,T;s_1,s_2,s_3)Y_t+C(t,T;s_1,s_2,s_3)}.$$

类似于 $f_1(u)$ 的求解,同样地,我们可以得到 $f_2(u)$ 的闭解形式:

$$f_2(u)=\mathrm{e}^{\mathrm{i}u[r(T-t)+X_t]-\frac{1}{2}\mathrm{i}u\rho[\sigma^{-1}Y_t^2+\sigma(T-t)]+\frac{1}{2}D(t,T;\hat{s}_1,\hat{s}_3)Y_t^2+B(t,T;\hat{s}_1,\hat{s}_2,\hat{s}_3)Y_t+C(t,T;\hat{s}_1,\hat{s}_2,\hat{s}_3)},$$

式中,$\hat{s}_1=\frac{1}{2}u^2(1-\rho^2)+\frac{1}{2}\mathrm{i}u(1-2\kappa\rho\sigma^{-1})$,$\hat{s}_2=\mathrm{i}u\kappa\theta\rho\sigma^{-1}$,$\hat{s}_3=\frac{1}{2}\mathrm{i}u\rho\sigma^{-1}$.

由此,定理得证.

由上述定理 2,我们分别得到了概率测度 Q_1,Q_2 的特征函数 $f_1(u)$ 和 $f_2(u)$ 的闭解形式,再根据傅里叶逆变换公式,我们就可以计算出 Q_1,Q_2 的概率分布函数:

$$F_{jl}=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left(f_j(u)\frac{\mathrm{e}^{-\mathrm{i}u\log K_l}}{\mathrm{i}u}\right)\mathrm{d}u,\quad j=1,2,\quad l=1,2,3.\tag{15}$$

最后,只要将式(15)代入式(9),我们便可以得到:一个期日为 T,标的资产价格服从均值回复 O-U 随机波动率模型式(7),且收益函数 h 由式(1)给出的蝶式期权,它的定价公式可表示为:

$$C^O(t,S_t,Y_t)=S_tF_{11}-\mathrm{e}^{-r(T-t)}K_1F_{21}-2[S_tF_{12}-\mathrm{e}^{-r(T-t)}K_2F_{22}]+S_tF_{13}-\mathrm{e}^{-r(T-t)}K_3F_{23}. \tag{16}$$

由此,我们就通过鞅方法,结合特征函数,得到了均值回复 Ornstein-Uhlenbeck 随机波动率模型下,蝶式期权价格的闭解形式.

2 结论

在本文中,为了与实际更加贴合,从推广波动率为随机过程入手,考虑了标的资产价格满足均值回复 Ornstein-Uhlenbeck 随机过程下蝶式期权的定价公式. 利用随机偏微分方程方法定价,得到了蝶式期权价格满足的微分方程,然后,根据鞅方法,结合特征函数以及傅里叶逆变换公式,求解出了蝶式期权价格的闭解形式.

[参考文献]

[1] BLACK F,SCHOLES M J. The pricing of options and corporate liabilities[J]. Journal of political economy,1973,81(3):637-654.

[2] HULL J C,WHITE A. The pricing of options on assets with stochastic volatilities[J]. Journal of finance,1987(42):281-300.

[3] SCOTT L O. Option pricing when the variance changes randomly:theory,estimation,and an application[J]. Journal of financial and quantitative analysis,1987,22(4):419-438.

[4] HESTON S L. A closed form-solution for options with stochastic volatility with applications to bond and currency options[J]. Review of financial studies,1933(6):327-343.

[5] RAINER S,ZHU J W. Stochastic volatility with an Ornstein-Uhlenbeck process:an extension[J]. European finance review,1999(3):23-46.

[6] 邓国和,杨向群. 随机波动率与双指数跳扩组合模型的美式期权定价[J]. 应用数学学报,2009(2):236-255.

[7] 梅正阳,刘次华. 一类随机波动率的美式期权定价[J]. 山西大学学报(自然科学版),2008(3):443-446.

[8] 姜礼尚,徐承龙,任学敏,等. 金融衍生产品定价的数学模型与案例分析[M]. 北京:高等教育出版社,2008.

[9] STEIN E M,STEIN J C. Stock price distributions with stochastic volatility:an analytic approach[J]. Review of financial studies,1991,4(4):727-752.

[10] OLDRICH V. An equilibrium characterization of the term structure[J]. Journal of financial economics,1977,5(2):177-188.

[11] POTERBA J M,SUMMERS L H. Mean reversion in stock prices:evidence and implications[J]. Journal of financial economics,1988,22(1):27-59.

[12] GEMAN H,KAROUI N E,ROCHET J C. Changes of numeraire,changes of probability measure and option pricing[J]. Journal of applied probability,1995(32):443-458.

[责任编辑:陆炳新]