Existence of Explosive Nonnegative Solutions for a Class of Quasilinear Ordinary Differential Equations

Yang Huisheng[1], Yang Zuodong[2]

(1. School of Mathematics and Information Science, Henan Normal University, 453002, Xinxiang, China)

(2. School of Mathematics and Computer Science, Nanjing Normal University, 210097, Nanjing, China)

Abstract: In this paper, the necessary and sufficient conditions of the existence of nonnegative solutions for the two point boundary value problem

$$-(\Phi_p(u'))' = \lambda f(u(x)); \quad 0 < x < 1$$

$$\lim_{x \to 0^+} u(x) = \infty = \lim_{x \to 1^-} u(x),$$

is established, where $\lambda > 0$ is a parameter and f is a smooth function.

Key words: explosive nonnegative solutions, two-point boundary value problems, existence

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Here we consider the two point boundary value problem

$$-(\Phi_p(u'))' = \lambda f(u(x)), \quad 0 < x < 1 \tag{1}$$

$$\lim_{x \to 0^+} u(x) = \infty = \lim_{x \to 1^-} u(x), \tag{2}$$

where $\lambda > 0$ is a positive parameter and f is a smooth function, and $\Phi_p(u) = |u|^{p-2}u$, $p > 1$.

Explosive solutions of the problem

$$-\Delta u(x) = f(u(x)), x \in \Omega \tag{3}$$

$$u|_{\partial\Omega} = \infty, \tag{4}$$

where Ω is bounded domain in $\mathbf{R}^N$ $(N \geqslant 1)$ have been extensively studied, see [3 - 9]. For general nonlinearities $f(u)$ and in one space dimension, very recently, Anuradha *et al*[1] and Wang Shin-Hwa[2] considered problem (3) ~ (4). In this paper, we proved the existence of explosive nonnegative solutions to (1) ~ (2) basing on building a quadrature method, the extends and complement part results of [1 - 2].

First, define

$$F(s) = \int_0^s f(t)\mathrm{d}t,$$

and

$$I = \{s \in R^+ \cup \{0\} : f(s) < 0 \text{ and } F(s) > F(u) \forall u > s\}.$$

Suppose that u is a nonnegative solution of Problem (1) ~ (2). Let

$$\rho = \min_{x \in (0,1)} u(x).$$

Now we shall prove that u is symmetric to $x = 1/2$. Let u be a positive solution of (1) ~ (2). Then u has only one minimum point in (0,1) (there is no local maximum point of u in (0,1)) and u is the unique solution of the problem

$$v'' = \frac{f(v)}{(p-1)|v'|^{p-2}}$$

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Biography: Yang Huisheng, born in 1953, assistant professor of Henan Normal University, Majored in nonlinear differential equation.

Corresponding author: Yang Zuodong, born in 1961, Professor of Nanjing Normal University, Majored in nonlinear differential equation, E - mail: yangzuodong@njnu.edu.cn

$$v(\epsilon) = v_0,\ v'(\epsilon) = u'(\epsilon),$$

in $[\epsilon, \xi_0)$, ϵ is given arbitrary constant and ξ_0 is the minimum point of u by the standard ordinary differential equation theory (suppose u has two minimum points in $(0,1)$, we can reach a contradiction by directly integrating (1). The similar argument implies that there is no local maximum point of u in $(0,1)$). Let $y = 1 - x$ for $x \in (\xi_0, 1-\epsilon]$ and $\bar{u}(y) = u(1-x)$. Then $\bar{u}(y)$ satisfies the problem

$$\bar{u}_{yy} = -\frac{f(\bar{u})}{(p-1)\mid \bar{u}' \mid^{p-2}} \text{ in } [\epsilon, 1-\xi_0)$$

$$\bar{u}(\epsilon) = v_0\ \bar{u}'(\epsilon) = u'(\epsilon).$$

Thus, $u(x)$ and $\bar{u}(y)$ satisfy the same initial value problem. Let η be the minimum point of $\bar{u}$, then $\eta = \xi_0$. Since $u(x) = \bar{u}(y)$ in (ϵ, ξ_0) and $u_x'(\xi_0) = -\bar{u}_y'(1-\xi_0) = 0$, then $\eta = 1 - \xi_0$. This implies $\xi_0 = 1/2$ and $u(x) = u(1-x)$ for $x \in (\epsilon, 1/2)$, from ϵ being arbitrary, therefore, u is symmetric to $x = 1/2$ and $u' < 0$ in $(0, 1/2)$ and $u' > 0$ in $(1/2, 1)$.

That is, $u(x)$ must achieve its minimum at $x = 1/2$. Multiplying (1) through by $u'(x)$, we obtain

$$-(\Phi_p(u'))'u'(x) = \lambda f(u)u'(x)$$

which can be integrated yielding

$$-(p-1)/p \mid u' \mid^p = \lambda F(u(x)) + C. \tag{5}$$

If $\rho = \inf_{x\in(0,1)} u(x)$, then $u(1/2) = \rho$. Substituting $x = 1/2$ in (5), we have

$$C = -\lambda F(u(1/2)) = -\lambda F(\rho).$$

Thus

$$u'(x) = -(p\lambda/(p-1))^{1/p}(F(\rho) - F(u))^{1/p},\ \forall x \in (0, 1/2) \tag{6}$$

and by symmetry,

$$u'(x) = (p\lambda/(p-1))^{1/p}(F(\rho) - F(u))^{1/p},\ \forall x \in (1/2, 1).$$

Dividing through by $(F(\rho) - F(u))^{1/p}$ and integrating (6) from 0 to x, we obtain

$$\int_{u(0)}^{u(x)} \frac{du}{(F(\rho) - F(u))^{1/p}} = -\left(\frac{p\lambda}{(p-1)}\right)^{1/p} x,\ \forall x \in (0, 1/2) \tag{7}$$

Substituting $x = 1/2$ in (7), we see that $G(\rho)$ must exist and λ and ρ must satisfy

$$G(\rho) = 2\left(\frac{p-1}{p}\right)^{1/p} \int_\rho^\infty \frac{du}{(F(\rho) - F(u))^{1/p}} = \lambda^{1/p}. \tag{8}$$

In fact we have the following lemma.

Lemma 1　Given $\lambda > 0$ and $f \in C^1$, there exists a unique solution to (1) ~ (2) with $\inf_{x\in(0,1)} u(x) = \rho$ if and only if $G(\rho) = \lambda^{1/p}$, $\rho \in I$.

By a modification of the method given in [1 - 2], we establish the following results.

Theorem 2　If there exists any solution to (1) ~ (2) for any $\lambda > 0$, then

$$\limsup_{u\to\infty}\left(-\frac{f(u)}{(u \underbrace{\ln\ln\cdots\ln}_{n-1} u \underbrace{\ln\ln\cdots\ln}_{n-2} u \cdots \ln u)^{p-1} (\underbrace{\ln\ln\cdots\ln}_{n} u)^p} \right) = \infty,$$

where $n \in \mathbf{N} = \{1, 2, \cdots\}$.

Proof　Assume that

$$\limsup_{u\to\infty}\left(-\frac{f(u)}{(u \underbrace{\ln\ln\cdots\ln}_{n-1} u \underbrace{\ln\ln\cdots\ln}_{n-2} u \cdots \ln u)^{p-1} (\underbrace{\ln\ln\cdots\ln}_{n} u)^p} \right) \neq \infty .$$

Then $\exists K > 0$, $M_3 > 0$ such that

$$-f(u) \leqslant K(u \underbrace{\ln\ln\cdots\ln}_{n-1} u \underbrace{\ln\ln\cdots\ln}_{n-2} u \cdots \ln u)^{p-1} (\underbrace{\ln\ln\cdots\ln}_{n} u)^p),\ \forall u > M_3.$$

So

$$-f(u) < K((u \underbrace{\ln\ln\cdots\ln}_{n-1} u \underbrace{\ln\ln\cdots\ln}_{n-2} u \cdots \ln u)^{p-1} (\underbrace{\ln\ln\cdots\ln}_{n-1} u \underbrace{\ln\ln\cdots\ln}_{n-2} u \cdots \ln u$$

$$+ \underbrace{\ln\ln\cdots\ln u}_{n-1} + \underbrace{\ln\ln\cdots\ln u}_{n-2} + \cdots + \ln u + 1)(\underbrace{\ln\ln\cdots\ln u}_{n})^{p}$$
$$+ (u\,\underbrace{\ln\ln\cdots\ln u}_{n-1}\,\underbrace{\ln\ln\cdots\ln u}_{n-2}\cdots\ln u)^{p-1}(\underbrace{\ln\ln\cdots\ln u}_{n})^{p-1}), \text{ for } u > M_3 > p.$$

This implies that

$$-F(u) = \int_0^u -f(w)\,\mathrm{d}w = \int_0^{M_3} -f(w)\,\mathrm{d}w + \int_{M_3}^u -f(w)\,\mathrm{d}w \leqslant$$
$$-F(M_3) + \frac{K}{p}(u\,\underbrace{\ln\ln\cdots\ln u}_{n-1}\,\underbrace{\ln\ln\cdots\ln u}_{n-2}\cdots\ln u)^{p}(\underbrace{\ln\ln\cdots\ln u}_{n})^{p}$$
$$-\frac{K}{p}(M_3\,\underbrace{\ln\ln\cdots\ln M_3}_{n-1}\,\underbrace{\ln\ln\cdots\ln M_3}_{n-2}\cdots\ln M_3)^{p}(\underbrace{\ln\ln\cdots\ln M_3}_{n})^{p}, \text{ for } u > M_3.$$

Let $\rho \in I$ and

$$K_1 = F(\rho) - F(M_3) - (K/p)(M_3\,\underbrace{\ln\ln\cdots\ln M_3}_{n-1}\,\underbrace{\ln\ln\cdots\ln M_3}_{n-2}\cdots\ln M_3)^{p}(\underbrace{\ln\ln\cdots\ln M_3}_{n})^{p},$$

then we obtain

$$F(\rho) - F(u) \leqslant K_1 + K/p(u\,\underbrace{\ln\ln\cdots\ln u}_{n-1}\,\underbrace{\ln\ln\cdots\ln u}_{n-2}\cdots\ln u)^{p}(\underbrace{\ln\ln\cdots\ln u}_{n})^{p}. \tag{9}$$

Now, $\exists\, M_4 > p$ such that

$$K/p(u\,\underbrace{\ln\ln\cdots\ln u}_{n-1}\,\underbrace{\ln\ln\cdots\ln u}_{n-2}\cdots\ln u)^{p}(\underbrace{\ln\ln\cdots\ln u}_{n})^{p} \geqslant K_1, \text{ for } u > M_4. \tag{10}$$

For $M = \max\{M_3, M_4\}$, (9) and (10) imply

$$F(\rho) - F(u) < 2K/p(u\,\underbrace{\ln\ln\cdots\ln u}_{n-1}\,\underbrace{\ln\ln\cdots\ln u}_{n-2}\cdots\ln u)^{p}(\underbrace{\ln\ln\cdots\ln u}_{n})^{p}, \text{ for } u > M. \tag{11}$$

Without loss of generality, we may assume $M > \max\{\rho, p\}$ and obtain from Eq.(11) that

$$G(\rho) = 2((p-1)/p)^{1/p}\int_\rho^\infty \frac{\mathrm{d}u}{(F(\rho)-F(u))^{1/p}} \geqslant 2((p-1)/p)^{1/p}\int_M^\infty \frac{\mathrm{d}u}{(F(\rho)-F(u))^{1/p}}$$
$$\geqslant 2^{(p-1)/p}\left(\frac{p-1}{K}\right)^{1/p}\int_M^\infty \frac{\mathrm{d}u}{u\,\underbrace{\ln\ln\cdots\ln u}_{n-1}\,\underbrace{\ln\ln\cdots\ln u}_{n-2}\cdots\ln u(\underbrace{\ln\ln\cdots\ln u}_{n})}$$
$$= 2^{(p-1)/p}\left(\frac{p-1}{K}\right)^{1/p}\underbrace{\ln\ln\cdots\ln u}_{n+1}\Big|_M^\infty = \infty.$$

So, $G(\rho)$ does not exist if

$$\limsup_{u\to\infty}(-f(u)/(u\,\underbrace{\ln\ln\cdots\ln u}_{n-1}\,\underbrace{\ln\ln\cdots\ln u}_{n-2}\cdots\ln u)^{p-1}(\underbrace{\ln\ln\cdots\ln u}_{n})^{p}) \neq \infty$$

and Theorem 2 follows from Lemma 1.

Theorem 3 Assume that there exists $\alpha > p-1$ such that $f(u)$ satisfies

$$\lim_{u\to\infty}\frac{-f(u)}{u^{\alpha}} = \infty, \tag{12}$$

then there exist solutions to problem (1) ~ (2) for some $\lambda > 0$.

To prove Theorem 3, we need a technical lemma.

Lemma 4 Let f satisfy Eq.(12) and $\rho \in [\rho_1, \rho_2] \subset I$. Then there exists $M > 0$ such that

$$F(\rho) - F(u) \geqslant K u^{\alpha+1}, \text{ for } u > M \quad \rho \in [\rho_1, \rho_2].$$

Proof If f satisfies Eq.(12), it is easy to see that there exists a constant $M_1 > 0$ such that

$$-f(u) \geqslant u^{\alpha}, \text{ for } u \geqslant M_1,$$

which implies

$$-F(u) = -F(M_1) + \int_{M_1}^u -f(w)\,\mathrm{d}s \geqslant -F(M_1) + \frac{u^{\alpha+1}}{\alpha+1} - \frac{M_1^{\ \alpha+1}}{\alpha+1} \qquad \forall\, u \geqslant M_1.$$

Letting $K_1 = -F(M_1) - \dfrac{M_1^{\ \alpha+1}}{\alpha+1} + \inf_{\rho\in[\rho_1,\rho_2]}F(\rho)$ and $K_2 = \dfrac{1}{\alpha+1} > 0$, we obtain

$$F(\rho)-F(u)\geqslant K_1+K_2u^{\alpha+1}\qquad\forall u\geqslant M_1,\forall\rho\in[\rho_1,\rho_2].\tag{13}$$

Now, $\exists M_2>0$ such that $(K_2/2)u^{\alpha+1}\geqslant -K_1,\forall u\geqslant M_2$ which implies

$$K_1+K_2u^{\alpha+1}\geqslant 1/2K_2u^{\alpha+1},\forall u\geqslant M_2.\tag{14}$$

Letting $M=\max\{M_1,M_2\}$ and $K=1/2K_2$, we have by (13) ~ (14) that

$$F(\rho)-F(u)\geqslant Ku^{\alpha+1},\forall u\geqslant M,\forall\rho\in[\rho_1,\rho_2]$$

and the lemma is proved.

Proof of Theorem 3 Suppose that f satisfies Eq.(12), let $\rho\in I$. Since I is open, $\exists\rho_1,\rho_2\in I$ such that $\rho\in(\rho_1,\rho_2)$ and $[\rho_1,\rho_2]\subset I$. Let K and M be as in Lemma 4. Note that

$$G(\rho)=2((p-1)/p)^{1/p}\int_\rho^\infty\frac{\mathrm{d}u}{(F(\rho)-F(u))^{1/p}}.$$

Converges if and only if

$$\int_\rho^{\rho+\delta}\frac{\mathrm{d}u}{(F(\rho)-F(u))^{1/p}}<\infty;\ 0<\delta<\rho_2-\rho$$

and

$$\int_M^\infty\frac{\mathrm{d}u}{(F(\rho)-F(u))^{1/p}}<\infty$$

where we assume without loss of generality that $M>\rho_2$. Now, $[\rho_1,\rho_2]\subset I$ implies $L=\inf_{z\in[\rho_1,\rho_2]}(-f(z))>0$. This combined with the mean value theorem implies

$$F(\rho)-F(u)=-f(z)(u-\rho)\geqslant L(u-\rho)\qquad\forall u\in[\rho_1,\rho_2]$$

Since $\rho+\delta<\rho_2$, we have

$$\int_\rho^{\rho+\delta}\frac{\mathrm{d}u}{(F(\rho)-F(u))^{1/p}}\leqslant\frac{1}{L^{1/p}}\int_\rho^{\rho+\delta}\frac{\mathrm{d}u}{(u-\rho)^{1/p}}=\frac{p\delta^{(p-1)/p}}{L^{1/p}(p-1)}<\infty.$$

Also, using Lemma 4 we have

$$\int_M^\infty\frac{\mathrm{d}u}{(F(\rho)-F(u))^{1/p}}\leqslant\frac{1}{K^{1/p}}\int_M^\infty\frac{\mathrm{d}u}{u^{(\alpha+1)/p}}=\frac{\alpha-p+1}{pK^{1/p}M^{(\alpha-p+1)/p}}<\infty.$$

Hence $G(\rho)$ is well defined $\forall\rho\in I$, and by Lemma 1 there exists a solution to (1) ~ (2) for $\lambda=[G(\rho)]^p$ given and any $\rho\in I$.

Example Consider the problem

$$\begin{cases}-(\Phi_p(u'))'=-\lambda u^\alpha\\ \lim\limits_{x\to0^+}u(x)=\infty=\lim\limits_{x\to1^-}u(x),\end{cases}$$

where $\alpha>p-1$ is a given constant. Since $f(u)=-u^\alpha$ is decreasing for $u>0$ and $f(0)=0$. Note that $F(u)=-u^{\alpha+1}/(\alpha+1)$ which implies

$$G(\rho)=2((p-1)/p)^{1/p}(\alpha+1)^{1/p}\int_\rho^\infty\frac{\mathrm{d}u}{(u^{\alpha+1}-\rho^{\alpha+1})^{1/p}}.$$

Letting $v=(u^{\alpha+1}-\rho^{\alpha+1})^{1/p}$ we obtain

$$\frac{\mathrm{d}u}{(u^{\alpha+1}-\rho^{\alpha+1})^{1/p}}=\frac{pv^{p-2}\mathrm{d}v}{(\alpha+1)(v^p+\rho^{\alpha+1})^{\alpha/(\alpha+1)}}$$

so that

$$G(\rho)=2(\frac{(p-1)(\alpha+1)}{p})^{1/p}\int_0^\infty\frac{pv^{p-2}\mathrm{d}v}{(\alpha+1)(v^p+\rho^{\alpha+1})^{\alpha/(\alpha+1)}}.$$

Now, Letting $v=\rho^{(\alpha+1)/p}\tan^{2/p}\theta$ we obtain

$$G(\rho)=4(\frac{p-1}{p(\alpha+1)^{p-1}\rho^{\alpha-p+1}})^{1/p}\int_0^{\pi/2}\tan^{(p-2)/p}\theta(\sec\theta)^{2/(\alpha+1)}\mathrm{d}\theta$$

$$=4(\frac{p-1}{p(\alpha+1)^{p-1}\rho^{\alpha-p+1)}})^{1/p}\frac{\Gamma(\frac{\alpha-p+1}{p(\alpha+1)})\Gamma(\frac{p-1}{p})}{2\Gamma(\frac{\alpha}{\alpha+1})}.$$

Letting

$$M_\alpha = 4\left(\frac{p-1}{p(\alpha+1)^{p-1}}\right)^{1/p} \frac{\Gamma\left(\frac{\alpha-p+1}{p(\alpha+1)}\right)\Gamma\left(\frac{p-1}{p}\right)}{2\Gamma\left(\frac{\alpha}{\alpha+1}\right)},$$

then, we have

$$G(\rho) = \frac{M_\alpha}{\rho^{(\alpha-p+1)^{1/p}}}.$$

Here M_α is finite since f satisfies (12) for $\alpha > p-1$ and thus, $G(\rho)$ is finite. From $\lim\limits_{\rho\to\infty} G(\rho) = 0^+$, $\lim\limits_{\rho\to 0^+} G(\rho) = \infty$, and $G(\rho)$ is strictly decreasing on $(0,\infty)$. These results imply that G is a bijective mapping from $(0,\infty)$ onto $(0,\infty)$. Thus, given $\lambda>0$ there exists a unique $\rho>0$ such that $G(\rho) = \lambda^{1/p}$. Since $I=(0,\infty)$, $G(\rho)$ is not defined for $\rho\leqslant 0$. Hence, by lemma 1, there is a unique nonnegative solution to (1) ~ (2) for each $\lambda>0$. Hence, $\lambda = M_\alpha^p\rho^{p-1-\alpha}$.

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一类拟线性常微分方程爆破解的存在性

杨会生[1],杨作东[2]

(1.河南师范大学数学与信息科学学院,453002,河南,新乡)

(2.南京师范大学数学与计算机科学学院,210097,江苏,南京)

[摘要] 本文得到了两点边值问题

$$-(\Phi_p(u'))' = \lambda f(u(x));\ 0<x<1$$

$$\lim_{x\to 0^+} u(x) = \infty = \lim_{x\to 1^-} u(x),$$

非负解存在的必要条件和充分条件,这里 $\lambda>0$ 是参数,f 是一个光滑函数.

[关键词] 爆破非负解,两点边值问题,存在性

[责任编辑:陆炳新]