

Closed Subspaces of Star-lindelöf Spaces

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Abstract In this paper , we prove the following statements : (1) Every $1-\frac{1}{2}$ -star-Lindelöf space can be embedded in a star-Lindelöf space as a closed G_δ subset . (2) A $1-\frac{1}{2}$ -star-Lindelöf space X can be embedded as a regular closed subset into a starLindelöf space if and only if there exists a nowhere dense closed subset F of X such that any closed subset C disjoint with F is a relative star-Lindelöf subset of X .

Key words absolutely star-Lindelöf , star-Lindelöf , $1-\frac{1}{2}$ -starLindelöf

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0 Introduction

By a space , we mean a topological space . A space X is absolutely star-Lindelöf^[1] if for every open cover \mathcal{U} of X and every dense subset D of X , there exists a countable subset $F \subseteq D$ such that $S(\ F , \mathcal{U}) = X$, where $S(\ F , \mathcal{U}) = \bigcup \{ U \in \mathcal{U} : U \cap F \neq \emptyset \}$. It is clear that every Lindelöf space is absolutely star-Lindelöf .

A space X is star-Lindelöf^[2] if for every open cover \mathcal{U} of X , there exists a countable subset $F \subseteq X$ such that $St(\ F , \mathcal{U}) = X$.

A space X is $1-\frac{1}{2}$ -starLindelöf^[2] if for every open cover \mathcal{U} of X , there exists a countable subset ν of \mathcal{U} such that $S(\ \bigcup \nu , \mathcal{U}) = X$. From the above definitions , it is not difficult to see that every absolutely star-Lindelöf space is star-Lindelö and every star-Lindelö space is $1-\frac{1}{2}$ -starLindelöf .

In [3] , a star-Lindelöf space is called strongly star-Lindelöf and a $1-\frac{1}{2}$ -star-Lindelöf space is called starLindelöf . In [4] , Song and Shi showed that every star-Lindelöf space can be represented as a closed G_δ -subset in a Tychonff star-Lindelö space ; In [5] , Song showed that there exists a star-Lindelöf Tychonoff space having a zero-set which is not $1-\frac{1}{2}$ -starLindelöf . Thus , it is natural for us to consider the following two questions :

Question 1 Is it true that every $1-\frac{1}{2}$ -starLindelöf space can be embedded into some star Lindelöf space as a closed subset ? as a closed G_δ -subset ?

Question 2 Characterize those $1-\frac{1}{2}$ -starLindelöf spaces which can be represented as regular closed subsets in star Lindelöf spaces ?

The purpose of this paper is to prove the two statements stated in the abstract which give answers to the above two questions .

Moreover , let ω denote the first infinite cardinal and c the cardinality of the continuum . Other terms and symbols which we do not define will be used as in [6] .

1 Closed Subsets of Star-Lindelöf Spaces

Example 1.1 There exists a $1/\frac{1}{2}$ -star-Lindelöf Tychonoff space which is not star-Lindelöf.

Proof Let D be a discrete space of cardinality c . let

$$X = (\beta(D) \times (\omega + 1)) \setminus ((\beta(D) \setminus D) \times \{\omega\}).$$

To show that X is $1/\frac{1}{2}$ -starLindelöf. Let \mathcal{U} be an open cover of X . Since $\beta(D) \times \omega$ is a σ -compact dense subset of X , then there exists a countable subfamily ν of \mathcal{U} such that $\beta(D) \times \omega \subseteq \bigcup \nu$, hence $X = St(\bigcup \nu, \mathcal{U})$. This shows that X is $1/\frac{1}{2}$ -starLindelöf.

To show S_1 is not star-Lindelöf. Since $|D| = c$, we can enumerate D as $\{d_\alpha : \alpha < c\}$. For each $\alpha < c$, let $U_\alpha = [0, \omega] \times \{d_\alpha\}$ and $V = \beta(D) \times [0, \omega)$. Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < c\} \cup \{V\}$$

of X . Let F be a countable subset of X . Then, there exists a $\alpha < c$ such that $U_\alpha \cap F = \emptyset$. Thus, $d_\alpha, \omega \notin St(F, \mathcal{U})$, which shows that X is not star Lindelöf.

In the following, we give a construction that can embed a $1/\frac{1}{2}$ -starLindelöf space into a star-Lindelöf space as a G_δ -closed subset. For a Tychonoff space X , let $\beta(X)$ denote the Čech-Stone compactification of X .

Theorem 1.2 Every $1/\frac{1}{2}$ -starLindelöf space can be embedded in a star-Lindelöf space as a G_δ -colsed subset.

Proof The proof is based on the idea of that Bonanzinga and Matveev [7, Theorem 1]. Let X be a $1/\frac{1}{2}$ -starLindelöf space and $\tau(X)$ the topology of X . For each $U \in \tau(X)$, let $\mathcal{U}_U = \{V \in \mathcal{U} : V \cap U \neq \emptyset\}$. Let $\mathcal{L} = \{\mathcal{U}_U : U \in \mathcal{U}\}$, and consider \mathcal{L} the discrete space. Let $A = \beta(\mathcal{L}) \times \omega$ and define $\mathcal{S}(X) = X \cup A$. We topologize $\mathcal{S}(X)$ as follows: The subspace A has the usual product topology and is an open subspace of $\mathcal{S}(X)$; and a basic neighborhood of $x \in X$ in $\mathcal{S}(X)$ is a set of the form

$$G_{U,n} = U \cup (\text{cl}_{\beta(\mathcal{L})} \mathcal{U}(U) \times \{m : n < m < \omega\})$$

for an open neighborhood U of x in X and $n < \omega$, where

$$\mathcal{U}(U) = \{\mathcal{U}_V \in \mathcal{L} : \exists V \in \mathcal{U} \text{ s.t. } V \cap U \neq \emptyset\}.$$

Then, it is not difficult to check that X is a closed G_δ subset of $\mathcal{S}(X)$ by the definition of the topology. Thus, it suffices to show that $\mathcal{S}(X)$ is star-Lindelöf. Let \mathcal{U} be an open cover of $\mathcal{S}(X)$. Without loss of generality, we can assume that \mathcal{U} consists of basic open sets. Since A is σ -compact, there exists a countable subset $F' \subseteq A$ such that $A \subseteq St(F', \mathcal{U})$. On the other hand, if we put $\mathcal{U}_X = \{U \cap X : U \in \mathcal{U}\}$, then \mathcal{U}_X is an open cover of X . Since X is $1/\frac{1}{2}$ -starLindelöf, then there exists a countable subset ν of \mathcal{U}_X such that $St(\bigcup \nu, \mathcal{U}_X) = X$. Let $F'' = \{U_{V,n} : n < \omega, V \in \nu\} \subseteq A$. To show that $X \subseteq St(F'', \mathcal{U})$, let $x \in X$ be fixed. Then, there exist $U \in \mathcal{U}_X, n < \omega$ and $V \in \nu$ such that

$$x \in U \subseteq G_{U,n} \in \mathcal{U} \text{ and } V \cap U \neq \emptyset.$$

Since $U_{V,n+1} \in F'' \cap G_{U,n}, x \in St(F'', \mathcal{U})$. Hence, $X \subseteq St(F'', \mathcal{U})$. If we put $F = F' \cup F''$, then $\mathcal{S}(X) = St(F, \mathcal{U})$, which proves that $\mathcal{S}(X)$ is star-Lindelöf.

Bonanzinga^[1] showed that a regular-closed subset of a star-Lindelöf space need not be star-Lindelöf. Moreover, Song^[5] showed that a regular colsed subset of a star-Lindelöf space need not be $1/\frac{1}{2}$ -starLindelöf. Thus, we consider the above question 2 naturally. For answering the question, we introduce the following definition:

Definition For a topological space X , a subset F of X is called a relative star-Lindelöf of X provided that if for

every open cover \mathcal{U} of X , there exists a countable subset C of X such that $F \subseteq St(C, \mathcal{U})$.

Theorem 1.3 A $1/\frac{1}{2}$ -starLindelöf space X can be embedded as a regular closed subset into a starLindelöf space if and only if there exists a nowhere dense closed subset F of X such that any closed subset C disjoint with F is a relative star-Lindelöf subset of X .

Proof Necessity. Suppose that X is a $1/\frac{1}{2}$ -starLindelöf space which is embedded as a regular closed subset into a star-Lindelöf space Y . Then, there exists an open subset U of Y such that $X = \overline{U}$. Let $F = X \setminus U$. Then, F is a nowhere dense closed subset of X . Suppose that E is a closed subset of X disjoint with F and \mathcal{U} is an open cover of X . Put $\nu = \mathcal{U} \setminus (X \setminus F) \cup \{Y \setminus E\}$. Then ν is an open cover of Y . Hence, there exists a countable subset C' of Y such that $St(C', \nu) = Y$, since Y is star-Lindelöf. Let $C = C' \cap X$. Then C is a countable subset of X and $E \subseteq St(C, \mathcal{U})$. Thus, E is a relative star-Lindelöf subset of X .

Sufficiency. Suppose that X is a $1/\frac{1}{2}$ -starLindelöf space and there exists a nowhere dense closed subset F such that any closed C with $F \cap C = \emptyset$ is a relative star-Lindelöf subset of X . Put $Y' = (X \times \{0\}) \cup S(X)$, where $S(X)$ is defined as in the proof of Theorem 1.2. Let Y be the quotient space of Y' by identifying every pair of points $x \cdot 0$ in $X \times \{0\}$ and x in X for every $x \in F$. Let $\pi: Y' \rightarrow Y$ be the quotient map. It is easy check that $\pi(X \times \{0\})$ is a regular-closed subspace of Y .

Next, it suffices to show that Y is star-Lindelöf. Let \mathcal{U} be an open cover of Y . Then, there exists a countable subset E_0 of $\pi(S(X))$ such that $\pi(S(X)) \subseteq St(E_0, \mathcal{U})$, since $\pi(S(X))$ is homeomorphic to $S(X)$ and $S(X)$ is star-Lindelöf by Theorem 1.2. Let $G = Y \setminus St(E_0, \mathcal{U})$. Then, G is a close subset of $\pi(X)$. Hence, G is a relative star-Lindelöf subset of $\pi(X)$, since $\pi(F)$ is homeomorphic to F and $G \cap \pi(F) = \emptyset$. There exists a countable subset E_1 of G such that $G \subseteq St(E_1, \mathcal{U})$. If we put $E = E_0 \cup E_1$, then $Y = St(E, \mathcal{U})$, hence, Y is star-Lindelöf, which completes the proof.

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Star-Lindelöf 空间的闭子空间

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[摘要] 证明了下面两个结论 (1) 每个 $1/\frac{1}{2}$ -star-Lindelöf 空间都能作为一个闭子集 G_0 被嵌入在一个 star-Lindelöf 空间中.(2) 一个 $1/\frac{1}{2}$ -star-Lindelöf 空间 X 能被作为一个正则的闭子集嵌入到一个 star-Lindelöf 空间中当且仅当存在一个 X 的处处不稠密的闭子集 F 使得任何与 F 不相交的闭子集 C 都是 X 的一个相对 star-Lindelöf 子集.

[关键词] 绝对 star-Lindelöf , star-Lindelöf , $1/\frac{1}{2}$ -star-Lindelöf [责任编辑 : 陆炳新]