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Weak Convergence Theorem for Nonexpansive Semigroups

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Abstract: Let X be a Banach space which satisfies Opial's condition, let C be a nonempty weakly compact subset of X. Let S be a nonexpansive semigroup on C. We prove that if $X \in C$, and $\lim_{t\to\infty} \| T(t+h)x - T(t)x\| = 0$ for all $h \geqslant 0$. Then $T(t)x$ converges weakly to some $y \in F(S)$.

Key words: nonexpansive; semigroup

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Let C be a nonempty closed subset of a Banach space X. Let $S = \{T(t): t \geqslant 0\}$ be a nonexpansive semigroup on C, i.e. $S = \{T(t): t \geqslant 0\}$ be a family mappings of C into itself such that $T(t+s) = T(t) \cdot T(s)$, $\| T(t)x - T(t)y\| \leqslant \|x - y\|$ for $t, s \in [0, \infty]$ and $x, y \in C$. We set $F = \{x \in C: T(t)x = x$ for all $t \geqslant 0\}$.

We denote by $\omega_\omega(x)$ the set of weak limit points of $T(t)x$ as $t \to \infty$. X is said to have the Opial property if $\{x_n\}$ convergences to x weakly implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \neq x$.

Recently Lau [1,2], Miyadera and Kobayasi [3] consider the problem of weak convergence of the $\{T(t)x\}$, $x \in C$, to a common fixed point of S when S is nonexpansive semigroup and C is a closed convex subset of a uniformly convex Banach space. However, the proofs emplyed by Lau [1] (Lemma 2.1, Lemma 2.2 and Theorem 2.3) do not extend beyond uniformly convex Banach space satisfying Opial's condition and the proof emplyed by Lau [2] (Theorem 3) do not extend beyond uniformly convex Banach space.

In this paper, we prove the weakly convergence theorem for nonexpansive semigroups in a Banach space satisfying Opial's condition.

First, we need the following two lemmas.

Lemma 1 Let C be a nonempty set of a Banach space X, $S = \{T(t): t \geqslant 0\}$ be a nonexpansive semigroup on C. Then for every $p \in F$, $x \in C$, $\|T(t)x - p\|$ converges as $t \to \infty$.

Proof Since $\|T(t+s)x - p\| \leqslant \|T(s)x - p\|$ for all $t, s \in [0, \infty]$. For fixed $s \geqslant 0$, taking the

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lim sup as $t\rightarrow\infty$, we obtain $\limsup_{t\rightarrow\infty}\|T(t)x-p\|\leqslant\|T(s)x-p\|$, which implies that $\lim_{t\rightarrow\infty}\|T(t)x-p\|=:\inf\{\|T(s)x-p\|:s\geqslant 0\}$.

Lemma 2 Let C be a nonempty weakly compact subset of a Banach space X that satisfies Opial's condition, and let $S=\{T(t)0:t\geqslant 0\}$ be a nonexpansive semigroup on C. If $x\in C$ and $T(t)x$ convergence to y weakly, then y is fixed point of S.

Proof For any $s\geqslant 0$, since $\|T(t+s)x-T(s)y\|\leqslant\|T(t)x-y\|$, taking the lim sup as $t\rightarrow\infty$, we have $\limsup_{t\rightarrow\infty}\|T(t)x-T(s)y\|\leqslant\limsup_{t\rightarrow\infty}\|T(t)x-y\|$, The Opial's condition implies that $T(s)y=y$, and hence $y\in F$.

Theorem 1 Let C be a nonempty weakly compact subset of a Banach space X that satisfies Opial's condition, and let $S=\{T(t):t\geqslant 0\}$ be a nonexpansive semigroup on C. If $x\in C$ and $\lim_{t\rightarrow\infty}\|T(t+h)x-T(t)x\|=0$ for all $h\geqslant 0$. Then $T(t)x$ converges weakly to a fixed point.

Proof Let $y\in\omega_\omega(x)$, and $t_n\uparrow\infty$ such that $\omega-\lim_{n\rightarrow\infty}T(t_n)x=y$. We first show that $y\in F(S)$. By lemma 2, it is enough to show that $T(t)y$ converges weakly to y. Let z is a weak limit point of $T(t)y$, for any $\varepsilon>0$, there exist an integer p and nonnegative $s_0,s_1,\cdots,s_p,a_0,a_1,\cdots,a_p$ such that $\sum_{i=0}^{p}a_i=1$ and

$$\left\|z-\sum_{i=1}^{p}a_iT(s_i)y\right\|<\varepsilon \tag{1}$$

Since $\lim_{n\rightarrow\infty}\|T(t_n+s_i)x-T(t_n)x\|=0$ for all $0\leqslant i\leqslant p$, we can choose N such that

$$\|T(t_n+s_i)x-T(t_n)x\|<\varepsilon \tag{2}$$

for all $0\leqslant i\leqslant p$, and $n>N$. Then we have

$$\begin{aligned}\|T(t_n)x-z\| &\leqslant\left\|T(t_n)x-\sum_{i=1}^{p}a_iT(t_n+s_i)y\right\|+\left\|\sum_{i=1}^{p}a_iT(t_n+s_i)y-z\right\|\\ &\leqslant\sum_{i=1}^{p}a_i\|T(t_n)x-T(t_n+s_i)y\|+\left\|\sum_{i=1}^{p}a_iT(t_n+s_i)y-\sum_{i=1}^{p}a_iT(s_i)y\right\|+\varepsilon\\ &\leqslant\|T(t_n)x-y\|+3\varepsilon.\end{aligned}$$

This implies that $\limsup_{n\rightarrow\infty}\|T(t_n)x-z\|\leqslant\limsup_{n\rightarrow\infty}\|T(t_n)x-y\|+3\varepsilon$.

Since $\varepsilon>0$ is arbitrary, we have $\limsup_{n\rightarrow\infty}\|T(t_n)x-z\|\leqslant\limsup_{n\rightarrow\infty}\|T(t_n)x-y\|$.

Now the Opial's condition implies that $z=y$, and hence $T(t)y$ converges weakly to y. Let v is a weak limit point of $T(t)x$, then we have $v,y\in F$, suppose that $v\neq y$, by lemma 1 the Opial's condition we have $\lim_{n\rightarrow\infty}\|T(t)x-y\|=\lim_{n\rightarrow\infty}\|T(t_n)x-y\|<\lim_{n\rightarrow\infty}\|T(t_n)x-v\|=\lim_{n\rightarrow\infty}\|T(t)x-v\|$

But, by the same way we also have $\lim_{n\rightarrow\infty}\|T(t)x-v\|<\lim_{n\rightarrow\infty}\|T(t)x-y\|$.

This is a contradiction, and hence we have $v=y$. This completes the proof.

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关于非扩张半群的弱收敛定理

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［摘要］ 设 X 是满足 Opial 条件的巴拿赫空间，C 是 X 的一个弱紧致子集，S 是 C 上的一个非扩张半群，本文证明了如果 $X\in C$，并且对于一切 $h\geqslant 0$，$\lim\limits_{t\to\infty}\|T(t+h)x-T(t)x\|=0$，则 $T(t)x$ 弱收敛于某个 $y\in F(S)$（S 的不动点集全体）。

［关键词］ 非扩张；半群

［责任编辑：陆炳新］